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On Some Properties of Interval Matrices

K. Ganesan

Abstract**—**By using a new set of arithmetic operations on interval numbers, we discuss some arithmetic properties of interval matrices which intern helps us to compute the powers of interval matrices and to solve the system of interval linear equations.

Keywords**—**Interval arithmetic, Interval matrix, linear equations.

I. Introduction

LET $\tilde{a} = [a_1, a_2] = \{x : a_1 \le x \le a_2, x \in R\}$.

If $\tilde{a} = a_1 = a_2 = a$, then $\tilde{a} = [a, a] = a$ is a real number (or a degenerate interval). We shall use the terms *interval* and *interval number* interchangeably. We use IR to denote the set of all interval numbers on the real line R. The mid-point and width (or half-width) of an interval number $\tilde{a} = [a_1, a_2]$ are defined as $m(\tilde{a}) = \left(\dfrac{a_1 + a_2}{2}\right)$ and $w(\tilde{a}) = \left(\dfrac{a_2 - a_1}{2}\right)$.

It is well known, that matrices play major role in various areas such as mathematics, statistics, physics, engineering, social sciences and many others. In real life, due to the inevitable measurement inaccuracy, we do not know the exact values of the measured quantities; we know, at best, the intervals of possible values. Consequently, we can not successfully use traditional classical matrices and hence the use of interval matrices is more appropriate.

Hansen and Smith [4] started the use of interval arithmetic in matrix computations. After this motivation and inspiration, several authors such as Alefeld and Herzberger [1], Hansen et al ([5], [6]), Jaulin et al [9], Neumaier [10] and Rohn ([12], [13]) etc have studied interval matrices.

Consider a system of interval linear equations: $\tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$ where $\tilde{A}, \tilde{\mathbf{b}}$ and $\tilde{\mathbf{x}}$ are (m×n), (m×1), (n×1) interval matrices respectively. In the existing literature, several methods available for computing the smallest box $\tilde{\mathbf{x}}$ containing the exact solution of the system. In contrast to the problem of solving system of interval linear equations, the concept of determinant of interval matrices has been given less attention. In this paper we discuss some of the arithmetic properties of interval matrices which intern helps us to compute the powers of interval matrices and to solve the system of interval linear equations.

II. Preliminaries

The aim of this section is to present some notations, notions and results which are of useful in our further considerations.

A. Comparing Interval Numbers

Sengupta and Pal [2] proposed the following simple and efficient index for comparing any two intervals on the real line through decision maker's satisfaction.

Let $\preceq$ be an extended order relation between the interval numbers $\tilde{a} = [a_1, a_2]$ and $\tilde{b} = [b_1, b_2]$ in IR, then for $m(\tilde{a}) < m(\tilde{b})$, we construct a premise $(\tilde{a} \preceq \tilde{b})$ which implies that $\tilde{a}$ is inferior to $\tilde{b}$ (or $\tilde{b}$ is superior to $\tilde{a}$).

An *acceptability function* $A_{\preceq} : IR \times IR \to [0, \infty)$ is defined as: $A_{\preceq}(\tilde{a}, \tilde{b}) = A(\tilde{a} \preceq \tilde{b}) = \dfrac{m(\tilde{b}) - m(\tilde{a})}{w(\tilde{b}) + w(\tilde{a})}$, where $w(\tilde{b}) + w(\tilde{a}) \ne 0$.

$A_{\preceq}$ may be interpreted as the grade of acceptability of the *"first interval number to be inferior to the second interval number"*.

For any two interval numbers $\tilde{a}$ and $\tilde{b}$ in IR, either $A(\tilde{a} \preceq \tilde{b}) > 0$ or $A(\tilde{b} \preceq \tilde{a}) > 0$ or $A(\tilde{a} \preceq \tilde{b}) = A(\tilde{b} \preceq \tilde{a}) = 0$ and $A(\tilde{a} \preceq \tilde{b}) + A(\tilde{b} \preceq \tilde{a}) = 0$. Also the proposed *A*-index is transitive; for any three interval numbers $\tilde{a}, \tilde{b}, \tilde{c}$ in IR, if $A(\tilde{a} \preceq \tilde{b}) \ge 0$ and $A(\tilde{b} \preceq \tilde{c}) \ge 0$, then $A(\tilde{a} \preceq \tilde{c}) \ge 0$. But it does not mean that $A(\tilde{a} \preceq \tilde{c}) \ge \max\{A(\tilde{a} \preceq \tilde{b}), A(\tilde{b} \preceq \tilde{c})\}$. If $A(\tilde{a} \preceq \tilde{b}) = 0$, then we say that the interval numbers $\tilde{a}$ and $\tilde{b}$ are equivalent (or non-inferior to each other) and we denote it by $\tilde{a} \approx \tilde{b}$. In particular, whenever $A(\tilde{a} \preceq \tilde{b}) = 0$ and $w(\tilde{b}) = w(\tilde{a})$, then $\tilde{a} = \tilde{b}$. Also if $A(\tilde{a} \preceq \tilde{b}) \ge 0$, then we say that $\tilde{a} \preceq \tilde{b}$ and if $A(\tilde{b} \preceq \tilde{a}) \ge 0$, then we say that $\tilde{b} \preceq \tilde{a}$.

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K. Ganesan is with the Department of Mathematics, S. R. M. University, Kattankulathur, Chennai – 603 203, India. (e-mail: gansan_k@yahoo.com, ganesan@ma.srmuniv.ac.in).

B. A New Interval Arithmetic

We recall a new type of arithmetic operations on interval numbers introduced in [3]: For $\tilde{x}$ and $\tilde{y}$ in IR and for $* \in \{+,-,\cdot,\div\}$, we define $\tilde{x} * \tilde{y} = [m(\tilde{x}) * m(\tilde{y}) - k, m(\tilde{x}) * m(\tilde{y}) + k]$, where $k = \min\{(m(\tilde{x}) * m(\tilde{y})) - \alpha, \beta - (m(\tilde{x}) * m(\tilde{y}))\}$, α and β are the end points of the interval $\tilde{x} \circledast \tilde{y}$ under the existing interval arithmetic. In particular

(i). Addition: $\tilde{x} + \tilde{y} = [x_1, x_2] + [y_1, y_2]$
$= [m(\tilde{x}) + m(\tilde{y}) - k, m(\tilde{x}) + m(\tilde{y}) + k]$,
where $k = \left(\dfrac{(y_2 + x_2) - (y_1 + x_1)}{2}\right)$.

(ii). Subtraction: $\tilde{x} - \tilde{y} = [x_1, x_2] - [y_1, y_2]$
$= [(m(\tilde{x}) - m(\tilde{y})) - k, (m(\tilde{x}) - m(\tilde{y})) + k]$,
where $k = \left(\dfrac{(y_2 + x_2) - (y_1 + x_1)}{2}\right)$.

(iii). Multiplication: $\tilde{x}\tilde{y} = [x_1, x_2]\,[y_1, y_2]$
$= [m(\tilde{x})m(\tilde{y}) - k, m(\tilde{x})m(\tilde{y}) + k]$, where
$k = \min\{(m(\tilde{x})m(\tilde{y})) - \alpha, \beta - (m(\tilde{x})m(\tilde{y}))\}$,
$\alpha = \min(x_1y_1, x_1y_2, x_2y_1, x_2y_2)$ and
$\beta = \max(x_1y_1, x_1y_2, x_2y_1, x_2y_2)$.

(iv). Inverse: $\tilde{x}^{-1} = [x_1, x_2]^{-1} = \left[\dfrac{1}{m(\tilde{x})} - k, \dfrac{1}{m(\tilde{x})} + k\right]$, where

$k = \min\left\{\dfrac{1}{x_2}\left(\dfrac{x_2 - x_1}{x_1 + x_2}\right), \dfrac{1}{x_1}\left(\dfrac{x_2 - x_1}{x_1 + x_2}\right)\right\}$ and $0 \notin [x_1, x_2]$.

