Unsteady Laminar Boundary Layer Forced Flow in the Region of the Stagnation Point on a Stretching Flat Sheet

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Abstract—This paper analyses the unsteady, two-dimensional stagnation point flow of an incompressible viscous fluid over a flat sheet when the flow is started impulsively from rest and at the same time, the sheet is suddenly stretched in its own plane with a velocity proportional to the distance from the stagnation point. The partial differential equations governing the laminar boundary layer forced convection flow are non-dimensionalised using semi-similar transformations and then solved numerically using an implicit finite-difference scheme known as the Keller-box method. Results pertaining to the flow and heat transfer characteristics are computed for all dimensionless time, uniformly valid in the whole spatial region without any numerical difficulties. Analytical solutions are also obtained for both small and large times, respectively representing the initial unsteady and final steady state flow and heat transfer. Numerical results indicate that the velocity ratio parameter is found to have a significant effect on skin friction and heat transfer rate at the surface. Furthermore, it is exposed that there is a smooth transition from the initial unsteady state flow (small time solution) to the final steady state (large time solution).

Keywords—Forced flow, Keller-box method, Stagnation point, Stretching flat sheet, Unsteady laminar boundary layer, Velocity ratio parameter.

I. INTRODUCTION

THE study of laminar flow and heat transfer due to a stretching surface has gained considerable interest because of its extensive applications in the field of engineering and technology. This type flows has been initiated by Sakiadis who, in his pioneering work [1], developed the flow field due to a flat surface which is moving with a constant velocity in a quiescent fluid. Due to the higher viscosity of the fluid near the sheet one can assume that the fluid is affected by the sheet but not vice versa. Thus the dynamic problem can be idealized to the case of a fluid disturbed by a tangentially moving boundary. The two-dimensional steady-state flow due to stretching of a sheet where the surface velocity is proportional to the distance from the orifice has been obtained by Crane [2] and, the same was experimentally confirmed by Vleggaar [3]. The steady-state solutions for the associated two-dimensional flows, belonging to an important class of exact solutions of the Navier-Stokes equations, were undertaken by Wang [4], Brady and Acrivos [5] and, Banks [6]. On the other hand, Pop and

Na [7] and Nazar et al. [8] have studied the time-dependent boundary layer flow due to an impulsively stretching surface while; Wang et al. [9] have considered the problem of unsteady two-dimensional boundary layer flow due to a suddenly stretched plane surface in a viscous fluid. Awang Kechil and Hashim [10] presented series solutions for unsteady boundary-layer flows due to impulsively stretched plate.

Stagnation-point flow of an incompressible viscous fluid over a stretching flat sheet has important practical applications in engineering and manufacturing processes such as continuous casting, glass fibre production, metal extrusion, hot-rolling paper and wire drawing. Motivated by this, the present paper being different from the above-mentioned investigations, considers the problem of the two-dimensional laminar boundary layer flow of a viscous and incompressible fluid in the region of the stagnation point on a stretching sheet. The unsteadiness in the flow field is caused by impulsively creating motion in the free stream and at the same time sudden stretching the surface. The governing equations are transformed using semi-similar coordinates originated by Williams and Rhyne [11]. The boundary layer structure of the present problem is found to depend on the parameter λ which defines the ratio of the velocity of the stretching surface to that of the frictionless potential flow in the neighborhood of the stagnation point [10] and numerical solutions of the transformed boundary layer equation for a wide range of values of the parameter λ have been computed using Keller-box method [12], [13], for the whole transient regime. The steady-state counterpart/s of the problem under consideration have been studied by Chaim [14] and, Mahapatra and Gupta [15]. Particular cases of the present results are compared with those of [10] and [15] and the agreement is very good.

