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Numerical solution for elliptical crack with developing cusps subject to shear loading

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Abstract—This paper study the behavior of the solution at the crack edges for an elliptical crack with developing cusps, Ω in the plane elasticity subjected to shear loading. The problem of finding the resulting shear stress can be formulated as a hypersingular integral equation over Ω and it is then transformed into a similar equation over a circular region, *D*, using conformal mapping. An appropriate collocation points are chosen on the region *D* to reduce the hypersingular integral equation into a system of linear equations with $(2N+1)(N+1)$ unknown coefficients, which will later be used in the determination of shear stress intensity factors and maximum shear stress intensity. Numerical solution for the considered problem are compared with the existing asymptotic solution, and displayed graphically. Our results give a very good agreement to the existing asymptotic solutions.

Keywords—elliptical crack, stress intensity factors, hypersingular integral equation, shear loading, conformal mapping.

I. Introduction

Natural crack occurring in practice are often initiated at corners and edges. Thus, the solution of plane cracks of arbitrary shape inside an infinite isotropic elastic medium has become an important topic in fracture mechanic. Elliptical crack as an example of crack geometry becomes a challenge for researchers in Fracture Mechanics and considerable attention have been devoted to that crack configuration due to its fundamental role in the studies of fracture susceptibility of flawed, three dimensional and engineering structure.

The problem of flat elliptical crack subject to a uniform shear loading can be found in Eshelby [1]. The problem was formulated with sufficient generality to include the flat elliptical crack subjected to constant normal or shear stress applied in arbitrary direction along the crack surface. Kassir and Sih [2] formulated the elliptical crack problem subjected to uniform shear as a mixed boundary problem and obtained the stress intensity factors along the crack border whilst the potential functions proposed by Segedin [3] was implemented by Smith and Sorensen [4] to generalize the solution to account for the effect of the non-uniform shear loading onto the stress intensity factors of an embedded flat elliptical crack. The shear stress are expressed in the form of a third degree polynomial and the sliding mode and tearing mode stress intensity factor along the crack border are obtained analytically. Later, Sorensen and Smith [5] adopted the alternative method to analyze the semi-elliptical crack in a plate under uniform tension and the stress field due to the free surface while Martin [6; 7] obtained the stress intensity factor for an elliptical crack with various polynomial stress field subject to normal and shear loadings. The stress intensity factor were obtained for constant, linear and quadratic stress fields. Roy and Chatterjee [8; 9] developed a new integral equation method for solving the integro-differential equation in the elliptical crack problem subjected to normal and shear loading. Lee [10] advocated the double Fourier transform to determine the stress intensity factors for an embedded elliptical crack in a plate of finite thickness subjected to shear stress. Further, Chen et al. [11; 12] also applied the integro-differential equation to evaluate the stress intensity factors for an elliptical crack with polynomial traction under shear loading. The solution is written as a power function where it is directly related to the crack opening displacement and the corresponding stress intensity factor along the crack border. Theotokoglou [13] employed the boundary integral equation method for solving the stress intensity factors of elliptical crack under shear loading while singular integral equation method was discussed by Noda and Kagita [14] to evaluate the stress intensity factor of a semi-elliptical crack. Meanwhile, Hachi et al. [15] applied the hybrid weight function technique and point weight function method to calculate the stress intensity factor for elliptical and semi elliptical cracks. Recently, Fang et al. [16] adopted the conformal mapping technique and the complex variable method to analyze the elastic interaction between an edge dislocation and a sharp crack emanating from a surface semi-elliptic hole. Moreover, the stress intensity factor at the tip of the crack and the image force acting on the edge dislocation are derived.

In present paper, we obtain the numerical solutions for an elliptical crack subject to shear loading via the solution of hypersingular integral equation by taking into consideration an appropriate Gauss quadrature rule for finite part integrals. Our numerical results give a very good agreement compared to the existing asymptotic solutions.

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II. Statement of Problem and Basic Equations

Suppose an elliptical crack enclosed in three dimensional infinite isotropic elastic body with material constants E and

ν, occupies the region, Ω which governed by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}<1; \quad z=0. \tag{1}$$

Assume that elliptical crack lie in the central plane and the center of the crack as the origin of the Cartesian coordinates, the z axes is perpendicular to the surface crack and the x and y axes aligned along the major, a and minor, b semi-axes of crack, respectively.