From (iii), it is clear tha $\lambda\tilde{x} = \begin{cases} [\lambda x_1, \lambda x_2], \text{ for } \lambda \geq 0 \\ [\lambda x_2, \lambda x_1], \text{ for } \lambda < 0. \end{cases}$

It is to be noted that $\tilde{x} * \tilde{y} \subseteq \tilde{x} \circledast \tilde{y} = \{x * y : x \in \tilde{x}, y \in \tilde{y}\}$, where $\circledast \in \{\oplus, \ominus, \otimes, \oslash\}$ is the existing interval arithmetic. For example if we take $\tilde{x} = [-1, 2]$ and $\tilde{y} = [3, 5]$, then $\tilde{x} \otimes \tilde{y} = [\min(-3, -5, 6, 10), \max(-3, -5, 6, 10)] = [-5, 10]$ and $\tilde{x} \cdot \tilde{y} = \tilde{x}\,\tilde{y} = [-1, 2]\,[3, 5] = [-5, 9]$ so that $\tilde{x} * \tilde{y} \subseteq \tilde{x} \circledast \tilde{y}$.

It is also to be noted that we use $\circledast$ to denote the existing interval arithmetic and $*$ to denote the modified interval arithmetic. But wherever there is no confusion we use the same notation for both the cases. We require the following results to prove the results in the subsequent section.

Proposition 2.1: For any $\tilde{x} = [x_1, x_2]$ and $\tilde{y} = [y_1, y_2]$ in IR, we have

(i). $m(\tilde{x} + \tilde{y}) = m(\tilde{x}) + m(\tilde{y})$ and
$w(\tilde{x} + \tilde{y}) = w(\tilde{x}) + w(\tilde{y})$.

(ii). $m(\tilde{x} - \tilde{y}) = m(\tilde{x}) - m(\tilde{y})$ and
$w(\tilde{x} - \tilde{y}) = w(\tilde{x}) + w(\tilde{y})$.

(iii). $m(\tilde{x}\tilde{y}) = m(\tilde{x})m(\tilde{y})$ and $w(\tilde{x}\tilde{y}) = 0$ if and only if $\tilde{x} = [x_1, x_2] = 0$ or $\tilde{y} = [y_1, y_2] = 0$.

(iv). $m\left(\dfrac{1}{\tilde{x}}\right) = \dfrac{1}{m(\tilde{x})}$ and $w\left(\dfrac{1}{\tilde{x}}\right) = \dfrac{w(\tilde{x})}{x_1x_2}$,
provided $0 \notin [x_1, x_2]$.

(v). $m(\alpha\tilde{x} + \beta\tilde{y}) = \alpha\, m(\tilde{x}) + \beta m(\tilde{y})$ and
$w(\alpha\tilde{x} + \beta\tilde{y}) = |\alpha|\, w(\tilde{x}) + |\beta|\, w(\tilde{y})$, where
$\alpha, \beta \in R$.

Remark: Without loss of generality, we assume that for any interval number $\tilde{a} = [a_1, a_2]$ with $m(\tilde{a}) \neq 0$ and $0 \in \tilde{a}$, there exist $\tilde{b} = [m(\tilde{a}) - k, m(\tilde{a}) + k]$, where $0 < k < h$ and $h = \min\{|a_1|, |a_2|\}$, such that $\tilde{b} \approx \tilde{a}$ and $0 \notin \tilde{b}$. Hence if $\dfrac{\tilde{x}}{\tilde{a}}$ with $m(\tilde{a}) \neq 0$ and $0 \in \tilde{a}$, then we replace $\dfrac{\tilde{x}}{\tilde{a}}$ by $\dfrac{\tilde{x}}{\tilde{b}}$, where $\tilde{b} \approx \tilde{a}$ and $0 \notin \tilde{b}$.

An interval vector $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots\ldots, \tilde{x}_n)$ is a vector whose components are interval numbers, where $\tilde{x}_i$, i =1,2,3,.....,n is the i[th] component of $\tilde{\mathbf{x}}$. We use IR^n to denote the set of all n-component interval vectors. The midpoint of an interval vector $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots\ldots, \tilde{x}_n)$ is the vector of midpoints of its interval components,
i. e. $m(\tilde{\mathbf{x}}) = (m(\tilde{x}_1), m(\tilde{x}_2), m(\tilde{x}_3), \ldots\ldots, m(\tilde{x}_n))$ and the width of interval vector $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots\ldots, \tilde{x}_n)$ is the vector of widths of its interval components,
i. e. $w(\tilde{\mathbf{x}}) = (w(\tilde{x}_1), w(\tilde{x}_2), w(\tilde{x}_3), \ldots\ldots, w(\tilde{x}_n))$.

We define the sum, difference and scalar multiplication of interval vectors as in the case of real classical vectors except that the components are interval numbers.

III. MAIN RESULTS

An *interval matrix* $\tilde{A}$ is a matrix whose elements are interval numbers. An interval matrix $\tilde{A}$ will be written as

$$\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \ldots & \tilde{a}_{1n} \\ \ldots & \ldots & \ldots \\ \tilde{a}_{m1} & \ldots & \tilde{a}_{mn} \end{pmatrix} = (\tilde{a}_{ij})_{(m\times n)}$$, where each $\tilde{a}_{ij} = [\underline{a}_{ij}, \overline{a}_{ij}]$

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(or) $\tilde{A} = [\underline{A}, \overline{A}]$ for some $\underline{A}, \overline{A}$ satisfying $\underline{A} \le \overline{A}$. We use IR^{mxn} to denote the set of all (mxn) interval matrices. The midpoint of an interval matrix $\tilde{A}$ is the matrix of midpoints of its interval elements defined as $m(\tilde{A}) = \begin{pmatrix} m(\tilde{a}_{11}) & .... & m(\tilde{a}_{1n}) \\ ... & ... & ... \\ m(\tilde{a}_{m1}) & ... & m(\tilde{a}_{mn}) \end{pmatrix}$. The width of an interval matrix $\tilde{A}$ is the matrix of widths of its interval elements defined as $w(\tilde{A}) = \begin{pmatrix} w(\tilde{a}_{11}) & .... & w(\tilde{a}_{1n}) \\ ... & ... & ... \\ w(\tilde{a}_{m1}) & ... & w(\tilde{a}_{mn}) \end{pmatrix}$ which is always nonnegative. We use O to denote the *null matrix* $\begin{pmatrix} 0 & .... & 0 \\ ... & ... & ... \\ 0 & ... & 0 \end{pmatrix}$ and $\tilde{O}$ to denote the *null interval matrix* $\begin{pmatrix} \tilde{0} & .... & \tilde{0} \\ ... & ... & ... \\ \tilde{0} & ... & \tilde{0} \end{pmatrix}$. Also we use I to denote the *identity matrix* $\begin{pmatrix} 1 & .... & 0 \\ ... & 1 & ... \\ 0 & ... & 1 \end{pmatrix}$ and $\tilde{I}$ to denote the *identity interval matrix* $\begin{pmatrix} \tilde{1} & .... & \tilde{0} \\ ... & \tilde{1} & ... \\ \tilde{0} & ... & \tilde{1} \end{pmatrix}$.

