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Certain Conditions for Strongly Starlike and Strongly Convex Functions

Sukhwinder Singh Billing, Sushma Gupta and Sukhjit Singh Dhaliwal

Abstract—In the present paper, we investigate a differential subordination involving multiplier transformation related to a sector in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$. As special cases to our main result, certain sufficient conditions for strongly starlike and strongly convex functions are obtained.

Keywords—Analytic function, Multiplier transformation, Strongly starlike function, Strongly convex function.

I. Introduction

LET $\mathcal{H}$ be the class of functions analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$ and for $a \in \mathbb{C}$ (set of complex numbers) and $n \in \mathbb{N} = \{1, 2, \cdots\}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions f of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots .$$

Let $\mathcal{A}$ be the class of functions f, analytic in $\mathbb{E}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$.

A function $f \in \mathcal{A}$ is said to be strongly starlike of order α, $0 < \alpha \leq 1$, if

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2},$$

equivalently

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}.$$

A function $f \in \mathcal{A}$ is said to be strongly convex of order α, $0 < \alpha \leq 1$, if

$$\left|\arg \left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\alpha\pi}{2},$$

equivalently

$$1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}.$$

For two analytic functions f and g in the open unit disk $\mathbb{E}$, we say that f is subordinate to g in $\mathbb{E}$ and write as $f \prec g$ if there exists a Schwarz function w analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1,\ z \in \mathbb{E}$ such that $f(z) = g(w(z))$.

In case the function g is univalent, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

Sukhwinder Singh Billing is with the Deaprtment of Applied Sciences, Baba Banda Singh Bahadur Engineering College, Fatehgarh Sahib-140 407, Punjab, India.
E-mail: ssbilling@gmail.com
Sushma Gupta and Sukhjit Singh Dhaliwal are with the Department of Mathematics, Sant Longowal Institute of Engineering & Technology, Longowal - 148 106 (Deemed University), Sangrur, Punjab, India.
E-mails: sushmagupta1@yahoo.com , sukhjit_d@yahoo.com
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Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k,\ p \in \mathbb{N},$$

which are analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$. We note that $\mathcal{A}_1 = \mathcal{A}$.

For $f \in \mathcal{A}_p$, we define the multiplier transformation $I_p(n, \lambda)$ as

$$I_p(n,\lambda)[f](z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k,\ (\lambda \geq 0, n \in \mathbb{Z}).$$

The operator $I_p(n, \lambda)$ has been recently studied by Aghalary et al. [1]. Earlier, the operator $I_1(n, \lambda)$ was investigated by Cho and Kim [2] and Cho and Srivastava [3], whereas the operator $I_1(n, 1)$ was studied by Uralegaddi and Somanatha [9]. $I_1(n, 0)$ is the well-known Sălăgean [8] derivative operator D^n, defined as:

$$D^n[f](z) = z + \sum_{k=2}^{\infty} k^n a_k z^k,\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

where $f \in \mathcal{A}$.

In 1989, the operator $I_1(n, 0)$ has been studied by Owa, Shen and Obradovič [7]. Recently, Li and Owa [4] studied the operator $I_1(n, 0)$. Many significant results regarding the operator $I_p(n, \lambda)$ have been obtained by different authors.

In the present paper, we study a differential subordination involving multiplier transformation in a sector. As special cases to our main result, we derive some simple sufficient conditions for members of the class $\mathcal{A}$ to be strongly starlike and strongly convex functions.

II. Preliminaries

We shall need the following lemma to prove the main result.

Lemma 2.1: ([5]). Let $\mu > 0$ be a real number and let $\beta_0 = \beta_0(\mu, n), n \in \mathbb{N}$ be the root of the equation $\beta\pi = \frac{3\pi}{2} - \arctan(n\mu\beta)$.

Let

$$\alpha = \alpha(\beta, \mu, n) = \beta + \frac{2}{\pi}\arctan(n\mu\beta),\ 0 < \beta \leq \beta_0. \quad (1)$$

If $P \in \mathcal{H}[1, n]$, then $P(z) + \mu z P'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha}$ implies $P(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}$.

III. Main Result

Theorem 3.1: If $f \in \mathcal{A}_p$ satisfies

$$(1-\gamma)\frac{I_p(n,\lambda)[f](z)}{z^p} + \gamma\ \frac{I_p(n+1,\lambda)[f](z)}{z^p} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}, \tag{2}$$

then

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta},$$

where $\alpha = \alpha\left(\dfrac{\gamma}{p+\lambda}, \delta\right)$ satisfies the equation

$$2\arctan\left[\frac{\gamma}{p+\lambda}(\delta-\alpha)\right] + \pi(\delta - 2\alpha) = 0, \tag{3}$$

and γ and δ are real numbers such that $\gamma \geq 1,\ 0 < \delta \leq 1$.