Now, let the faces Ω^+ and Ω^- of the elliptic crack be subjected to a constant force directed along the x axes acting tangentially in opposite direction at the arbitrary crack point (x_0, y_0), then, the problem is antisymmetric about $z=0$. Without lost of generality, suppose that the stress vanish at infinity and the body force are absent. Thus, the stress field can be found by considering the half-space $z\geq 0$ subjected to the following mixed boundary condition on its surface $z=0$:

$$\begin{aligned} \tau_{zz}(x,y) &= \tau_{yz}(x,y)=0 \\ \tau_{xz}(x,y) &= \frac{\mu}{1-\nu}q(x,y)\in\Omega \\ u_x(x,y,0) &= u_y(x,y,0)=0,(x,y)\in\partial\Omega \end{aligned} \tag{2}$$

where τ_{zz}, τ_{yz} and τ_{xz} are stress tensors, μ is shear modulus, ν is Poisson's ratio, $q(x,y)$ is the resultant force in the x direction and $\partial\Omega$ is the boundary of elliptical crack. Using the Sogmigliana formula, followed by integration by part gives [17]

$$\frac{-1}{4\pi}\int_\Omega\left\{\alpha\frac{\partial}{\partial x}\Big(\frac{1}{R}\Big)+\beta\frac{\partial}{\partial y}\Big(\frac{1}{R}\Big)\right\}d\Omega = q_x(x_0,y_0) \tag{3}$$

$$\frac{-1}{4\pi}\int_\Omega\left\{\alpha\frac{\partial}{\partial y}\Big(\frac{1}{R}\Big)-\beta\frac{\partial}{\partial x}\Big(\frac{1}{R}\Big)\right\}d\Omega = q_y(x_0,y_0) \tag{4}$$

for$(x_0,y_0)\in\Omega$, where $[u_x(x,y)]$ and $[u_y(x,y)]$ is the displacement discontinuity in u_x and u_y across the crack,

$$\alpha=\frac{\partial[u_x]}{\partial x}+\frac{\partial[u_y]}{\partial y} \quad\text{and}\quad \beta=(1-\nu)\left(\frac{\partial[u_x]}{\partial y}-\frac{\partial[u_y]}{\partial x}\right)$$

and R is the distance,

$$R=\sqrt{(x-x_0)^2+(y-y_0)^2}; \quad (x_0,y_0)\in\Omega$$

The resulting boundary terms involve the $[u_x(x,y)]$ and $[u_y(x,y)]$ are evaluated at the crack edge, and these equations are to be solved subject to

$$[u_x(x,y)]=0 \quad\text{and}\quad [u_y(x,y)]=0 \quad\text{for}\quad (x,y)\in\partial\Omega \tag{5}$$

where $\partial\Omega$ is boundary of Ω.

Integrating equations (3) and (4) by parts and using condition (5), making use of the relationship of Cauchy principle value and hypersingular equation [18] yields [19]

$$\neq\!\!\!\!\!\!\int_\Omega \frac{(2-\nu+3\nu\cos 2\Theta)[u_x]+3\nu\sin 2\Theta[u_y]}{R^3}d\Omega = 8\pi q_x(x_0,y_0) \tag{6}$$

$$\neq\!\!\!\!\!\!\int_\Omega \frac{3\nu\sin 2\Theta[u_x]+(2-\nu-3\nu\cos 2\Theta)[u_y]}{R^3}d\Omega = 8\pi q_y(x_0,y_0) \tag{7}$$

where the angle Θ is defined by

$$x-x_0=R\cos\Theta \qquad\text{and}\qquad y-y_0=R\sin\Theta.$$

The cross on the integral means the hypersingular, and it must be interpreted as a Hadamard finite part integral. Simplify (6) and (7), then, the problem of determine the displacement of crack faces subjected to (2) can be written as the hypersingular integral equation [20; 21]

$$\frac{1}{8\pi}\neq\!\!\!\!\!\!\int_\Omega \frac{2-\nu+3\nu e^{2j\Theta}}{R^3}w(x,y)\,d\Omega = q(x_0,y_0), \quad (x_0,y_0)\in\Omega \tag{8}$$

where $q(x,y)=q_x(x_0,y_0)$, $w(x,y)=[u_x]$ is the unknown crack opening displacement. Equation (8) is to be solved subject to $w=0$ on $\partial\Omega$. In next section, we will discuss on the transformation of equation (8) into a similar equation over a circular region, D and solved numerically for the physical problem.

