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Some Geodesics in Open Surfaces Classified by Clairaut's Relation

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Abstract—In this paper, we studied some properties of geodesic on some open surfaces: Hyperboloid, Paraboloid and Funnel Surface. Geodesic equation in the v-Clairaut parameterization was calculated and reduced to definite integral. Some geodesics on some open surfaces as mention above were classified by Clairaut's relation.

Keywords—Geodesic, Surface of revolution, Clairaut's relation, Clairaut parameterization.

I. Introduction

GEODESICS are curves in surfaces that plays a role analogous to straight line in the plane. Geometrically, a geodesic in a surface is an embedded simple curve such that the portion of the curve between any two points is the shortest curve on the surface. It has been known that great circles are geodesics on a sphere. A geodesic can be obtained as the solution of the non-linear system of second-order ordinary differential equation with the given points and its tangent direction for the initial conditions. Reference [3] was shown numerical solution of a geodesic by iterative method. Generally, the geodesic equations are very complicated for solving explicitly. However, there are two cases where the solution can be reduced to definite integral. A solution of geodesic equation on some surfaces of revolution may be obtained from the v-Clairaut parameterization by reduced to compute integrals [5],[8]. Moreover, families of geodesics are classified by Clairaut's relation [8]. In this paper, we studied some properties of geodesic on some open surface: Hyperboloid, Paraboloid and Funnel Surface. Geodesic equation is calculated in the v-Clairaut parameterization form which can be reduced to definite integral and some geodesics on the open surfaces are classified by Clairaut's relation.

II. Basic Theory

We begin by recalling the basic theory about surface of revolution and its geodesic, most of which can be found in [1], [2], [6], [7], [8], [9] and [11]. Let D denote an open set in the plane $\mathbf{R}^2$. Typically, the open set D will be an open disk or an open rectangle. Let

$$\mathbf{x} : D \to \mathbf{R}^3, (u,v) \mapsto (x(u,v), y(u,v), z(u,v))$$

denote a mapping of D into 3-space. If we fix $v = v_0$ and let u vary, then $\mathbf{x}(u, v_0)$ depends on one parameter, that is a curve. It is called a u-parameter curve. Similarly, if we fix $u = u_0$, the curve is a v-parameter curve. The tangent vectors for the u -parameter and v-parameter curves are given by differentiating the component functions of $\mathbf{x}$ with respect to u and v respectively. For each point on surface S, let $T_p(S)$ denotes the tangent plane at p . Let $\alpha : [a,b] \to S$ be a smooth curve on a surface S. The first fundamental form for S is

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2, \qquad (1)$$

where $E(u,v) = \mathbf{x}_u \cdot \mathbf{x}_u, F(u,v) = \mathbf{x}_u \cdot \mathbf{x}_v, G(u,v) = \mathbf{x}_v \cdot \mathbf{x}_v$.

A. Surface of Revolution

Let $I \subseteq \mathbf{R}$ be an interval and let $\alpha(v) = (0, f(v), g(v)), v \in I$ be a regular parametrized plane curve with $f(v) > 0$. Then the surface of revolution obtained by rotating α about the z-axis is parametrized by

$$\mathbf{x} = (f(v)\cos u, f(v)\sin u, g(v)), \qquad (2)$$

where $v \in I, 0 \le u \le 2\pi$.

The u-parameter curves are generating curve α called meridians and The v -parameter curves are circles, called parallels (Fig.1).

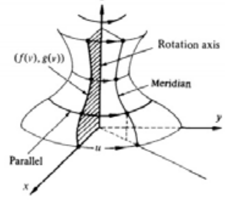

Fig. 1. Surface of revolution [7]

B. Geodesics

Definition 1: Let $\alpha : [a,b] \to S$ be a smooth curve on a surface S . The curve α is called a geodesic on S if $\alpha''(s)$ is orthogonal to the tangent space $T_{\alpha(s)}(S)$ for each $s \in (a,b)$.

That is, the acceleration α' of a geodesic is orthogonal to $T_{\alpha(s)}(S)$, or orthogonal to the velocity α' of α . Thus, geodesics have constant speed, since differentiation of $\|\alpha'\|^2 = \alpha' \cdot \alpha'$ gives $2\alpha' \cdot \alpha'' = 0$

Theorem 1: Let $\alpha(s) = \mathbf{x}(u_1(s), u_2(s))$ be a smooth curve on a surface S . Then α is a geodesic if and only if α satisfies the following differential equations

$$u_i''(s) + \sum_{j,k=1}^{2} \Gamma_{ij}^k (u_1(s), u_2(s)) u_j'(s) u_k'(s) = 0, (i = 1, 2). \qquad (3)$$

