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Implicit Two Step Continuous Hybrid Block Methods with Four Off-Steps Points for Solving Stiff Ordinary Differential Equation

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***Abstract*— In this paper, a self starting two step continuous block hybrid formulae (CBHF) with four Off-step points is developed using collocation and interpolation procedures. The CBHF is then used to produce multiple numerical integrators which are of uniform order and are assembled into a single block matrix equation. These equations are simultaneously applied to provide the approximate solution for the stiff ordinary differential equations. The order of accuracy and stability of the block method is discussed and its accuracy is established numerically.

***Keywords*— Collocation and Interpolation, Continuous Hybrid Block Formulae, Off-Step Points, Stability, Stiff ODEs.

I. Introduction

CONSIDER the initial value problem of ordinary differential equation:

$$y' = f(x,y) \quad , \qquad y(x_o) = y_0 \tag{1}$$

We seek a solution in the range $a \le x \le b$, where a and b are finite, and we assume that f satisfies the conditions which guarantee that the problem has a unique continuously differentiable solution, which we shall indicate by $y(x)$. Consider the sequence of points $\{x_n\}$ defined by $x_n = a + nh$, $n = 0, 1, 2, \dots, \frac{b-a}{h}$. Where the parameter h *is* constant.

An essential property of the majority of computational methods for the solution of (1) is that of discretization, that is: we seek an approximate solution, not on the continuous interval $a \le x \le b$ but on the discrete point set $\{x_n\}$. The k-step linear multistep method (LMM) for the solution of (1) is generally written as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j \; f_{n+j} \tag{2}$$

Which has $2k+1$ unknown $\alpha's$ and $\beta's$ and therefore can be of order $2k$. But according to Dahlquist [5], the order of (2) cannot exceed k + 1 (k is odd) or k + 2 (k is even) for the method to be stable. Several authors such as Gear [7], Gragg and Stetter [9], and Butcher [3] proposed modified forms of (2) which were shown to overcome the Dahlquist barrier theorem. These methods, known as hybrid methods were obtained by incorporating off-step points in the derivation process.

We define a k-step continuous hybrid formula to be of the type:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = \; h \sum_{j=0}^{k+v} \beta_{rj} \;\; f_{n+jr} \tag{3}$$

Where h is the stepsize ,k is the step number, v is the number of off-points , $\alpha_k = 1$, α_j, β_{rj} are unknown constants which are uniquely determined. Hybrid methods have been considered by Gupta [10], Lambert [14], and Kohfeld and Thompson [13] and are of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j \; f_{n+j} + h \beta_v \; f_{n+1-r} \quad , r \in (0,1) \tag{4}$$

which were shown to be of order up to 2k+2. However, Gupta [10] noted that the design of algorithms for (4) is more tedious due to the occurrence of f_{n+1-r} in (4) which increases the number of predictors needed to implement the method. The hybrid method proposed in this paper as in [11] does not share this disadvantage since it is self-starting.

The algorithm is based on interpolation and collocation, see Lie and Norsett [15], Atkinson [1], Onumanyi et al [17], and Gladwell and Sayers [8].

We observe that block methods were first introduced by Milne [16] for use only as a means of obtaining starting values for predictor-corrector algorithms and has since then been developed by several researchers (see [18,19,20]), for general use.

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To this end, our continuous representation generates a main discrete hybrid method and four additional methods which are combined and implemented as a block method which simultaneously generates approximations y_{n+2-jr} to the exact solutions $y(x_{n+2-jr})$, $j = 0, , 1, 2, ..., 6$ and $r = \frac{1}{3}$. Without loss of generality, $y_{n+\frac{7}{3}}$, $y_{n+\frac{8}{3}}$, y_{n+3}, $y_{n+\frac{10}{3}}$, $y_{n+\frac{11}{3}}$, y_{n+4} are obtained in the next block using y_{n+2} as the starting value and this does not affect the subsequent points. Thus, the order of the algorithm is maintained.