II. PROBLEM FORMULATION AND BASIC EQUATIONS

Let us consider the unsteady, laminar incompressible flow of a viscous fluid near the stagnation point of a flat sheet coinciding with the plane $y = 0$, the flow being confined to $y > 0$. Prior to the time t = 0, the surface is at rest in an unbounded quiescent fluid with uniform temperature T_∞. At time $t > 0$, the surface is suddenly stretched with the local tangential velocity $u_w = bx$ (b is a positive constant) keeping

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the origin fixed, as shown schematically in Fig. 1, where x is the coordinate measured along the stretching surface from the stagnation point O.
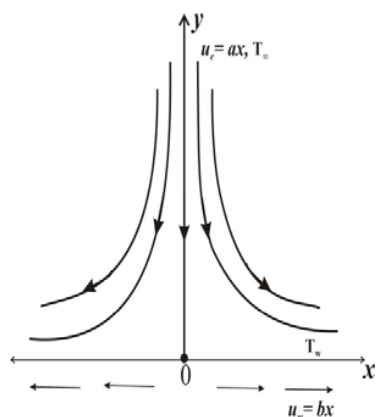

Fig. 1 Physical model and coordinate system

It is also assumed that for $t > 0$, the velocity distribution in the potential flow (free stream velocity), given by $u_e = ax$ (a is a positive constant), starts impulsively in motion from rest. The impulsive change in the surface velocity gives rise to unsteadiness in the flow field. The stretching surface is maintained at constant temperature T_w and is assumed to be greater than ambient temperature T_∞. The fluid is assumed to have constant physical properties and viscous dissipation effects are neglected. Under the aforesaid assumptions, the boundary-layer equations based on conservation of mass, momentum and energy governing the unsteady, two-dimensional stagnation point forced convection flow are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = a^2 x + \nu\frac{\partial^2 u}{\partial y^2} \qquad (2)$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \qquad (3)$$

subject to the initial and boundary conditions:

$$t < 0: u(x,y,t) = v(x,y,t) = 0, \ T(x,y,t) = T_\infty \qquad for all \ x,y$$

$$\left.\begin{array}{l} t \geq 0: u(x,y,t) = u_w = bx, v(x,y,t) = 0, T(x,y,t) = T_w \ at \ y = 0 \\ u(x,y,t) = u_e = ax, T(x,y,t) = T_\infty \ as \ y \to \infty \end{array}\right\} \quad (4)$$

Here u and v are velocity components along x and y-directions, respectively; T is the temperature; ν and α denote, respectively, kinematic viscosity and thermal diffusivity and subscripts e, w, and ∞ denote the conditions at the edge of the boundary-layer, on the wall and in the free stream, respectively.

To solve (1)-(3) for $t \geq 0$, it is convenient to choose a new time scale ξ so that the region of time integration may become finite. Accordingly, introducing the following transformations [11]

$$\eta = (b/\nu)^{1/2}\xi^{-1/2}y, \ \xi = 1 - \exp(-t^*), t^* = bt$$

$$u = \frac{\partial\psi}{\partial y}, \ v = -\frac{\partial\psi}{\partial x}, \ \psi(x,y,t) = (b\nu)^{1/2}\xi^{1/2}xf(\eta,\xi)$$

$$u = bxf'(\eta,\xi), v = -(b\nu)^{1/2}\xi^{1/2}f(\eta,\xi),$$

$$\Pr = \frac{\nu}{\alpha}, \ T = T_\infty + \left(T_w - T_\infty\right)G(\eta,\xi) \qquad (5)$$

to (1)-(3), we find that (1) is identically satisfied and (2) and (3) reduce, respectively, to the system of non linear partial differential equations:

$$\frac{\partial^3 f}{\partial\eta^3} + \frac{\eta}{2}(1-\xi)\frac{\partial^2 f}{\partial\eta^2} + \xi\left[\lambda^2 + f\left(\frac{\partial^2 f}{\partial\eta^2}\right) - \left(\frac{\partial f}{\partial\eta}\right)^2\right] = \xi(1-\xi)\frac{\partial^2 f}{\partial\eta\partial\xi} \quad (6)$$

$$\frac{1}{\Pr}\frac{\partial^2 G}{\partial\eta^2} + \frac{\eta}{2}(1-\xi)\frac{\partial G}{\partial\eta} + \xi f\frac{\partial G}{\partial\eta} = \xi(1-\xi)\frac{\partial G}{\partial\xi} \qquad (7)$$

which must be solved over the range of ξ $(0 \leq \xi \leq 1)$, subject to the boundary conditions

$$\left.\begin{array}{l} f(0,\xi) = f'(0,\xi) = 1. \ f'(\infty,\xi) = \lambda \\ G(0,\xi) = 1 \quad G(\infty,\xi) = 0 \end{array}\right\} \quad (8)$$

Here $\lambda = a/b$ is a positive constant denoting velocity ratio parameter; η and ξ are the transformed dimensionless independent variables; t^* is the dimensionless time; ψ is the stream function; f is the dimensionless stream function; f' is the dimensionless velocity; G is dimensionless temperature and Pr is the Prandtl number. The prime (') denotes derivatives with respect to η.