III. Method Of Solution For Elliptical Crack

Assume that the plate Ω is elliptical shaped with small eccentricity, c,

$$c=\frac{\sqrt{a^2-b^2}}{a} \tag{9}$$

such that the Ω is written as

$$\Omega=\{(x,y): x^2+(1+c)^2y^2<a^2\}$$

with the major and minor axes of length are $2a(1+c)$ and $2a$, respectively. Now, the elliptical domain is transformed into a circular domain, D though the transformation [22]

$$(x,y)=(aX,bY) \tag{10}$$

and the Cartesian system is transformed to a polar coordinate system using

$$(X,Y)=(s\cos\varphi, s\sin\varphi) \quad (0\leq s<1, 0\leq\varphi<2\pi). \tag{11}$$

Employed pertubation method as described by [23], the unit circular, $D=|\zeta|<1$ where $\zeta=se^{i\varphi}$ is mapped conformally onto the the interior of ellipse, Ω

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad a^2-b^2=1, \quad z=x+iy$$

using $z=af(\zeta)$, [24, page 265]

$$f(\zeta)=a(\zeta+A(\zeta)) \tag{12}$$

where

$$A(\zeta)=\frac{1}{2}c\zeta(1+\zeta^2).$$

This gives the perturbation magnitude at the given position ζ and the boundary of Ω is given by $r=\partial\Omega$. Equation (12) was also adopted by Borodachev [25] in calculating the elliptical crack problem subjected to a point load applied at the center of crack. It is not difficult to see that the domain is circular if $c=0$ and tends to the elliptical shaped if the eccentricity is greater than zero, see Figure 1 for various choice of c. Substitute (12) into (8) gives [21]

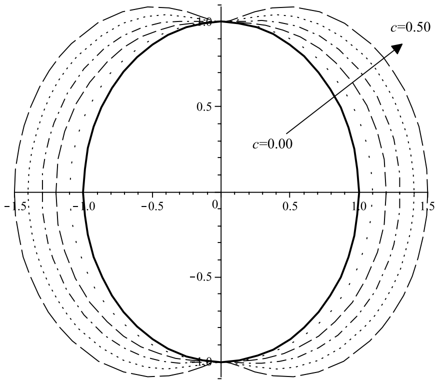

Fig. 1: The domain elliptical, Ω with different choices of c.

$$\begin{aligned}\frac{2-\nu+3\nu e^{2j\Theta}}{8\pi}\times\!\!\!\!\!\!\int_D \frac{W(\xi,\eta)}{S^3}d\xi d\eta \\ +\frac{2-\nu}{8\pi}-\!\!\!\!\!\!\int_D W(\xi,\eta)K^{(1)}(\zeta,\zeta_0)d\xi d\eta \\ +\frac{3\nu}{8\pi}\int_D W(\xi,\eta)K^{(2)}(\zeta,\zeta_0)d\xi d\eta \\ =Q(\xi_0,\eta_0);\ (\xi_0,\eta_0)\in D \end{aligned} \quad (13)$$

where the $K^{(1)}(\zeta,\zeta_0)$ and $K^{(2)}(\zeta,\zeta_0)$ are

$$\begin{aligned} K^{(1)}(\zeta,\zeta_0) &= \frac{|f'(\zeta)|^{\frac{3}{2}}|f'(\zeta_0)|^{\frac{3}{2}}}{|f(\zeta)-f(\zeta_0)|^3}e^{j(\delta-\delta_0)}-\frac{1}{|\zeta-\zeta_0|^3} \\ K^{(2)}(\zeta,\zeta_0) &= \frac{|f'(\zeta)|^{\frac{3}{2}}|f'(\zeta_0)|^{\frac{3}{2}}}{|f(\zeta)-f(\zeta_0)|^3}e^{j(2\Theta-\delta-\delta_0)}- \\ & \quad \frac{1}{|\zeta-\zeta_0|^3}e^{2j\Phi}. \end{aligned}$$

This hypersingular integral equation over a circular disc D is to be solved subject to $W=0$ on $s=1$ and $K^{(1)}$ is a Cauchy-type singular kernel while $K^{(2)}$ is a weakly singular kernel.