In this paper our aim is to generate a two step implicit continuous hybrid block Methods with four off-steps Points and to demonstrate the efficiency in its implementation on stiff ODEs

II. Derivation of the Method

In this section, our objective is to derive the main hybrid block method of the form

$$\sum_{j=1}^{2} \alpha_j y_{n+j} = h \sum_{j=0}^{6} \beta_{jr} f_{n+jr} \tag{5}$$

α_j, β_{rj} are unknown constants and r is chosen from the interval
(0, 1). In order to obtain (5), we proceed by seeking an approximate of the exact solution y(x) by assuming a continuous solution Y(x) of the form

$$Y(x) = \sum_{j=0}^{p+q-1} l_j \varphi_j(x) \tag{6}$$

Such that $x \in [x_0, X_n]$, l_j are unknown coefficients and $\varphi_j(x)$ are polynomial basis function of degree $p+q-1$, where the number of interpolation points p and the distinct collocation point q are respectively chosen to satisfy $1 \le p < k$ and $q > 0$.

The integer $k \ge 1$ denotes the step number of the method. We thus construct a k-step block multistep method by imposing the following conditions

$$Y(x_{n+j}) = y_{n+j}, \quad j = 1, 2, 3 \dots p-1 \tag{7}$$

$$Y'(x_{n+2-jr}) = f_{n+2-jr}, \quad j = 0, 1, 2, 3 \dots q-1 \tag{8}$$

Where $r \in (0, 1)$, y_{n+2-jr} is the numerical solution for the exact solution $y(t_{n+2-jr})$, $f_{n+2-jr} = f(x_{n+2-jr})$ and n is the grid index. It should be noted that equations (7) and (8) leads to a system of $p+q$ equations which must be solved to obtain the coefficient l_j. Our k step continuous hybrid method is obtained by substituting these values of l_j into equation (4). After some algebraic computation, our method yields the expressed in the form

$$Y(t) = \sum_{j=1}^{p-1} \alpha_j y_{n+j} + h \sum_{j=0}^{q} \beta_{jr} f_{n+jr} \tag{9}$$

which is used to generate the main discrete hybrid block method of the form (3) .

For k=2, taking $r = \frac{1}{3}$, $p = 2$, $q = 6$, $\varphi_i(x) = x^i$, $i = 0, 1, \dots, 7$ and thus intepolating (9) at $x = \left\{ x_n, x_{n+\frac{1}{3}}, x_{n+\frac{2}{3}}, x_{n+\frac{4}{3}}, x_{n+\frac{5}{3}}, x_{n+2} \right\}$ we generate the following continuous hybrid block method

$$\left.\begin{aligned}
y_n &= y_{n+1} + \frac{1}{6720}\left[-685h f_n - 3240h f_{n+\frac{1}{3}} - 1161h f_{n+\frac{2}{3}} - 2176h f_{n+1} + 729h f_{n+\frac{4}{3}} - 216h f_{n+\frac{5}{3}} + 29h f_{n+2}\right] \\
y_{n+\frac{1}{3}} &= y_{n+1} + \frac{1}{11340}\left[37h f_n - 1398h f_{n+\frac{1}{3}} - 4863h f_{n+\frac{2}{3}} - 1328h f_{n+1} - 33h f_{n+\frac{4}{3}} + 30h f_{n+\frac{5}{3}} - 5h f_{n+2}\right] \\
y_{n+\frac{2}{3}} &= y_{n+1} + \frac{1}{181440}\left[-271h f_n + 2760h f_{n+\frac{1}{3}} - 30819h f_{n+\frac{2}{3}} - 37504h f_{n+1} + 6771h f_{n+\frac{4}{3}} - 1608h f_{n+\frac{5}{3}} + 191h f_{n+2}\right] \\
y_{n+\frac{4}{3}} &= y_{n+1} + \frac{1}{181440}\left[-191h f_n + 1608h f_{n+\frac{1}{3}} - 6771h f_{n+\frac{2}{3}} + 37504h f_{n+1} + 30819h f_{n+\frac{4}{3}} - 2760h f_{n+\frac{5}{3}} + 271h f_{n+2}\right] \\
y_{n+\frac{5}{3}} &= y_{n+1} + \frac{1}{11340}\left[5h f_n - 30h f_{n+\frac{1}{3}} + 33h f_{n+\frac{2}{3}} + 1328h f_{n+1} + 4863h f_{n+\frac{4}{3}} + 1398h f_{n+\frac{5}{3}} - 37h f_{n+2}\right] \\
y_{n+2} &= y_{n+1} + \frac{1}{6720}\left[-29h f_n + 216h f_{n+\frac{1}{3}} - 729h f_{n+\frac{2}{3}} + 2176h f_{n+1} + 1161h f_{n+\frac{4}{3}} + 3240h f_{n+\frac{5}{3}} + 685h f_{n+2}\right]
\end{aligned}\right\} \tag{10}$$