The quantities of engineering interest are the skin friction coefficient (which indicates physically the surface shear stress) and heat transfer coefficient in the form of Nusselt number, which are defined as

$$C_f\left(\text{Re}_x\right)^{\frac{1}{2}} = \frac{\tau_w}{\rho(bx)^2} = -\frac{1}{\sqrt{\xi}}f''(0,\xi), \ 0 \leq \xi \leq 1, \quad \tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \quad (9)$$

and

$$Nu(\mathrm{Re}_x)^{-\frac{1}{2}} = -\frac{x\left(\frac{\partial T}{\partial y}\right)_{y=0}}{T_w - T_\infty} = -\frac{1}{\sqrt{\xi}}G'(0,\xi) \qquad (10)$$

where τ_w is the wall shear stress and $\mathrm{Re}_x = \left(bx^2/\nu\right)$ is the local Reynolds number.

III. ANALYTICAL SOLUTION

Analytical (exact) solution of the problem under consideration can be obtained, by dividing the unsteady phenomena into ensuing two cases:

1. Initial unsteady state flow ($\xi = 0$):

When $\xi = 0$, which corresponds to $t^* = 0$, (6) and (7) reduce to the set of ordinary differential equations viz.,

$$\frac{\partial^3 f}{\partial \eta^3} + \left(\frac{\eta}{2}\right)\frac{\partial^2 f}{\partial \eta^2} = 0 \qquad (11)$$

$$\frac{1}{\mathrm{Pr}}\frac{\partial^2 G}{\partial \eta^2} + \left(\frac{\eta}{2}\right)\frac{\partial G}{\partial \eta} = 0 \qquad (12)$$

and corresponding boundary conditions (8) become

$$\left.\begin{array}{l} f(0,0) = f'(0,0) = 1. \quad f'(\infty,0) = \lambda \\ G(0,0) = 1 \quad G(\infty,0) = 0 \end{array}\right\} \qquad (13)$$

Equations (11) and (12) with boundary conditions (13) admit closed form (exact) solutions and they are given by:

$$\left.\begin{array}{l} f(\eta,0) = \lambda\eta + (1-\lambda)\,\eta\,erfc(\eta/2) + \left(2/\sqrt{\pi}\right)\left[1-\exp(-\eta^2/4)\right] \\ G(\eta,0) = erfc\left(\frac{\eta\sqrt{\mathrm{Pr}}}{2}\right) \end{array}\right\} \qquad (14)$$

where

$$erfc(z) = \left(2/\sqrt{\pi}\right)\int_\eta^{+\infty}\exp(-z^2)dz \qquad (15)$$

is the complementary error function.
Hence

$$f''(0) = \frac{1}{\sqrt{\pi}}\ ,\ G'(0) = -\sqrt{\frac{\mathrm{Pr}}{\pi}} \qquad (16)$$

2. Final steady state flow ($\xi = 1$):-

When $\xi = 1$, which corresponds to $t^* \to +\infty$, (6) and (7) reduce to the set of ordinary differential equations viz.

$$\frac{\partial^3 f}{\partial \eta^3} + f\left(\frac{\partial^2 f}{\partial \eta^2}\right) - \left(\frac{\partial f}{\partial \eta}\right)^2 = 0 \qquad (17)$$

$$\frac{1}{\mathrm{Pr}}\frac{\partial^2 G}{\partial \eta^2} + f\frac{\partial G}{\partial \eta} = 0 \qquad (18)$$

and boundary conditions (8) become

$$\left.\begin{array}{l} f(0,1) = f'(0,1) = 1. \quad f'(\infty,1) = 0 \\ G(0,1) = 1 \quad G(\infty,1) = 0 \end{array}\right\} \qquad (19)$$

when $\lambda = 0$.