IV. Numerical Solution

Write W as

$$W(\xi,\eta)=\sum_{n,k}W_k^n A_k^n(s,\varphi) \quad (14)$$

where $A_k^n(s,\varphi)$ is defined by

$$A_k^n(s,\varphi)=s^{|n|}C_{2k+1}^{|n|+\frac{1}{2}}(\sqrt{1-s^2})e^{jn\varphi}. \quad (15)$$

Introduce

$$L_h^m(s,\varphi)=s^{|m|}C_{2h+1}^{|m|+\frac{1}{2}}(\sqrt{1-s^2})\cos m\varphi. \quad (16)$$

The relationship of these two functions, $A_k^n(s,\varphi)$ and $L_h^m(s,\varphi)$ can be expressed as

$$\int_\Omega A_k^n(s,\varphi)L_h^m(s,\varphi)\frac{sdsd\varphi}{\sqrt{1-s^2}}=B_k^n\delta_{kh}\delta_{mn} \quad (17)$$

where δ_{ij} is Kronecker delta and

$$B_k^n=\begin{cases}\dfrac{\pi^2\Gamma(2k+2)}{\left(2k+\frac{3}{2}\right)(2k+1)!\left[\Gamma\left(\frac{1}{2}\right)\right]^2} & n=0 \\ \dfrac{\pi^2\Gamma(2k+2n+2)}{2^{2n+1}\left(2k+n+\frac{3}{2}\right)(2k+1)!\left[\Gamma\left(n+\frac{1}{2}\right)\right]^2} & n\neq 0.\end{cases}$$

Substitute equation (14) into (13) and making use of Krenk's formula [26] yields

$$\sum_{n=-N_1}^{N_1}\sum_{k=0}^{N_2}\mathcal{F}_k^n(s_0,\varphi_0)W_k^n=Q(\xi_0(s_0,\varphi_0),\eta_0(s_0,\varphi_0)) \quad (18)$$

where

$$\begin{aligned}\mathcal{F}_k^n(s_0,\varphi_0)&=-E_k^n\frac{(2-\nu+3\nu e^{2j\Theta})A_k^n(s_0,\varphi_0)}{2\sqrt{1-s_0^2}} \\ &+\frac{2-\nu}{8\pi}\int_D A_k^n(s,\varphi)K^{(1)}(\zeta,\zeta_0)d\xi d\eta \\ &+\frac{3\nu}{8\pi}\int_D A_k^n(s,\varphi)K^{(2)}(\zeta,\zeta_0)d\xi d\eta; \\ & 0\le s\le 1,\ 0\le\varphi<2\pi. \end{aligned} \quad (19)$$

Multiply (18) by $L_h^m(s_0,\varphi_0)$, integrate of the domain D and using (17) gives

$$\sum_{n,k}\tilde{W}_k^n\left(-\frac{2-\nu+3\nu e^{2j\Theta}}{2}\delta_{hk}\delta_{|m||n|}+S_{hk}^{mn}\right)=Q_h^m; \\ -N_1\le m\le N_1,\quad 0\le h\le N_2 \quad (20)$$

where

$$\begin{aligned} S_{hk}^{mn} &= \frac{2-\nu}{8\pi\sqrt{E_h^nB_h^n}\sqrt{E_k^mB_k^m}}T_{hk}^{mn} \\ T_{hk}^{mn} &= \int_D L_h^m(\zeta_0)\int_D A_k^n(\zeta)H(\zeta,\zeta_0)d\zeta d\zeta_0 \\ Q_h^m &= \frac{1}{\sqrt{E_h^mB_h^m}}\int_D L_h^m(\zeta_0)Q(\zeta_0)d\zeta_0 \\ W_h^m &= -\tilde{W}_h^mG_{2h+1}^{m+\frac{1}{2}}\sqrt{\frac{E_h^m}{B_h^m}}. \end{aligned}$$