A. Arithmetic Operations on Interval Matrices

We define arithmetic operations on interval matrices as follows: If $\tilde{A}, \tilde{B} \in IR^{m\times n}$, $\tilde{\mathbf{x}} \in IR^{n}$ and $\alpha \in IR$, then

(i). $\tilde{\alpha}\tilde{A} = (\tilde{\alpha}\, \tilde{a}_{ij})_{1 \le i \le m,\, 1 \le j \le n}$

(ii). $(\tilde{A} + \tilde{B}) = (\tilde{a}_{ij} + \tilde{b}_{ij})_{1 \le i \le m,\, 1 \le j \le n}$

(iii). $(\tilde{A} - \tilde{B}) = (\tilde{a}_{ij} - \tilde{b}_{ij})_{1 \le i \le m,\, 1 \le j \le n}$

(iv). $\tilde{A}\tilde{B} = \sum_{k=1}^{n} \left(\tilde{a}_{ik}\, \tilde{b}_{kj}\right)_{1 \le i \le m,\, 1 \le j \le n}$

(v). $\tilde{A}\tilde{\mathbf{x}} = \left(\sum_{j=1}^{n} \tilde{a}_{ij}\, \tilde{x}_j \right)_{1 \le i \le m}$

If $m(\tilde{A}) = m(\tilde{B})$, then the interval matrices $\tilde{A}$ and $\tilde{B}$ are said to be equivalent and is denoted by $\tilde{A} \approx \tilde{B}$. In particular if $m(\tilde{A}) = m(\tilde{B})$ and $w(\tilde{A}) = w(\tilde{B})$, then $\tilde{A} = \tilde{B}$. If $m(\tilde{A}) = O$, then we say that $\tilde{A}$ is a *zero interval matrix* and is denoted by $\tilde{O}$. In particular if $m(\tilde{A}) = O$ and $w(\tilde{A}) = O$, then $\begin{pmatrix} [0,0] & .... & [0,0] \\ ... & ... & ... \\ [0,0] & ... & [0,0] \end{pmatrix}$. Also if $m(\tilde{A}) = O$ and $w(\tilde{A}) \ne O$, then $\begin{pmatrix} \tilde{0} & .... & \tilde{0} \\ ... & ... & ... \\ \tilde{0} & ... & \tilde{0} \end{pmatrix} \approx \tilde{O}$. If $\tilde{A} \not\approx \tilde{O}$ (i.e. $\tilde{A}$ is not equivalent to $\tilde{O}$), then $\tilde{A}$ is said to be a non-*zero interval matrix*. If $m(\tilde{A}) = I$ then we say that $\tilde{A}$ is an identity *interval matrix* and is denoted by $\tilde{I}$.

In particular if $m(\tilde{A}) = I$ and $w(\tilde{A}) = O$, then $\tilde{A} = \begin{pmatrix} [1,1] & .... & [0,0] \\ ... & [1,1] & ... \\ [0,0] & ... & [1,1] \end{pmatrix}$. Also, $m(\tilde{A}) = I$ and $w(\tilde{A}) \ne O$, then $\begin{pmatrix} \tilde{1} & .... & \tilde{0} \\ ... & \tilde{1} & ... \\ \tilde{0} & ... & \tilde{1} \end{pmatrix} \approx \tilde{I}$.

Proposition 3.1: *If* $\tilde{A}, \tilde{B} \in IR^{n\times n}$, then

(i). $m(\tilde{A} + \tilde{B}) = m(\tilde{A}) + m(\tilde{B})$ and
$w(\tilde{A} + \tilde{B}) = w(\tilde{A}) + w(\tilde{B})$.

(ii). $m(\tilde{A} - \tilde{B}) = m(\tilde{A}) - m(\tilde{B})$ and
$w(\tilde{A} - \tilde{B}) = w(\tilde{A}) + w(\tilde{B})$.

(iii). $m(\tilde{A}\tilde{B}) = m(\tilde{A})m(\tilde{B})$.

Proof: Let $\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \cdots & \tilde{a}_{nn} \end{pmatrix}$ and $\tilde{B} = \begin{pmatrix} \tilde{b}_{11} & \cdots & \tilde{b}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{b}_{n1} & \cdots & \tilde{b}_{nn} \end{pmatrix}$ so that $\tilde{A} + \tilde{B} = \begin{pmatrix} \tilde{a}_{11} + \tilde{b}_{11} & \cdots & \tilde{a}_{1n} + \tilde{b}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{n1} + \tilde{b}_{n1} & \cdots & \tilde{a}_{nn} + \tilde{b}_{nn} \end{pmatrix}$. Now

$$m(\tilde{A} + \tilde{B}) = \begin{pmatrix} m(\tilde{a}_{11} + \tilde{b}_{11}) & \cdots & m(\tilde{a}_{1n} + \tilde{b}_{1n}) \\ \vdots & \ddots & \vdots \\ m(\tilde{a}_{n1} + \tilde{b}_{n1}) & \cdots & m(\tilde{a}_{nn} + \tilde{b}_{nn}) \end{pmatrix}$$

$$= \begin{pmatrix} m(\tilde{a}_{11}) + m(\tilde{b}_{11}) & \cdots & m(\tilde{a}_{1n}) + m(\tilde{b}_{1n}) \\ \vdots & \ddots & \vdots \\ m(\tilde{a}_{n1}) + m(\tilde{b}_{n1}) & \cdots & m(\tilde{a}_{nn}) + m(\tilde{b}_{nn}) \end{pmatrix}$$

$$=\begin{pmatrix} m(\tilde{a}_{11}) & \cdots & m(\tilde{a}_{1n}) \\ \vdots & \ddots & \vdots \\ m(\tilde{a}_{n1}) & \cdots & m(\tilde{a}_{nn}) \end{pmatrix}+\begin{pmatrix} m(\tilde{b}_{11}) & \cdots & m(\tilde{b}_{1n}) \\ \vdots & \ddots & \vdots \\ m(\tilde{b}_{n1}) & \cdots & m(\tilde{b}_{nn}) \end{pmatrix}$$

$=m(\tilde{A})+m(\tilde{B}).$

Also $w(\tilde{A}+\tilde{B})=\begin{pmatrix} w(\tilde{a}_{11}+\tilde{b}_{11}) & \cdots & w(\tilde{a}_{1n}+\tilde{b}_{1n}) \\ \vdots & \ddots & \vdots \\ w(\tilde{a}_{n1}+\tilde{b}_{n1}) & \cdots & w(\tilde{a}_{nn}+\tilde{b}_{nn}) \end{pmatrix}$

$$=\begin{pmatrix} w(\tilde{a}_{11})+w(\tilde{b}_{11}) & \cdots & w(\tilde{a}_{1n})+w(\tilde{b}_{1n}) \\ \vdots & \ddots & \vdots \\ w(\tilde{a}_{n1})+w(\tilde{b}_{n1}) & \cdots & w(\tilde{a}_{nn})+w(\tilde{b}_{nn}) \end{pmatrix}$$

$$=\begin{pmatrix} w(\tilde{a}_{11}) & \cdots & w(\tilde{a}_{1n}) \\ \vdots & \ddots & \vdots \\ w(\tilde{a}_{n1}) & \cdots & w(\tilde{a}_{nn}) \end{pmatrix}+\begin{pmatrix} w(\tilde{b}_{11}) & \cdots & w(\tilde{b}_{1n}) \\ \vdots & \ddots & \vdots \\ w(\tilde{b}_{n1}) & \cdots & w(\tilde{b}_{nn}) \end{pmatrix}$$

$=w(\tilde{A})+w(\tilde{B}).$

(ii) As in (i), by using the result $m(\tilde{x}-\tilde{y})=m(\tilde{x})-m(\tilde{y})$ and $w(\tilde{x}-\tilde{y})=w(\tilde{x})+w(\tilde{y})$, we can prove $m(\tilde{A}-\tilde{B})=m(\tilde{A})-m(\tilde{B})$ and $w(\tilde{A}-\tilde{B})=w(\tilde{A})+w(\tilde{B})$.