The methods in (10) are combined and implemented as a block method which simultaneously generates approximations y_{n+2-jr} to the exact solutions $y(x_{n+2-jr})$, $j = 0, , 1, 2, ..., 6$ and $r = \frac{1}{3}$ on $I_N : a < x_0 < x_2 < ... < x_{N-2} < x_N = b$, $h = x_{n+1} - x_n$, $n = 0, 2, \dots N-2$. Where I_N is the partition of $[a, b]$ and h is the constant step size of the partition I_N. The integrators (10) is a one block two step hybrid methods of order $(7, 7, 7, 7, 7, 7)^T$ with error constants

$$C_{p+1} = C_8 = \left(-\frac{1}{653184}, \frac{1}{4960116}, -\frac{191}{793618560}, -\frac{191}{793618560}, \frac{1}{4960116}, -\frac{1}{653184} \right)^T$$

III. STABILITY ANALYSIS

Dahlquist [5] investigated the special stability problem connected with stiff equations, he introduced the concept of A-stability, and gave the following definition as follow:

Definition 3.1 [5,6] **:** The stability region *R* associated with a multistep formula is defined as
the set *R* = { $h\lambda$:A numerical formula applied to $y' = \lambda y$, $y(t_0) = \gamma$, with constant stepsize h > 0 , produce a sequence (y_n) satisfying that $y_n \to 0$ as $n \to \infty$ }.

Definition 3.2 [5,6]: A numerical method is said to be A-Stable if its region of absolute stability contains the half- plain $\mathrm{Re}(\lambda h) < 0$

Definition 3.3[5,6]: A numerical method is said to be $A(\alpha)$-Stable $\alpha \in \left(0, \pi/2\right)$ if $S \supset S_\alpha = \{z : |\arg(-z)| < \alpha , z \neq 0\}$

Definition 3.4 [4]: The Block Method (8) is said to be **zero-stable** if the roots $R_{i,j} = 1(1)k$ of the first characteristic polynomial $\rho(R) = \det\left[\sum_{i=0}^{k} A_j R^{k-i}\right] = 0$, $A_0 = -I$, satisfies $|R_j| \leq 1$. If one of the roots is +1, we say this root is the **principal root** of $\rho(R)$.

The linear stability properties of the block formula (10) are determined through the application to the test equation

$$y' = \lambda y , \qquad \lambda < 0 \tag{11}$$

Application of (11) to (10) then gives the stability polynomials of the proposed method as:

$$\begin{aligned} P(z, \varpi) = {} & \varpi^5 - \varpi^6 - \frac{673 z\varpi^5}{1890} + \frac{4453 z\varpi^6}{1890} - \frac{6097657 z^2\varpi^5}{16934400} - \frac{1898377 z^2\varpi^6}{1128960} - \frac{14651533 z^3\varpi^5}{228614400} \\ & + \frac{110764093 z^3\varpi^6}{160030080} + \frac{7688567 z^4\varpi^5}{1143072000} - \frac{159704729 z^4\varpi^6}{1463132160} + \frac{6612929 z^5\varpi^5}{1714608000} + \frac{5498438599 z^5\varpi^6}{2304433152\ 00} \\ & + \frac{81077 z^6\varpi^5}{171460800} - \frac{219635473 z^6\varpi^6}{1152216576\ 00} = 0 \end{aligned} \tag{12}$$

From (12) we obtain the usual property of A-stability in the spirit of [2] , which requires that for all $z = h\lambda \in C^-$ and Re(z) < 0, $\rho(\varpi , z)$ must have a dominant eigenvalue ϖ_6 such that $|\varpi_6| < 1$. From our analysis we have that the eigenvalues $\{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5, \varpi_6\} = (0,0,0,0,0,\varpi_6)$ and the dominant eigenvalue ϖ_6 is a function of z given by

$$\varpi_6 = \frac{336\left(\begin{array}{r} 6858432000 - 2442182400\ z - 2469551085\ z^2 - 439545990 z^3 + 46131402 z^4 \\ + 26451716\ z^5 + 3243080 z^6 \end{array}\right)}{5\left(\begin{array}{r} 4608866304\ 00 - 1085887918\ 080 z + 7749934264\ 80 z^2 - 3190005878\ 40 z^3 + 5030698963\ 5 z^4 \\ - 1099687719\ 8 z^5 + 878541892 z^6 \end{array}\right)} \tag{13}$$

Clearly from (13) $\mathrm{Re}(z) < 0$, $\varpi_6 < 1$. Hence, the block method (10) is A-stable since its region of absolute stability contains the left half-plane, $\{z \in C \mid \mathrm{Re}(z) < 0\}$.