In (20), we have used the following notations:

$$\begin{aligned}\zeta_0 &= \zeta_0(s_0,\varphi_0),\quad d\zeta_0=s_0ds_0d\varphi_0 \\ H(\zeta,\zeta_0) &= (2-\nu)K^{(1)}(\zeta,\zeta_0)+3\nu K^{(2)}(\zeta,\zeta_0), \\ Q(\zeta_0) &= Q(\xi_0,\eta_0)=Q(s_0\cos\varphi_0,s_0\sin\varphi_0).\end{aligned}$$

In evaluating the quadruple integrals in equation (20), we have used the Gaussian quadrature and trapezoidal formulas for the radial and angular directions, with appropriate choice of collocation points (s,φ) and (s_0,φ_0).

V. Stress Intensity Factors and Maximum Shear Stress Intensity

The sliding mode, $K_2(\theta)$ and the tearing mode, $K_3(\theta)$ stress intensity factors are defined as [27; 28]

$$K_j(\theta)=\lim_{r\to a}\sqrt{\frac{2\pi}{a-r}}V_jw(x,y);\qquad j=2,3 \quad (21)$$

while the maximum shear stress intensity, $M(\theta)$ is

$$M(\theta) = \sqrt{[K_2(\theta)]^2 + [K_3(\theta)]^2} \tag{22}$$

where V_j are constants.

Let $a(\varphi) = |f(e^{i\varphi})|$, $r = |f(se^{i\varphi})|$, and as s close to 1, we have

$$|f(e^{i\varphi}) - f(se^{i\varphi})| = (1-s)|f'(e^{i\varphi})|. \tag{23}$$

Substitutes equation (23) into (21) gives

$$K_j(\theta) = 2\sqrt{\pi}V_j|f'(e^{i\varphi})|^{-1}\sum_{n,k} W_k^n Y_k^n(\varphi) \tag{24}$$

where $Y_k^n(\varphi) = D_{2k+1}^{|n|+1/2}(0)\cos(n\varphi)$, the unknown coefficients, W_k^n are obtained from (20) and $D_m^\lambda(x)$ is defined recursively by

$$\begin{aligned} mD_m^\lambda(x) &= 2(m+\lambda-1)xD_{m-1}^\lambda(x) \\ &\quad -(m+2\lambda-2)D_{m-2}^\lambda(x),\ m = 2,3,4,\ldots \end{aligned}$$

with $D_0^\lambda(x) = 2\lambda$ and $D_1^\lambda(x) = 2\lambda x$.

VI. Results

Tables I and II show the sliding mode and tearing mode stress intensity factor elliptical crack at $c = 0.2$. It can be observe that the numerical scheme for $K_2(\theta)$ and $K_3(\theta)$ converges rapidly with a small value of $N = N_1 = N_2 = 9$ while Figures 2 to 4 indicate the stress intensity factors and maximum stress intensity versus degree at $c = 0.2$. From the figures, it can be seen that the numerical results have local extremal values when the crack front is at $\cos(\varphi) = \pm 1$ or $\sin(\varphi) = \pm 1$.

TABLE I: NUMERICAL CONVERGENCE FOR THE SLIDING MODE STRESS INTENSITY FACTOR, $K_2(\theta)$ FOR ELLIPTICAL CRACK WHEN $c = 0.2$.