The exact solution of the system (17) and (18) with boundary conditions (19) is given by

$$\left.\begin{array}{l} f(\eta,1) = [1 - \exp(-\eta)] \\ G(\eta,1) = e(e-1)^{-1}\left[1 - \exp(-e^{-\eta})\right] \end{array}\right\} \qquad (20)$$

when $Pr = 1.0$.

IV. NUMERICAL METHOD

The system of nonlinear partial differential equations (6) and (7) subject to boundary conditions (8) is solved numerically using an implicit finite-difference scheme known as Keller-box method, as described in [12], [13]. This method is unconditionally stable and has second order convergence property. To conserve the space, details of the entire solution procedure of Keller-box method used in the present study are not presented here.

Numerical computations were carried out for different values velocity ratio parameter λ. The step size $\Delta\eta$ in η-direction and the position of the edge of the boundary layer η_∞ have been adjusted to maintain the necessary accuracy. The values of $\Delta\eta$ between 0.001 and 0.1 were used so that numerical solutions obtained are independent of $\Delta\eta$ chosen, at least to four decimal places. However, a uniform grid $\Delta\eta=0.01$ was found to be satisfactory for a convergence criterion of 10^{-5} which gives accuracy to four decimal places. The boundary layer thickness η_∞ between 6 and 10 was chosen where the infinity boundary-conditions are achieved.

V. RESULTS AND DISCUSSION

In order to validate the accuracy of the numerical method used, the computed values of $f''(0,\xi)$ for the range $0 \le \xi \le 1$ obtained in this study taking $Pr = 1.0$ have been compared in Table I, with those of Awang Kechil and Hashim [10], and with those Mahapatra and Gupta [15], in Table II. The comparisons revealed good agreement and therefore the code that was developed can be used with high confidence to study the problem discussed in this paper.

TABLE I
COMPARISON OF $f''(0,\xi)$ WITH THOSE OF AWANG KECHIL AND HASHIM
[10] WHEN PR =1.0 AND $\lambda = 0.0$ (INITIAL UNSTEADY FLOW)

| ξ	Present Results	Awang Kechil and Hashim [8]
0.0	-0.564374	-0.5643740
0.1	-0.6106120	-0.6150550
0.3	-0.7115610	-0.7115696
0.5	-0.8004117	-0.8018198
0.7	-0.8873160	-0.8856581
0.8	-0.9200550	-0.9252701
0.9	-0.9623398	-0.9633761
1.0	-1.0000000	-1.0000000

TABLE II
COMPARISON OF $f''(0,\xi)$ FOR DIFFERENT VALUES OF λ WHEN PR =1.0
AND $\xi = 1.0$ (FINAL STEADY STATE FLOW) WITH THOSE OF MAHAPATRA
AND GUPTA [15]

	$\xi = 1.0$	
λ	Present Results	Mahapatra and Gupta [15]
0.05	-0.9876	-----
0.10	-0.9694	-0.9694
0.20	-0.9181	-0.9181
0.50	-0.6673	-0.6673
2.00	2.0176	2.0175
3.00	4.7296	4.7293
5.00	11.7537	-----