To determine zero stability, we substitute $z = h\lambda$ =0 to the equation (12) and we abtain

$$\varpi = \{0 , 0 , 0 , 0, 0, 1\}$$

Thus the continuous block hybrid formula (10) is zero stable. Since one of the roots is +1.

The absolute stability region S associated with the block method (10) are the set $S = \{z = h\lambda :$ for that z where the roots of the stability polynomial (12) are of moduli less than one $\}$

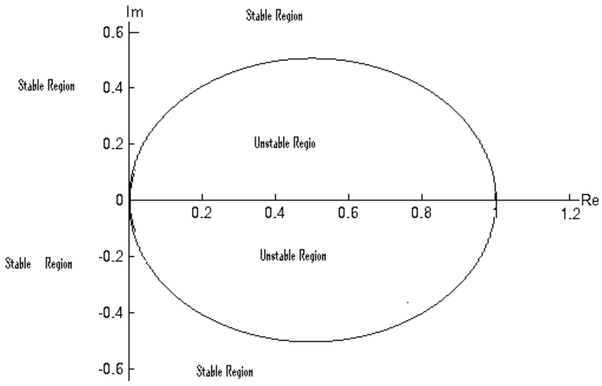

Fig 1. Region of absolute stability for two step continuous hybrid block formulae with two off-steps.

The stability region for the proposed methods lie outside the bounded region. Since all the region in the left half plane is in the stability region, the method is A-stable.

IV. NUMERICAL EXPERIMENT

This section deals with some numerical experiments, executed in MATLAB language with double precision arithmetic, which illustrate the result derived in the previous sections. For different choices of the constant step size h the absolute error at the endpoint X is obtained.

Example 1

Consider the stiffly linear problem

$$y_1' = -29998 y_1 - 59994 y_2 \qquad y_1(0) = 1$$
$$y_2' = 9999 y_1 + 19997 y_2 \qquad y_2(0) = 0$$

$$0 \leq t \leq 10$$

The eigenvalues of the systems are $\lambda_1 = -10000$ and $\lambda_2 = -1$ with exact solution

$$y_1(x) = \frac{1}{9999}(29997 e^{-10000x} - 19998 e^{-x})$$
$$y_2(x) = e^{-x} - e^{-10000x}$$

TABLE I EXAMPLE 1 ABSOLUTE ERROR FOR BLOCK HYBRID METHOD (1) WITH FOUR OFF-STEPS AT THE END POINT
T=10

h	error $i = \lvert y_i - y(X_i) \rvert$	
	$error1$	$error2$
0.01	8.26 x 10^{-15}	4.13 x10^{-15}
0.001	4.66 x 10^{-15}	2.33 x10^{-15}

Example 2

Consider the Stiffly nonlinear problem

$$y_1' = -\left(\varepsilon^{-1} + 2\right)y_1 + \varepsilon^{-1} y_2^2 \qquad y_1(0) = 1$$
$$y_2' = y_1 - y_2 - y_2^2 \qquad y_2(0) = 1$$
$$0 \le t \le 10$$

Where $\varepsilon = 10^{-6}$,the smaller ε is, the more serious the stiffness of the system. The exact solution is

$$y_1(x) = y_2^2(x) \quad , \qquad y_2(x) = e^{-x}$$

TABLE II EXAMPLE 2 ABSOLUTE ERROR FOR BLOCK HYBRID METHOD (11) WITH FOUR OFF-STEPS AT THE END POINT
T=10

h	error $i = \lvert y_i - y(X_i) \rvert$	
	$error1$	$error2$
0.1	4.5 x 10^{-15}	4.8 x 10^{-15}
0.01	1.4 x 10^{-16}	2.6 x10^{-15}

V. CONCLUSION

A continuous block hybrid formula with four off-step points has been proposed and implemented as a self starting method in block form for stiff ordinary differential equation. The good convergent and stability properties of our method therefore, makes it attractive for numerical solution of stiff problems We have demonstrated the accuracy of the block method on both linear and non linear problems as shown in tables 1 and 2 .

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