(iii) Let

$$\tilde{A}\tilde{B}=\begin{pmatrix} \tilde{a}_{11}\tilde{b}_{11}+....+\tilde{a}_{1n}\tilde{b}_{n1} & \cdots & \tilde{a}_{11}\tilde{b}_{1n}+....+\tilde{a}_{1n}\tilde{b}_{nn} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{n1}\tilde{b}_{11}+....+\tilde{a}_{nn}\tilde{b}_{n1} & \cdots & \tilde{a}_{n1}\tilde{b}_{1n}+....+\tilde{a}_{nn}\tilde{b}_{nn} \end{pmatrix}$$

Then

$$m(\tilde{A}\tilde{B})=\begin{pmatrix} m(\tilde{a}_{11}\tilde{b}_{11}+....+\tilde{a}_{1n}\tilde{b}_{n1}) & \cdots & m(\tilde{a}_{11}\tilde{b}_{1n}+....+\tilde{a}_{1n}\tilde{b}_{nn}) \\ \vdots & \ddots & \vdots \\ m(\tilde{a}_{n1}\tilde{b}_{11}+....+\tilde{a}_{nn}\tilde{b}_{n1}) & \cdots & m(\tilde{a}_{n1}\tilde{b}_{1n}+....+\tilde{a}_{nn}\tilde{b}_{nn}) \end{pmatrix}$$

$$=\begin{pmatrix} m(\tilde{a}_{11}\tilde{b}_{11})+....+m(\tilde{a}_{1n}\tilde{b}_{n1}) & \cdots & m(\tilde{a}_{11}\tilde{b}_{1n})+....+m(\tilde{a}_{1n}\tilde{b}_{nn}) \\ \vdots & \ddots & \vdots \\ m(\tilde{a}_{n1}\tilde{b}_{11})+....+m(\tilde{a}_{nn}\tilde{b}_{n1}) & \cdots & m(\tilde{a}_{n1}\tilde{b}_{1n})+....+m(\tilde{a}_{nn}\tilde{b}_{nn}) \end{pmatrix}$$

$$=\begin{pmatrix} m(\tilde{a}_{11})m(\tilde{b}_{11})+....+m(\tilde{a}_{1n})m(\tilde{b}_{n1}) & \cdots & m(\tilde{a}_{11})m(\tilde{b}_{1n})+....+m(\tilde{a}_{1n})m(\tilde{b}_{nn}) \\ \vdots & \ddots & \vdots \\ m(\tilde{a}_{n1})m(\tilde{b}_{11})+....+m(\tilde{a}_{nn})m(\tilde{b}_{n1}) & \cdots & m(\tilde{a}_{n1})m(\tilde{b}_{1n})+....+m(\tilde{a}_{nn})m(\tilde{b}_{nn}) \end{pmatrix}. \quad (1)$$

Also

$$m(\tilde{A})m(\tilde{B})=\begin{pmatrix} m(\tilde{a}_{11})m(\tilde{b}_{11})+....+m(\tilde{a}_{1n})m(\tilde{b}_{n1}) & \cdots & m(\tilde{a}_{11})m(\tilde{b}_{1n})+....+m(\tilde{a}_{1n})m(\tilde{b}_{nn}) \\ \vdots & \ddots & \vdots \\ m(\tilde{a}_{n1})m(\tilde{b}_{11})+....+m(\tilde{a}_{nn})m(\tilde{b}_{n1}) & \cdots & m(\tilde{a}_{n1})m(\tilde{b}_{1n})+....+m(\tilde{a}_{nn})m(\tilde{b}_{nn}) \end{pmatrix}. \quad (2)$$

From (3.1) and (3.2), we see that $m(\tilde{A}\tilde{B})=m(\tilde{A})m(\tilde{B})$.

Proposition 3.2: Let $\tilde{A}, \tilde{B}, \tilde{C}\in IR^{n\times n}$. Then multiplication of interval matrices is associative with respect to the modified interval arithmetic, that is $(\tilde{A}\,\tilde{B})\,\tilde{C}\approx\tilde{A}(\tilde{B}\tilde{C})$, provided either side is defined.

Proof: Let $\tilde{A}=\begin{pmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \cdots & \tilde{a}_{nn} \end{pmatrix}$, $\tilde{B}=\begin{pmatrix} \tilde{b}_{11} & \cdots & \tilde{b}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{b}_{n1} & \cdots & \tilde{b}_{nn} \end{pmatrix}$

and $\tilde{C}=\begin{pmatrix} \tilde{c}_{11} & \cdots & \tilde{c}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{c}_{n1} & \cdots & \tilde{c}_{nn} \end{pmatrix}$. Now

$$\tilde{A}\tilde{B}=\begin{pmatrix} \tilde{a}_{11}\tilde{b}_{11}+....+\tilde{a}_{1n}\tilde{b}_{n1} & \cdots & \tilde{a}_{11}\tilde{b}_{1n}+....+\tilde{a}_{1n}\tilde{b}_{nn} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{n1}\tilde{b}_{11}+....+\tilde{a}_{nn}\tilde{b}_{n1} & \cdots & \tilde{a}_{n1}\tilde{b}_{1n}+....+\tilde{a}_{nn}\tilde{b}_{nn} \end{pmatrix}$$

and

$(\tilde{A}\tilde{B})\tilde{C}=$

$$\begin{pmatrix} (\tilde{a}_{11}\tilde{b}_{11}+...+\tilde{a}_{1n}\tilde{b}_{n1})\tilde{c}_{11}+...+(\tilde{a}_{11}\tilde{b}_{1n}+...+\tilde{a}_{1n}\tilde{b}_{nn})\tilde{c}_{n1}\cdots & ...(\tilde{a}_{11}\tilde{b}_{11}+...+\tilde{a}_{1n}\tilde{b}_{n1})\tilde{c}_{1n}+...+(\tilde{a}_{11}\tilde{b}_{1n}+...+\tilde{a}_{1n}\tilde{b}_{nn})\tilde{c}_{nn} \\ \cdots\cdots & \cdots\cdots \\ (\tilde{a}_{n1}\tilde{b}_{11}+...+\tilde{a}_{nn}\tilde{b}_{n1})\tilde{c}_{11}+...+(\tilde{a}_{n1}\tilde{b}_{1n}+...+\tilde{a}_{nn}\tilde{b}_{nn})\tilde{c}_{n1}\cdots & ...(\tilde{a}_{n1}\tilde{b}_{11}+...+\tilde{a}_{nn}\tilde{b}_{n1})\tilde{c}_{1n}+...+(\tilde{a}_{n1}\tilde{b}_{1n}+...+\tilde{a}_{nn}\tilde{b}_{nn})\tilde{c}_{nn} \end{pmatrix}$$

$$\approx \begin{pmatrix} \tilde{a}_{11}\tilde{b}_{11}\tilde{c}_{11}+\ldots+\tilde{a}_{1n}\tilde{b}_{n1}\tilde{c}_{11}+\ldots+\tilde{a}_{11}\tilde{b}_{1n}\tilde{c}_{n1}+\ldots+\tilde{a}_{1n}\tilde{b}_{nn}\tilde{c}_{n1}\cdots & \cdots\tilde{a}_{11}\tilde{b}_{11}\tilde{c}_{1n}+\ldots+\tilde{a}_{1n}\tilde{b}_{n1}\tilde{c}_{1n}+\ldots+\tilde{a}_{11}\tilde{b}_{1n}\tilde{c}_{nn}+\ldots+\tilde{a}_{1n}\tilde{b}_{nn}\tilde{c}_{nn} \\ \cdots\cdots & \cdots\cdots \\ \tilde{a}_{n1}\tilde{b}_{11}\tilde{c}_{11}+\ldots+\tilde{a}_{nn}\tilde{b}_{n1}\tilde{c}_{11}+\ldots+\tilde{a}_{n1}\tilde{b}_{1n}\tilde{c}_{n1}+\ldots+\tilde{a}_{nn}\tilde{b}_{nn}\tilde{c}_{n1}\cdots & \cdots\tilde{a}_{n1}\tilde{b}_{11}\tilde{c}_{1n}+\ldots+\tilde{a}_{nn}\tilde{b}_{n1}\tilde{c}_{1n}+\ldots+\tilde{a}_{n1}\tilde{b}_{1n}\tilde{c}_{nn}+\ldots+\tilde{a}_{nn}\tilde{b}_{nn}\tilde{c}_{nn} \end{pmatrix}$$