Fig. 2 displays the effect of the velocity ratio parameter λ on the skin friction $\left[C_f\left(\mathrm{Re}_L\right)^{1/2}\right]$ and heat transfer $\left[Nu\left(\mathrm{Re}_L\right)^{-1/2}\right]$ coefficients in the entire regime of $\xi\,(0 \le \xi \le 1.0)$, taking $Pr = 0.7$ (air). It is observed that both $C_f\left(\mathrm{Re}_L\right)^{1/2}$ and $Nu\left(\mathrm{Re}_L\right)^{-1/2}$ increase with the increase of λ due to enhanced velocity and temperature gradient near the stretching surface and the effect of λ seems to be more significant as ξ increases. In fact, the percentage increase in $C_f\left(\mathrm{Re}_L\right)^{1/2}$ when λ increases from $\lambda = 2.0$ to $\lambda = 3.0$ is 259.22% while, it is about 972.30% for the increase of λ from $\lambda = 3.0$ to $\lambda = 5.0$ [See Fig. 2 (a)]. Similarly, in case of $Nu\left(\mathrm{Re}_L\right)^{-1/2}$ the percentage increase is about 10.68% when λ increases from $\lambda = 2.0$ to $\lambda = 3.0$ and it is 31.17% for the increase of λ from $\lambda = 3.0$ to $\lambda = 5.0$ [See Fig. 2 (b)].
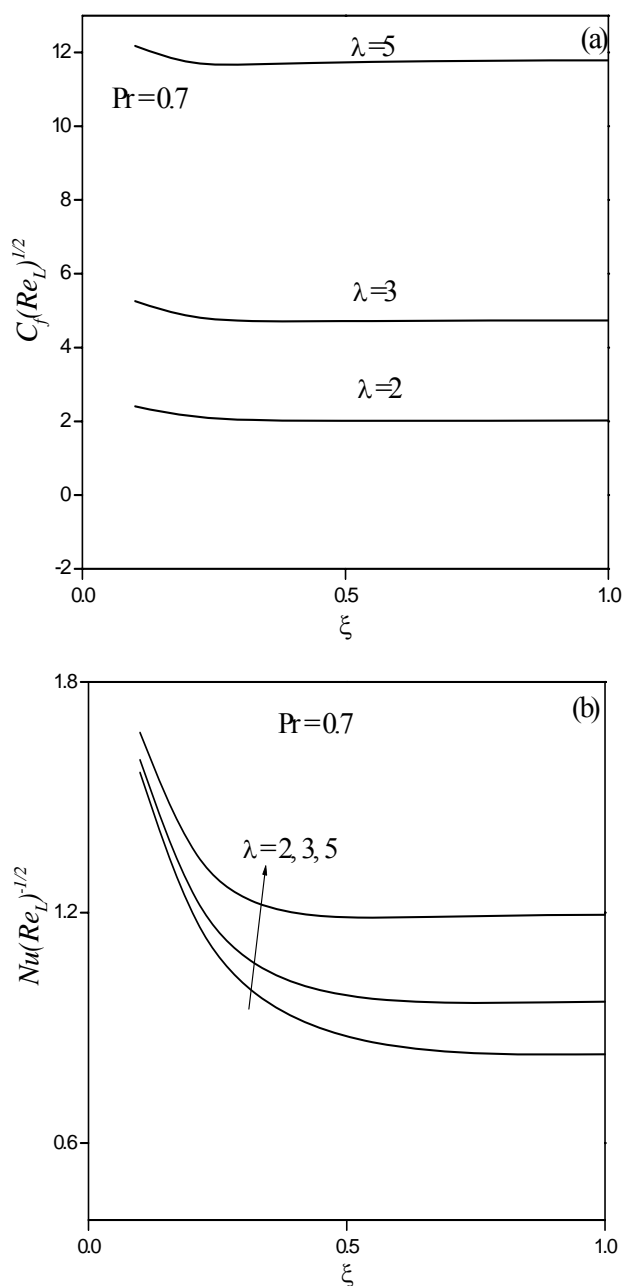
Fig. 2 Effect of the velocity ratio parameter λ on (a) skin friction
and (b) heat transfer coefficients

Also, it is observed that $C_f\left(\mathrm{Re}_L\right)^{1/2}$ and $Nu\left(\mathrm{Re}_L\right)^{-1/2}$ both strongly depending on ξ, [See (6) and (7)] are found to decrease rapidly in a small time interval $\left(0 < \xi < 0.3\right)$ after the start of the impulsive motion and reach the steady state near $\xi \cong 0.7$. Furthermore, the steady state solutions at $\xi = 1.0$ for different values of λ, substantiate the fact that

the transition from the initial unsteady state flow $(\xi = 0.0)$ to the final steady state flow $(\xi = 1.0)$ takes place smoothly and, without any singularity.