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Let $u_1 = u$ and $u_2 = v$. That is, the geodesic equations (3) can be simplified to

$$u'' + \Gamma^1_{11}(u')^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}(v')^2 = 0, \tag{4}$$

$$v'' + \Gamma^2_{11}(u')^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}(v')^2 = 0, \tag{5}$$

where the coordinate system $(u,v) = (u(s),v(s))$ is orthogonal, i.e.,$F(u,v) = 0$. Then, the Christoffel symbols $\Gamma^i_{jk}(i,j,k = 1,2)$ is given by

$$\Gamma^1_{11} = \frac{E_u}{2E}, \Gamma^1_{12} = \frac{E_v}{2E}, \Gamma^1_{22} = \frac{-G_u}{2E}, \tag{6}$$

$$\Gamma^2_{11} = \frac{-E_v}{2G}, \Gamma^2_{12} = \frac{G_u}{2G}, \Gamma^2_{22} = \frac{G_v}{G}. \tag{7}$$

By replacing (6) into (4) and (7) into (5) , a geodesic can be calculated by the system of non-linear differential equations :

$$u'' + \frac{E_u}{2E}(u')^2 + 2\frac{E_v}{2E}u'v' - \frac{G_u}{2E}(v')^2 = 0, \tag{8}$$

$$v'' - \frac{E_v}{2G}(u')^2 + 2\frac{G_u}{2G}u'v' + \frac{G_v}{G}(v')^2 = 0, \tag{9}$$

C. Clairaut Parametrization

The main classical tool used to get qualitative information about geodesics on surface of revolution is the Clairaut's relation. We say that an orthogonal patch $\mathbf{x}(u,v)$ is a v - Clairaut parametrization if $E_u = G_u = 0$ Thus, the geodesic equation with v-Clairaut parametrization is the system

$$u'' + \frac{E_v}{2E}u'v' = 0, \tag{10}$$

$$v'' - \frac{E_v}{2G}(u')^2 + \frac{G_v}{G}(v')^2 = 0, \tag{11}$$

Theorem 2: (Clairaut's relation) *Let $\mathbf{x} : D \to S$ be v-Clairaut parametrization and let $\alpha(s) = \mathbf{x}(u(s),v(s))$ be a geodesic on S . If θ is the angle from $\mathbf{x}_u$ to α' , then*

$$\sqrt{E}\cos\theta = c, \tag{12}$$

where c is called Clairaut's constant.

In general, the geodesic equation is difficult to solve explicitly. However, there are important cases where their solutions can be reduced to definite integrals. Thus, Geodesics equation for v-Clairaut parametrization with single integral is

$$u(v) = \pm\int_{v_0}^{v} \frac{c\sqrt{G}}{\sqrt{E}\sqrt{E-c^2}}dv. \tag{13}$$

We now recall two important classes of geodesics on surface of revolution.

Theorem 3: For a surface of revolution having parametrization $\mathbf{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$, any meridian is a geodesic and a parallel is a geodesic if and only if $f'(v_0) = 0$.

In other words, a necessary condition for a parallel of a surface of revolution to be a geodesic is that such a parallel is generated by the rotation of a point on the generating curve where the tangent is parallel to the axis of revolution. (Fig.2)

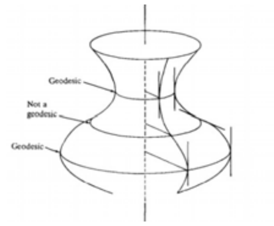

Fig. 2. Some properties of geodesic on surface of revolution [7]

III. THE MAIN RESULTS

A. Geodesic on Hyperboloid

The Hyperboloid [12] (also called a "Hyperboloid of one sheet") is a surface of revolution that can be obtained by rotating a hyperbola around an z-axis. Hyperboloid can be written in parameterization by

$$\mathbf{x}(u,v) = (a\sqrt{1+v^2}\cos u, a\sqrt{1+v^2}\sin u, bv), \tag{14}$$

where $u \in [0,2\pi), v > 0$ and $a > 0$. The coefficients E, F and G of the first fundamental form are given by

$$E = a^2(1+v^2), F = 0, G = \frac{a^2v^2}{1+v^2} + b^2, \tag{15}$$

So that the first fundamental form of Hyperboloid is

$$ds^2 = a^2(1+v^2)du^2 + (\frac{a^2v^2}{1+v^2} + b^2)dv^2. \tag{16}$$

Note that $E_u = G_u = 0$. Surfaces giving by parametrization with these properties are v-Clairaut parametrization. From (10) and (11), geodesic equation on Hyperboloid with v-Clairaut parametrization is satisfied by the following differential equations:

$$u'' + \frac{2v}{1+v^2}u'v' = 0, \tag{17}$$

$$v'' - \frac{a^2v(1+v^2)}{a^2v^2 + b^2(1+v^2)}u'^2 + \frac{a^2v}{(1+v^2)(a^2v^2b^2(1+v^2))}v'^2 = 0. \tag{18}$$