$$\approx \begin{pmatrix} \tilde{a}_{11}(\tilde{b}_{11}\tilde{c}_{11}+\ldots++\tilde{b}_{1n}\tilde{c}_{n1})+\ldots+\tilde{a}_{1n}(\tilde{b}_{n1}\tilde{c}_{11}+\ldots+\tilde{b}_{nn}\tilde{c}_{n1})\cdots & \cdots\tilde{a}_{11}(\tilde{b}_{11}\tilde{c}_{1n}+\ldots+\tilde{b}_{1n}\tilde{c}_{nn})+\ldots+\tilde{a}_{1n}(\tilde{b}_{n1}\tilde{c}_{1n}+\ldots+\tilde{b}_{nn}\tilde{c}_{nn}) \\ \cdots\cdots & \cdots\cdots \\ \tilde{a}_{n1}(\tilde{b}_{11}\tilde{c}_{11}+\ldots+\tilde{b}_{1n}\tilde{c}_{n1})+\ldots+\tilde{a}_{nn}(\tilde{b}_{n1}\tilde{c}_{11}+\ldots+\tilde{b}_{nn}\tilde{c}_{n1})\cdots & \cdots\tilde{a}_{n1}(\tilde{b}_{11}\tilde{c}_{1n}+\ldots+\tilde{b}_{1n}\tilde{c}_{nn})+\ldots+\tilde{a}_{nn}(\tilde{b}_{n1}\tilde{c}_{1n}+\ldots+\tilde{b}_{nn}\tilde{c}_{nn}) \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \cdots & \tilde{a}_{nn} \end{pmatrix} \begin{pmatrix} \tilde{b}_{11}\tilde{c}_{11}+\ldots+\tilde{b}_{1n}\tilde{c}_{n1} & \cdots & \tilde{b}_{11}\tilde{c}_{1n}+\ldots+\tilde{b}_{1n}\tilde{c}_{nn} \\ \cdots & \cdots & \cdots \\ \tilde{b}_{n1}\tilde{c}_{11}+\ldots+\tilde{b}_{nn}\tilde{c}_{n1} & \cdots & \tilde{b}_{n1}\tilde{c}_{1n}+\ldots+\tilde{b}_{nn}\tilde{c}_{nn} \end{pmatrix}$$

$= \tilde{A}(\tilde{B}\tilde{C})$. Hence $(\tilde{A}\,\tilde{B})\,\tilde{C} \approx \tilde{A}(\tilde{B}\tilde{C})$.

Example 3.1: It is to be noted that the associative law is not true with respect to the existing interval arithmetic.

Let $\tilde{A} = \begin{pmatrix} [-1, 0] & [1, 2] \\ [0, 1] & [2, 3] \end{pmatrix}$, $\tilde{B} = \begin{pmatrix} [1, 3] & [2, 3] \\ [1, 2] & [0, 2] \end{pmatrix}$ and

$\tilde{C} = \begin{pmatrix} [0, 1] & [1, 3] \\ [2, 3] & [-2, -1] \end{pmatrix}$ are (2×2) interval matrices in $IR^{n\times n}$.

By applying the existing interval arithmetic, we have

$\tilde{A}\tilde{B} = \begin{pmatrix} [-2, 4] & [-3, 4] \\ [2, 9] & [0, 9] \end{pmatrix}$ and

$(\tilde{A}\tilde{B})\tilde{C} = \begin{pmatrix} [-11, 16] & [-14, 18] \\ [0, 36] & [-16, 27] \end{pmatrix}$.

Also $\tilde{B}\tilde{C} = \begin{pmatrix} [4, 12] & [-5, 7] \\ [0, 8] & [-3, 6] \end{pmatrix}$ and

$\tilde{A}(\tilde{B}\tilde{C}) = \begin{pmatrix} [-12, 16] & [-13, 17] \\ [0, 36] & [-14, 25] \end{pmatrix}$. Here we see that

$m((\tilde{A}\tilde{B})\tilde{C}) = \begin{pmatrix} 2.5 & 2 \\ 18 & 4.5 \end{pmatrix}$ and $m(\tilde{A}(\tilde{B}\tilde{C})) = \begin{pmatrix} 2 & 2 \\ 18 & 5.5 \end{pmatrix}$, so that $m((\tilde{A}\,\tilde{B})\tilde{C}) \neq m(\tilde{A}(\tilde{B}\tilde{C}))$. Hence $(\tilde{A}\,\tilde{B})\,\tilde{C} \not\approx \tilde{A}(\tilde{B}\tilde{C})$.

Example 3.2: In many application problems, it is important to find the powers of interval matrices. Consider an example of computing the cube of an interval matrix.

Let $\tilde{A} = \begin{pmatrix} [1, 2] & [0, 3] \\ [3, 4] & [-2, 0] \end{pmatrix}$. By applying the existing interval arithmetic, we have

$\tilde{A}^2 = \tilde{A}\tilde{A} = \begin{pmatrix} [1, 2] & [0, 3] \\ [3, 4] & [-2, 0] \end{pmatrix}\begin{pmatrix} [1, 2] & [0, 3] \\ [3, 4] & [-2, 0] \end{pmatrix}$

$= \begin{pmatrix} [1, 16] & [-6, 9] \\ [-5, 8] & [0, 16] \end{pmatrix}$ and

$\tilde{A}^3 = \tilde{A}^2\tilde{A} = \begin{pmatrix} [-23, 68] & [-18, 60] \\ [-10, 80] & [-47, 24] \end{pmatrix}$. Also

$\tilde{A}^3 = \tilde{A}\tilde{A}^2 = \begin{pmatrix} [-14, 56] & [-12, 66] \\ [-13, 74] & [-56, 36] \end{pmatrix}$.

It is worth mentioning that we get different $\tilde{A}^3$ ($\tilde{A}^3$ are not even equivalent) for the same $\tilde{A}$ depending on the order in which we apply the matrix multiplication. It is to be noted that this is because of the non-associativity of interval matrices under the existing interval arithmetic and hence we are not able to proceed further in this direction.

Theorem 3.1: Let $\tilde{A}, \tilde{B}, \tilde{C} \in IR^{n\times n}$. Then multiplication of interval matrices is distributive with respect to addition of interval matrices, that is $\tilde{A}(\tilde{B}+\tilde{C}) \approx \tilde{A}\tilde{B}+\tilde{A}\tilde{C}$, provided either side is defined.