The corresponding velocity [$f'(\xi,\eta)$] and temperature [$G(\xi,\eta)$] profiles at $\xi = 0.5$, depicted in Fig. 3 for $\lambda > 0$, reveal that the flow has a boundary layer structure satisfying the boundary conditions, asymptotically. It is evident in these figures that momentum and thermal boundary layer thicknesses decrease with the increase in λ and, momentum boundary layer is relatively thicker than the thermal boundary layer. This is due to the fact that for a fixed value of b, corresponding to the stretching of the surface, increase in a in relation to b [such that $\lambda = (a/b) > 1$] implies increase in straining motion near the stagnation region resulting in increased acceleration of the external stream, which leads to the thinning of the boundary layer thickness, with the increase in λ. Further, when $\lambda < 1$, the flow field has an inverted boundary layer structure. It is a consequence from the fact that when $\lambda < 1$, the stretching velocity bx of the surface exceeds velocity ax of the external free stream.

IV. CONCLUSIONS

Unsteady boundary-layer flow of a viscous fluid in the stagnation region on a stretching flat sheet has been analyzed in the present study, where the unsteadiness is caused by the impulsively motion of the free steam velocity and by the suddenly stretched flat surface. The boundary layer structure of the present problem is found to depend on the parameter which delineates the ratio of the velocity of the stretching surface to that of the frictionless potential flow in the neighborhood of the stagnation point. Numerical results are obtained for all dimensionless time, uniformly valid in the whole spatial region without any numerical difficulties using Keller-box method. The variations of skin friction and heat transfer coefficients with the velocity ratio parameter are obtained and discussed. Analytical solutions are also obtained for both small and large times, respectively representing the initial unsteady and final steady state flow and heat transfer. A considerable advantage was found with the use of a transformed finite time scale in solving the governing partial differential equations, and it has been established that there is a smooth transition from the small time solution to the large time solution.
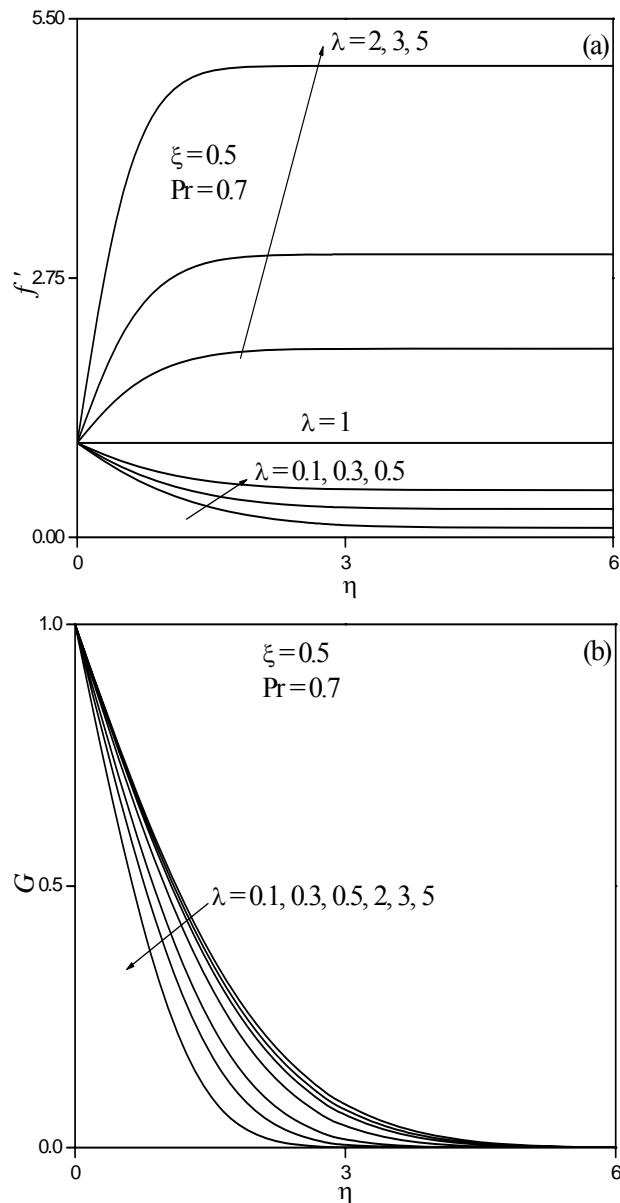

Fig. 3 Effect of the velocity ratio parameter λ on (a) velocity (b) temperature profile

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