N	$K_2(0.00)$	$K_2(\frac{\pi}{4})$	$K_2(\frac{\pi}{2})$	$K_2(\frac{3\pi}{4})$
0	-1.0590E-07	-8.2976E-08	-8.5777E-24	8.2976E-08
1	0.0000	-1.3681E-05	-3.2301E-21	4.1042E-05
2	-3.7817E-05	-1.3681E-05	-4.8132E-20	5.1559E-04
3	1.1721	0.8024	7.0959E-17	-0.8024
4	-2.5246E-04	6.7598E-04	6.8359E-21	6.0902E-04
5	1.1719	0.8024	7.0951E-17	-0.8024
6	1.2451	0.8024	8.7264E-14	-0.8024
7	1.3179	0.8024	7.0949E-17	-0.8024
8	1.3179	0.8024	7.0949E-17	-0.8024
9	1.3179	0.8024	7.0949E-17	-0.8024

VII. Conclusion

In the present paper, the elliptical crack is mapped conformally into a unit circle. Through this mapping, the equation is transformed into hypersingular integral equation over a circular region, which enable us to use the formula obtained by Krenk [26]. By choosing the appropriate collocation points, this equation is reduced into a system of linear equations and solved for the unknown coefficients, which are later used in finding the stress intensity factor and maximum shear stress

TABLE II: NUMERICAL CONVERGENCE FOR THE TEARING MODE STRESS INTENSITY FACTOR, $K_3(\theta)$ FOR ELLIPTICAL CRACK WHEN $c = 0.2$.

N	$K_3(0.00)$	$K_3(\frac{\pi}{4})$	$K_3(\frac{\pi}{2})$	$K_3(\frac{3\pi}{4})$
0	0.0000	6.9683E-08	1.17648E-07	6.9683E-08
1	0.0000	1.1489E-05	4.43026E-05	3.4467E-05
2	0.0000	1.7018E-04	6.6016E-04	4.330E-04
3	0.0000	-0.6738	-0.9732	-0.6738
4	0.0000	5.1146E-04	-9.3757E-05	5.1146E-04
5	0.0000	-0.6739	-0.9731	-0.6739
6	0.0000	-0.7159	-0.8597	-0.7159
7	0.0000	-0.8489	-1.0529	-0.8489
8	0.0000	-0.8489	-1.0529	-0.8489
9	0.0000	-0.8489	-1.0529	-0.8489

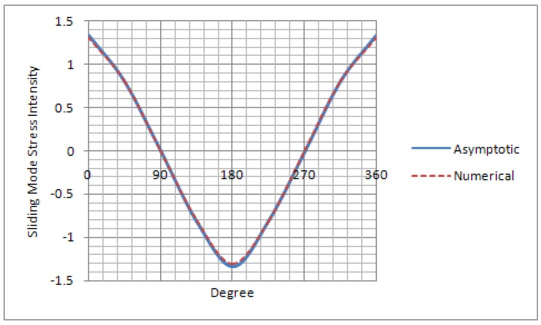

Fig. 2: The sliding mode stress intensity factor, $K_2(\theta)$ for $c = 0.2$

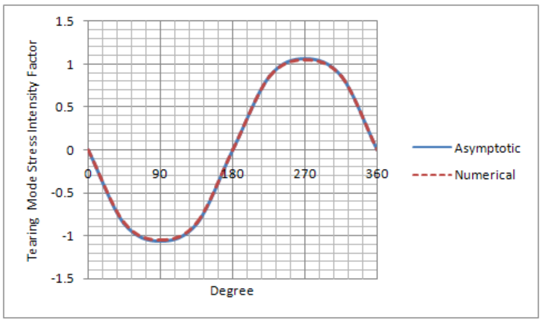

Fig. 3: The tearing mode stress intensity factor, $K_3(\theta)$ for $c = 0.2$

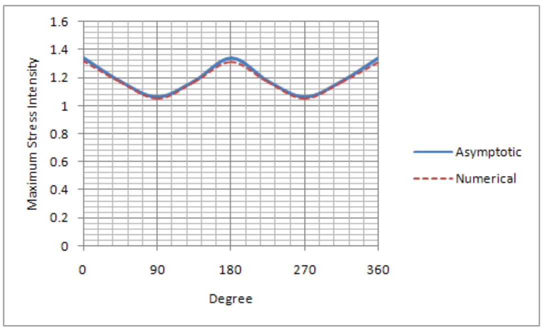

Fig. 4: The maximum stress intensity, $M(\theta)$ for $c = 0.2$

intensity. Through a careful analysis and comparison between the present solutions and existing solutions, it was shown that our numerical results agree with the existing asymptotic solution.

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