Proof: Let $\tilde{B}+\tilde{C} = \begin{pmatrix} \tilde{b}_{11}+\tilde{c}_{11} & \cdots & \tilde{b}_{1n}+\tilde{c}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{b}_{n1}+\tilde{c}_{n1} & \cdots & \tilde{b}_{nn}+\tilde{c}_{nn} \end{pmatrix}$ and

$$\tilde{A}(\tilde{B}+\tilde{C}) = \begin{pmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \cdots & \tilde{a}_{nn} \end{pmatrix} \begin{pmatrix} \tilde{b}_{11}+\tilde{c}_{11} & \cdots & \tilde{b}_{1n}+\tilde{c}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{b}_{n1}+\tilde{c}_{n1} & \cdots & \tilde{b}_{nn}+\tilde{c}_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \cdots\cdots\tilde{a}_{11}(\tilde{b}_{1n}+\tilde{c}_{1n})+\ldots+\tilde{a}_{1n}(\tilde{b}_{nn}+\tilde{c}_{nn}) \\ \cdots\cdots \\ \cdots\cdots\tilde{a}_{n1}(\tilde{b}_{1n}+\tilde{c}_{1n})+\ldots+\tilde{a}_{nn}(\tilde{b}_{nn}+\tilde{c}_{nn}) \end{pmatrix}$$

$$\approx \begin{pmatrix} \tilde{a}_{11}\tilde{b}_{11}+\tilde{a}_{11}\tilde{c}_{11}+\ldots+\tilde{a}_{1n}\tilde{b}_{n1}+\tilde{a}_{1n}\tilde{c}_{n1}\cdots\cdots \\ \cdots\cdots \\ \tilde{a}_{n1}\tilde{b}_{11}+\tilde{a}_{n1}\tilde{c}_{11}+\ldots+\tilde{a}_{nn}\tilde{b}_{n1}+\tilde{a}_{nn}\tilde{c}_{n1}\cdots\cdots \end{pmatrix}$$

$$\left.\begin{matrix} \ldots\ldots\tilde{a}_{11}\tilde{b}_{1n}+\tilde{a}_{11}\tilde{c}_{1n}+\ldots+\tilde{a}_{1n}\tilde{b}_{nn}+\tilde{a}_{1n}\tilde{c}_{nn} \\ \ldots\ldots \\ \ldots\ldots\tilde{a}_{n1}\tilde{b}_{1n}+\tilde{a}_{n1}\tilde{c}_{1n}+\ldots+\tilde{a}_{nn}\tilde{b}_{nn}+\tilde{a}_{nn}\tilde{c}_{nn} \end{matrix}\right)$$

$$=\begin{pmatrix} \tilde{a}_{11}\tilde{b}_{11}+\ldots+\tilde{a}_{1n}\tilde{b}_{n1} & \cdots & \tilde{a}_{11}\tilde{b}_{1n}+\ldots+\tilde{a}_{1n}\tilde{b}_{nn} \\ \ldots & \ldots & \ldots \\ \tilde{a}_{n1}\tilde{b}_{11}+\ldots+\tilde{a}_{nn}\tilde{b}_{n1} & \cdots & \tilde{a}_{n1}\tilde{b}_{1n}+\ldots+\tilde{a}_{nn}\tilde{b}_{nn} \end{pmatrix}+\begin{pmatrix} \tilde{a}_{11}\tilde{c}_{11}+\ldots+\tilde{a}_{1n}\tilde{c}_{n1} & \cdots & \tilde{a}_{11}\tilde{c}_{1n}+\ldots+\tilde{a}_{1n}\tilde{c}_{nn} \\ \ldots & \ldots & \ldots \\ \tilde{a}_{n1}\tilde{c}_{11}+\ldots+\tilde{a}_{nn}\tilde{c}_{n1} & \cdots & \tilde{a}_{n1}\tilde{c}_{1n}+\ldots+\tilde{a}_{nn}\tilde{c}_{nn} \end{pmatrix}$$

$=\tilde{A}\tilde{B}+\tilde{A}\tilde{C}$. Hence $\tilde{A}(\tilde{B}+\tilde{C})\approx\tilde{A}\tilde{B}+\tilde{A}\tilde{C}$.

Remark 3.1: It is to be noted that the distributive law for interval matrices is not true under the existing interval arithmetic, that is $\tilde{A}(\tilde{B}+\tilde{C})\not\approx\tilde{A}\tilde{B}+\tilde{A}\tilde{C}$.

Example 3.3: Let $\tilde{A}=\begin{pmatrix} [-1, 0] & [1, 2] \\ [0, 1] & [2, 3] \end{pmatrix}$,

$\tilde{B}=\begin{pmatrix} [1, 3] & [2, 3] \\ [1, 2] & [0, 2] \end{pmatrix}$ and $\tilde{C}=\begin{pmatrix} [0, 1] & [1, 3] \\ [2, 3] & [-2, -1] \end{pmatrix}$ are (2×2) interval matrices in $IR^{n\times n}$. By applying the existing interval arithmetic, we have $\tilde{A}\tilde{B}=\begin{pmatrix} [-2, 4] & [-3, 4] \\ [2, 9] & [0, 9] \end{pmatrix}$ and

$\tilde{A}\tilde{C}=\begin{pmatrix} [1, 6] & [-7, -1] \\ [4, 10] & [-6, 1] \end{pmatrix}$ so that

$\tilde{A}\tilde{B}+\tilde{A}\tilde{C}=\begin{pmatrix} [-1, 10] & [-10, 3] \\ [6, 19] & [-6, 10] \end{pmatrix}$ and hence

$m(\tilde{A}\tilde{B}+\tilde{A}\tilde{C})=\begin{pmatrix} 4.5 & -3.5 \\ 12.5 & 2 \end{pmatrix}$. Also

$(\tilde{B}+\tilde{C})=\begin{pmatrix} [1, 4] & [3, 6] \\ [3, 5] & [-2, 1] \end{pmatrix}$ and

$\tilde{A}(\tilde{B}+\tilde{C})=\begin{pmatrix} [-1, 10] & [-10, 2] \\ [6, 19] & [-6, 9] \end{pmatrix}$

Now $m(\tilde{A}(\tilde{B}+\tilde{C}))=\begin{pmatrix} 4.5 & -4 \\ 12.5 & 1.5 \end{pmatrix}$. Here we see that

$m(\tilde{A}(\tilde{B}+\tilde{C}))\neq m(\tilde{A}\tilde{B}+\tilde{A}\tilde{C})$. Hence $\tilde{A}(\tilde{B}+\tilde{C})\not\approx\tilde{A}\tilde{B}+\tilde{A}\tilde{C}$.

Theorem 3.2: The commutative law with respective scalar interval numbers under the modified interval arithmetic is true, that is $\tilde{\alpha}(\tilde{A}\tilde{\mathbf{x}})\approx\tilde{A}(\tilde{\alpha}\tilde{\mathbf{x}})$.

Proof: Let $\tilde{\alpha}\in IR$, $\tilde{\mathbf{x}}\in IR^{n}$ and $\tilde{A}\in IR^{mxn}$ with

$\tilde{\alpha}=[\alpha_1,\alpha_2]$, $\tilde{A}=\begin{pmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \cdots & \tilde{a}_{mn} \end{pmatrix}$ and $\tilde{\mathbf{x}}=\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \ldots \\ \tilde{x}_n \end{pmatrix}$. Now