From (13), we obtain a single integral which serves to characterize geodesies for a v-Clairaut parametrization. Thus, the geodesics equation on Hyperboloid for v-Clairaut parametrization with single integral may obtain from the following equation

$$u(v) = \pm\int_{v_0}^{v} \frac{c\sqrt{\frac{a^2v^2}{1+v^2} + b^2}}{\sqrt{a^2(1+v^2)}\sqrt{a^2(1+v^2) - c^2}}dv. \tag{19}$$

Hence, we classify some geodesics on the Hyperboloid by using Clairaut's relation and Theorem 3, thus

$$a\sqrt{(1+v^2)}\cos\theta = c. \tag{20}$$

By assuming that v is very small number, therefore we will consider into 3 cases:

1. Meridians on the Hyperboloid are geodesic which satisfies $c = 0$.

Let $\alpha(s) = \mathbf{x}(u(s),v(s))$ be a meridian on the Hyperboloid.

Since θ is the angle from $\mathbf{x}_u$ to α' , then $\theta = \pi/2$. By putting the meridian $\theta = \pi/2$ in Clairaut's relation, we have $a\sqrt{(1+v^2)}\cos(\pi/2) = c$. Since (Theorem 3.) any meridian is a geodesic, meridians on the Hyperboloid are geodesic (Fig3).

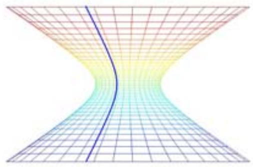

Fig. 3. Meridians on the Hyperboloid are geodesic.

2. Parallel on Hyperboloid at $v_0 = 0$ is geodesic which satisfies $c = a$.
Let $\alpha(s) = \mathbf{x}(u(s), v(s))$ be a parallel on the Hyperboloid. Since θ is the angle from $\mathbf{x}_u$ to α' , then $\theta = 0$. We obtain the parallel $\theta = 0$ in Clairaut's relation, we have $c = a\sqrt{1+v^2}$ Moreover, (Theorem 3.) a parallel is a geodesic if and only if $f'(v_0) = 0$ Since $f(v) = a\sqrt{1+v^2}$ then $f'(v) = 2av/\sqrt{1+v^2}$. Thus $f'(v_0) = 0$ where $v_0 = 0$. Therefore, the parallel on Hyperboloid at is geodesic which satisfies $c = a$ (Fig4).

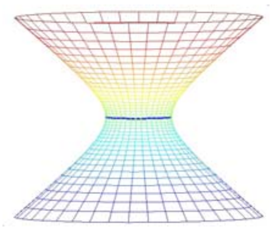

Fig. 4. Parallel on Hyperboloid at $v_0 = 0$ is geodesic which satisfies c=a.

3. Other geodesics satisfy $|c| < |a|$.
In this case, the geodesic is not perpendicular to any meridians which satisfy $a\sqrt{(1+v^2)}\cos\theta = c$. It is implies that $|c| < |a|$. We show geodesics curve on a Hyperboloid by using MATLAB [4], [10]. For example (Fig5), the starting point (u, v) and the direction $(du/ds, dv/ds)$ are given.

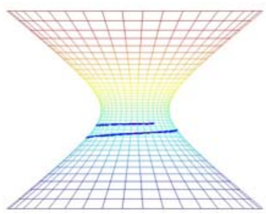

Fig. 5. Some geodesics on Hyperboloid.

With similarity to Hyperboloid, Paraboloid and Funnel surface are considered.

B. Geodesic on Paraboloid

The Paraboloid [12] is a surface of revolution which is an open surface where the generating curve intersects the axis of rotation. A Paraboloid can be parameterized by

$$\mathbf{x}(u, v) = (av\cos u, av\sin u, v^2), \tag{21}$$

where $u \in [0, 2\pi), v > 0$ and $a > 0$. The coefficients E, F and G of the first fundamental form are given by

$$E = a^2v^2, F = 0, G = a^2 + 4v^2, \tag{22}$$

So that the first fundamental form of Paraboloid is

$$ds^2 = a^2v^2du^2 + (a^2 + 4v^2)dv^2. \tag{23}$$

Note that $E_u = G_u = 0$. Surfaces giving by parametrization with these properties are v-Clairaut parametrization. From (10) and (11), geodesic equation on Paraboloid with v-Clairaut parametrization is satisfied by the following differential equations:

$$u'' + \frac{2}{v}u'v' = 0, \tag{24}$$

$$v'' - \frac{a^2v}{a^2+4v^2}u'^2 + \frac{4v}{a^2+4v^2}v'^2 = 0. \tag{25}$$

Geodesics equation on Paraboloid for v-Clairaut parametrization with single integral may obtain from the following equation

$$u(v) = \pm\int_{v_0}^{v} \frac{c\sqrt{a^2+4v^2}}{av\sqrt{a^2v^2-c^2}}dv. \tag{26}$$