$\tilde{\alpha}\tilde{\mathbf{x}}=\tilde{\alpha}\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \ldots \\ \tilde{x}_n \end{pmatrix}=\begin{pmatrix} \tilde{\alpha}\tilde{x}_1 \\ \tilde{\alpha}\tilde{x}_2 \\ \ldots \\ \tilde{\alpha}\tilde{x}_n \end{pmatrix}$. Also

$$\tilde{A}(\tilde{\alpha}\tilde{\mathbf{x}})=\begin{pmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \cdots & \tilde{a}_{mn} \end{pmatrix}\begin{pmatrix} \tilde{\alpha}\tilde{x}_1 \\ \tilde{\alpha}\tilde{x}_2 \\ \ldots \\ \tilde{\alpha}\tilde{x}_n \end{pmatrix}$$

$$=\begin{pmatrix} \tilde{a}_{11}\tilde{\alpha}\tilde{x}_1+\tilde{a}_{12}\tilde{\alpha}\tilde{x}_2+\ldots+\tilde{a}_{1n}\tilde{\alpha}\tilde{x}_n \\ \tilde{a}_{21}\tilde{\alpha}\tilde{x}_1+\tilde{a}_{22}\tilde{\alpha}\tilde{x}_2+\ldots+\tilde{a}_{2n}\tilde{\alpha}\tilde{x}_n \\ \ldots \\ \tilde{a}_{n1}\tilde{\alpha}\tilde{x}_1+\tilde{a}_{n2}\tilde{\alpha}\tilde{x}_2+\ldots+\tilde{a}_{nn}\tilde{\alpha}\tilde{x}_n \end{pmatrix}$$

$$=\tilde{\alpha}\begin{pmatrix} \tilde{a}_{11}\tilde{x}_1+\tilde{a}_{12}\tilde{x}_2+\ldots+\tilde{a}_{1n}\tilde{x}_n \\ \tilde{a}_{21}\tilde{x}_1+\tilde{a}_{22}\tilde{x}_2+\ldots+\tilde{a}_{2n}\tilde{x}_n \\ \ldots \\ \tilde{a}_{n1}\tilde{x}_1+\tilde{a}_{n2}\tilde{x}_2+\ldots+\tilde{a}_{nn}\tilde{x}_n \end{pmatrix}=\tilde{\alpha}(\tilde{A}\tilde{\mathbf{x}}).$$

That is $\tilde{\alpha}(\tilde{A}\tilde{\mathbf{x}})\approx\tilde{A}(\tilde{\alpha}\tilde{\mathbf{x}})$.

Remark 3.2: It is to be noted that the commutative law with respective scalars under the existing interval arithmetic is not true, that is $\tilde{\alpha}(\tilde{A}\tilde{\mathbf{x}})\not\approx\tilde{A}(\tilde{\alpha}\tilde{\mathbf{x}})$.

Example 3.4: Let $\tilde{A}=\begin{pmatrix} [-1, 0] & [1, 2] \\ [0, 1] & [2, 3] \end{pmatrix}$, $\tilde{\mathbf{x}}=\begin{pmatrix} [1,3] \\ [1,2] \end{pmatrix}$ are interval matrices and $\tilde{\alpha}=[2,3]$ be an interval number. By applying the existing interval arithmetic, we have

$\tilde{A}\tilde{\mathbf{x}}=\begin{pmatrix} [-1, 0] & [1, 2] \\ [0, 1] & [2, 3] \end{pmatrix}\begin{pmatrix} [1,3] \\ [1,2] \end{pmatrix}=\begin{pmatrix} [-2,4] \\ [2,9] \end{pmatrix}$ and

$\tilde{\alpha}(\tilde{A}\tilde{\mathbf{x}})=[2,3]\begin{pmatrix} [-2,4] \\ [2,9] \end{pmatrix}=\begin{pmatrix} [-6,12] \\ [4,27] \end{pmatrix}$ so that

$m(\tilde{\alpha}(\tilde{A}\tilde{\mathbf{x}}))=\begin{pmatrix} 3 \\ 15.5 \end{pmatrix}$. Also

$\tilde{\alpha}\tilde{\mathbf{x}}=\begin{pmatrix} [2,9] \\ [2,6] \end{pmatrix}$ and $\tilde{A}(\tilde{\alpha}\tilde{\mathbf{x}})=\begin{pmatrix} [-7,12] \\ [4,27] \end{pmatrix}$ so that

$m(\tilde{A}(\tilde{\alpha}\tilde{\mathbf{x}}))=\begin{pmatrix} 2.5 \\ 15.5 \end{pmatrix}$. From these wee see that

$m(\tilde{\alpha}(\tilde{A}\tilde{\mathbf{x}}))\neq m(\tilde{A}(\tilde{\alpha}\tilde{\mathbf{x}}))$ and hence $\tilde{\alpha}(\tilde{A}\tilde{\mathbf{x}})\not\approx\tilde{A}(\tilde{\alpha}\tilde{\mathbf{x}})$.

B. Determinant of an Interval Matrix

Consider an interval matrix $\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{pmatrix}$ of order (2 × 2).

Let us define the determinant of $\tilde{A}$ as

$|\tilde{A}| = \begin{vmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{vmatrix} = \tilde{a}_{11}\tilde{a}_{22} - \tilde{a}_{12}\tilde{a}_{21}$. From this we see that defining the determinant of a square interval matrix of order (2 × 2) is not a difficult task under the existing interval arithmetic.

Now we shall consider an interval matrix

$\tilde{B} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{a}_{23} \\ \tilde{a}_{31} & \tilde{a}_{32} & \tilde{a}_{33} \end{pmatrix}$ of order (3 × 3). Now we find $|\tilde{B}|$ by applying the existing interval arithmetic as

$\begin{vmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{a}_{23} \\ \tilde{a}_{31} & \tilde{a}_{32} & \tilde{a}_{33} \end{vmatrix} = \tilde{a}_{11}\tilde{A}_{11} + \tilde{a}_{12}\tilde{A}_{12} + \tilde{a}_{13}\tilde{A}_{13} =$

$\tilde{a}_{11}(\tilde{a}_{22}\tilde{a}_{33} - \tilde{a}_{32}\tilde{a}_{23})$ - $\tilde{a}_{12}(\tilde{a}_{21}\tilde{a}_{32} - \tilde{a}_{31}\tilde{a}_{23}) +$ $\tilde{a}_{13}(\tilde{a}_{21}\tilde{a}_{32} - \tilde{a}_{31}\tilde{a}_{22})$, which is not even equivalent to $\tilde{a}_{11}\tilde{a}_{22}\tilde{a}_{33} - \tilde{a}_{11}\tilde{a}_{32}\tilde{a}_{23}$ - $\tilde{a}_{12}\tilde{a}_{21}\tilde{a}_{32} + \tilde{a}_{12}\tilde{a}_{31}\tilde{a}_{23}$ $+ \tilde{a}_{13}\tilde{a}_{21}\tilde{a}_{32} - \tilde{a}_{13}\tilde{a}_{31}\tilde{a}_{22}$. (Here $\tilde{A}_{ij}$ is the cofactor of $\tilde{a}_{ij}$ in the usual sense.)

This is because the distributive law is not true under the existing interval arithmetic.

On the other hand if we apply the modified interval arithmetic to evaluate $|\tilde{B}|$, we have

$\begin{vmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{a}_{23} \\ \tilde{a}_{31} & \tilde{a}_{32} & \tilde{a}_{33} \end{vmatrix} = \tilde{a}_{11}\tilde{A}_{11} + \tilde{a}_{12}\tilde{A}_{12} + \tilde{a}_{13}\tilde{A}_{13}$

$= \tilde{a}_{11}(\tilde{a}_{22}\tilde{a}_{33} - \tilde{a}_{32}\tilde{a}_{23})$ - $\tilde{a}_{12}(\tilde{a}_{21}\tilde{a}_{32} - \tilde{a}_{31}\tilde{a}_{23}) +$ $\tilde{a}_{13}(\tilde{a}_{21}\tilde{a}_{32} - \tilde{a}_{31}\tilde{a}_{22}) \approx \tilde{a}_{11}\tilde{a}_{22}\tilde{a}_{33} - \tilde{a}_{11}\tilde{a}_{32}\tilde{a}_{23}$ - $\tilde{a}_{12}\tilde{a}_{21}\tilde{a}_{32} + \tilde{a}_{12}\tilde{a}_{31}\tilde{a}_{23} + \tilde{a}_{13}\tilde{a}_{21}\tilde{a}_{32} - \tilde{a}_{13}\tilde{a}_{31}\tilde{a}_{22}$

≈ an interval number.

By induction, we define the determinant of an interval matrix $\tilde{A} = (\tilde{a}_{ij})$ of order (n × n) as:

$\det\tilde{A} = |\tilde{A}| = \sum \tilde{a}_{ij}\tilde{A}_{ij},$ where $\tilde{A}_{ij}$ is the cofactor of $\tilde{a}_{ij}$ in the usual sense.

It is easy to see that most of the properties of determinants of classical matrices are hold good (up to equivalent) for the determinants of interval matrices under the modified interval arithmetic.

Definition 3.1: A square interval matrix $\tilde{A}$ is said to be *invertible* if $|\tilde{A}|$ is invertible (i.e. $|\tilde{A}| \not\approx \tilde{0}$) and is denoted by $\tilde{A}^{-1} = \dfrac{\operatorname{adj}(\tilde{A})}{|\tilde{A}|}$. Here $\operatorname{adj}(\tilde{A})$ is with usual meaning.

Theorem 3.5: Let $\tilde{A}\tilde{\mathbf{x}} \approx \tilde{\mathbf{b}}$ be a system of linear equations involving interval numbers. If the (n × n) interval matrix $\tilde{A}$ is invertible, then it is possible to find a smallest box $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots\ldots, \tilde{x}_n)$ which containing the exact solution of the system $\tilde{A}\tilde{\mathbf{x}} \approx \tilde{\mathbf{b}}$, where each $\tilde{x}_i = \dfrac{|\tilde{A}^{(i)}|}{|\tilde{A}|}$, $\tilde{A}^{(i)}$ is the interval matrix obtained when the ith column of $\tilde{A}$ is replaced by the vector $\tilde{\mathbf{b}} = (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \ldots\ldots, \tilde{b}_n)$.

Example 3.5: Let us consider an example given in Ning et al [11].

The system of interval equations $\tilde{A}\tilde{\mathbf{x}} \approx \tilde{\mathbf{b}}$ be given with

$\begin{pmatrix} [3.7, 4.3] & [-1.5, -0.5] & [0, 0] \\ [-1.5, -0.5] & [3.7, 4.3] & [-1.5, -0.5] \\ [0, 0] & [-1.5, -0.5] & [3.7, 4.3] \end{pmatrix}$ and

$\tilde{\mathbf{b}} = \begin{pmatrix} [-14, 0] \\ [-9, 0] \\ [-3, 0] \end{pmatrix}$. Here

$|\tilde{A}| = \begin{vmatrix} [3.7, 4.3] & [-1.5, -0.5] & [0, 0] \\ [-1.5, -0.5] & [3.7, 4.3] & [-1.5, -0.5] \\ [0, 0] & [-1.5, -0.5] & [3.7, 4.3] \end{vmatrix}$

= [37.103, 74.897] and $|\tilde{A}| \not\approx \tilde{0}$.

Now $|\tilde{A}^{(1)}| = \begin{vmatrix} [-14, 0] & [-1.5, -0.5] & [0, 0] \\ [-9, 0] & [3.7, 4.3] & [-1.5, -0.5] \\ [-3, 0] & [-1.5, -0.5] & [3.7, 4.3] \end{vmatrix}$

≈ [−14, 0] ([3.7, 4.3] [3.7, 4.3] − [−1.5, −0.5][−1.5, −0.5]) - [-1.5, -0.5] ([-9, 0] [3.7, 4.3] - [-3, 0] [-1.5, -0.5]) = [-14, 0] ([13.69, 18.31]-[0.25, 1.75]) - [-1.5, -0.5] ([-36, 0]-[0, 3]) = [-14, 0][11.94, 18.06] + [0.5, 1.5] [-6.15, -1.85] = [-210, 0] + [-39, 0] = [-249, 0] and

$|\tilde{A}^{(2)}| = \begin{vmatrix} [3.7, 4.3] & [-14, 0] & [0, 0] \\ [-1.5, -0.5] & [-9, 0] & [-1.5, -0.5] \\ [0, 0] & [-3, 0] & [3.7, 4.3] \end{vmatrix}$

≈ [3.7, 4.3] ([−9, 0] [3.7, 4.3] − [−3, 0] [−1.5, −0.5]) - [-14, 0] ([-1.5, -0.5] [3.7, 4.3]-[0, 0] [-1.5, -0.5]) = [3.7, 4.3] ([-36, 0]

-[0, 3]) - [-14, 0] ([-6.15, -1.85]-[0, 0]) =[3.7, 4.3][-39, 0] +[0, 14] [-6.15,-1.85] =[-156,0] + [-56,0] = [-212, 0].

Also $|\tilde{A}^{(3)}| = \begin{vmatrix} [3.7, 4.3] & [-1.5, -0.5] & [-14, 0] \\ [-1.5, -0.5] & [3.7, 4.3] & [-9, 0] \\ [0, 0] & [-1.5, -0.5] & [-3, 0] \end{vmatrix}$

≈ [3.7, 4.3] ([3.7, 4.3] [−3, 0] − [−1.5, −0.5] [−9, 0]) – [-1.5,-0.5] ([-1.5,-0.5] [-3, 0]-[0, 0] [-9, 0]) + [-14, 0] ([-1.5,-0.5][-1.5,-0.5] - [0, 0][3.7, 4.3]) = [3.7, 4.3]([-12, 0] -[0, 9])-[-1.5, -0.5][0, 3] + [-14, 0] [0.25, 1.75] = [3.7, 4.3] [-21, 0] - [-1.5, -0.5] [0, 3] = + [-14, 0] [0.25, 1.75] = [-184, 0] + [0, 3] + [-14, 0] = [-98, 3].

Then by the above theorem we see that

$$\tilde{x}_1 = \frac{|\tilde{A}^{(1)}|}{|\tilde{A}|} = \frac{[-249,0]}{[37.103,74.897]} = [-4.482,0],$$

$$\tilde{x}_2 = \frac{|\tilde{A}^{(2)}|}{|\tilde{A}|} = \frac{[-212,0]}{[37.103,74.897]} = [-3.816,0] \quad \text{and}$$

$$\tilde{x}_3 = \frac{|\tilde{A}^{(3)}|}{|\tilde{A}|} = \frac{[-98,0]}{[37.103,74.897]} = [-1.776,0.006].$$

In this case, we obtain the solution set (box) $\mathbf{\tilde{x}} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} [-4.482,0] \\ [-3.816,0] \\ [-1.776,0.006] \end{pmatrix}$. Using interval Gaussian elimination with existing interval arithmetic, Ning et al [11] obtained the solution set (box) $\begin{pmatrix} [-6.38,0] \\ [-6.40,0] \\ [-3.40,0] \end{pmatrix}$. Using Hansen's technique of [7] or Rohn's reformulation of [14], Ning et al [11] obtained the solution set (wider box) $\begin{pmatrix} [-6.38,1.12] \\ [-6.40,1.54] \\ [-3.40,1.40] \end{pmatrix}$. Using their technique, Ning et al [11] obtained the solution set (much wider box) $\begin{pmatrix} [-6.38,1.67] \\ [-6.40,2.77] \\ [-3.40,2.40] \end{pmatrix}$. It is to be noted that the solution set (box) obtained by our method is sharper then the solution sets obtained by other techniques.

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K. Ganesan was born in Perambalur, India in 1962 .He received B.Sc degree in Mathematics from the University of Madras in 1983, M.Sc degree in Mathematics from Bharathidasan University in 1986, M.Phil degree in Mathematics from the University of Madras in 1987 and Ph.D degree in Fuzzy and Interval Linear Programming Problems from the University of Madras in 2006. He is working as a Professor in the Department of Mathematics, S. R. M. University, Kattankulathur, Chennai – 603 203, India. His areas of interests are fuzzy set theory, interval analysis, operations research and Fuzzy Optimization.