Clairaut's relation of Paraboloid is

$$av\cos\theta = c. \tag{27}$$

By assuming that v is very small number, therefore we will consider into 3 cases:

1. Meridians on the Paraboloid are geodesics which satisfies $c = 0$.

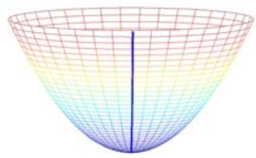

Fig. 6. Meridians on the Paraboloid are geodesics.

2. Parallel on Paraboloid at $v_0 = 0$ is not geodesic which satisfies $c = a$ since $f'(v_0) = a \neq 0$. 3. Other geodesics satisfy $|c| < |a|$.

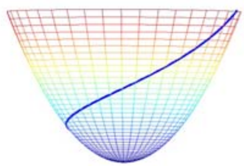

Fig. 7. Meridians on the Paraboloid are geodesics.

C. Geodesic on Funnel Surface

The Funnel Surface [12] is a surface of revolution of the curve $\ln v$ which is an open surface where the generating curve do not intersect the axis of rotation. A Funnel Surface can be parameterized by

$$\mathbf{x}(u, v) = (av \cos u, av \sin u, \ln v), \tag{28}$$

where $u \in [0, 2\pi), v > 0$ and $a > 0$. The coefficients E, F and G of the first fundamental form are given by

$$E = a^2v^2, F = 0, G = a^2 + \frac{1}{v^2}, \tag{29}$$

So that the first fundamental form of Funnel Surface is

$$ds^2 = a^2v^2du^2 + (a^2 + \frac{1}{v^2})dv^2. \tag{30}$$

Note that $E_u = G_u = 0$. Surfaces giving by parametrization with these properties are v-Clairaut parametrization. From (10) and (11), geodesic equation on Funnel Surface with v-Clairaut parametrization is satisfied by the following differential equations:

$$u'' + \frac{2}{v}u'v' = 0, \tag{31}$$

$$v'' + \frac{a^2v^3}{a^2v^2+1}u'^2 - \frac{1}{v(a^2v^2+1)}v'^2 = 0. \tag{32}$$

Geodesics equation on Funnel Surface for v-Clairaut parametrization with single integral may obtain from the following equation

$$u(v) = \pm \int_{v_0}^{v} \frac{c\sqrt{a^2v^2+1}}{av^2\sqrt{a^2v^2-c^2}}dv. \tag{33}$$

Clairaut's relation of Funnel Surface is

$$av \cos\theta = c. \tag{34}$$

By assuming that v is very small number, therefore we will consider into 3 cases:
1. Meridians on the Funnel Surface are geodesics which satisfies $c = 0$.

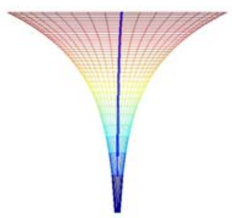

Fig. 8. Meridians on the Funnel Surface are geodesics.

2. Parallel on Funnel Surface at $v_0 = 0$ is not geodesic which satisfies $c = a$ since $f'(v_0) = a \neq 0$.
3. Other geodesics satisfy $|c| < |a|$ (Fig.9).

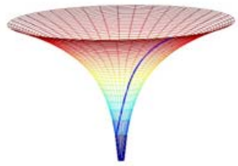

Fig. 9. Some geodesics on Funnel.

IV. Conclusion

We show how to compute the geodesics on some given surfaces for v-Clairaut parametrization in definite integral. We find that if there exists a point on the surface that cut the rotating exist (Paraboloid) or almost cut the rotating exist (Funnel Surface), then all parallels are not geodesic as shown in TABLE I.

TABLE I
The Classification of geodesic by Clairaut's relation

Surface of Revolution	Meridian	Parallel	Other geodesics
Hyperboloid	$c = 0$	$c = a$	$\|c\| < \|a\|$
Paraboloid	$c = 0$	-	$\|c\| < \|a\|$
Funnel Surface	$c = 0$	-	$\|c\| < \|a\|$

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