Proof: Let us define

$$\frac{I_p(n,\lambda)[f](z)}{z^p} = u(z). \tag{4}$$

Differentiate (4) logarithmically, we obtain

$$\frac{zI_p'(n,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} - p = \frac{zu'(z)}{u(z)}. \tag{5}$$

In view of the equality

$$zI_p'(n,\lambda)[f](z) = (p+\lambda)I_p(n+1,\lambda)[f](z) - \lambda I_p(n,\lambda)[f](z),$$

(5) reduces to

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} = 1 + \frac{zu'(z)}{(p+\lambda)u(z)}.$$

A little calculation yields

$$u(z) + \frac{\gamma}{p+\lambda}zu'(z) = (1-\gamma)\frac{I_p(n,\lambda)[f](z)}{z^p} + \gamma\ \frac{I_p(n+1,\lambda)[f](z)}{z^p}.$$

Therefore, in view of (2), we have

$$u(z) + \frac{\gamma}{p+\lambda}zu'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha}. \tag{6}$$

We note that for $\alpha + \beta = \delta$ and $\mu = \dfrac{\gamma}{p+\lambda}$, the condition (3) corresponds to the condition (1) of Lemma 2.1. Therefore, in view of Lemma 2.1, we have

$$u(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta} \tag{7}$$

where β satisfies the condition (1) of Lemma 2.1.

Let us, now, write $u(z) + \dfrac{\gamma}{p+\lambda}zu'(z) = v(z)$ and therefore, we have

$$\frac{I_p(n+1,\lambda)[f](z)}{z^p} = \left(1 - \frac{1}{\gamma}\right)u(z) + \frac{1}{\gamma}\ v(z).$$

Obviously, $\dfrac{I_p(n+1,\lambda)[f](z)}{z^p}$ is a convex combination of $u(z)$ and $v(z)$.

In view of condition (1) of Lemma 2.1, we conclude that $\alpha > \beta$, thus, from (6) and (7), we have

$$\frac{I_p(n+1,\lambda)[f](z)}{z^p} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}. \tag{8}$$

Write $w(z) = \dfrac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}$, obviously $w \in \mathcal{H}[1,1]$ and we can rewrite w as

$$w(z) = \frac{I_p(n+1,\lambda)[f](z)/z^p}{u(z)}.$$

From (7) and (8),we obtain

$$|\arg w(z)| \leq \left|\arg\frac{I_p(n+1,\lambda)[f](z)}{z^p}\right| + |\arg u(z)|$$
$$< \alpha\frac{\pi}{2} + \beta\frac{\pi}{2} = (\alpha+\beta)\frac{\pi}{2} = \delta\frac{\pi}{2}.$$

Hence, we have

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta},\ z \in \mathbb{E}.$$

■

IV. Applications to Univalent Functions

In this section, using Theorem 3.1, we derive certain sufficient conditions for strongly starlike and strongly convex functions.

On writing $p = 1$ and $\lambda = 0$ in Theorem 3.1. We have the following result.

Corollary 4.1: If $f \in \mathcal{A}$ satisfies

$$(1-\gamma)\frac{D^n[f](z)}{z} + \gamma\ \frac{D^{n+1}[f](z)}{z} \prec \left(\frac{1+z}{1-z}\right)^{\alpha},$$

then

$$\frac{D^{n+1}[f](z)}{D^n[f](z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta},$$

where $\alpha = \alpha(\gamma, \delta)$ satisfies the equation

$$2\arctan[\gamma(\delta-\alpha)] + \pi(\delta - 2\alpha) = 0,$$

and γ and δ are real numbers with $\gamma \geq 1,\ 0 < \delta \leq 1$.

When we select $p = 1, n = 0$ and $\lambda = 0$ in Theorem 3.1. We obtain the following result of Oros [6].

Corollary 4.2: If $f \in \mathcal{A}$ satisfies

$$(1-\gamma)\frac{f(z)}{z} + \gamma\ f'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha},$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta},$$

where $\alpha = \alpha(\gamma, \delta)$ satisfies the equation

$$2\arctan[\gamma(\delta-\alpha)] + \pi(\delta - 2\alpha) = 0,$$

and γ and δ are real numbers with $\gamma \geq 1,\ 0 < \delta \leq 1$.

By taking $p = 1$, $n = 1$ and $\lambda = 0$ in Theorem 3.1. We obtain the following result.

Corollary 4.3: If $f \in \mathcal{A}$ satisfies

$$f'(z) + \gamma\ zf''(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha},$$

then

$$1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta},$$

where $\alpha = \alpha(\gamma, \delta)$ satisfies the equation

$$2\arctan[\gamma(\delta - \alpha)] + \pi(\delta - 2\alpha) = 0,$$

and γ and δ are real numbers with $\gamma \geq 1,\ 0 < \delta \leq 1$.

By setting $p = 1$, $n = 0$ and $\lambda = 1$ in Theorem 3.1. We have the following result.

Corollary 4.4: If $f \in \mathcal{A}$ satisfies

$$\left(1 - \frac{\gamma}{2}\right)\frac{f(z)}{z} + \frac{\gamma}{2}f'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha},$$

then

$$\frac{1}{2}\left(1 + \frac{zf'(z)}{f(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^{\delta},$$

where $\alpha = \alpha\left(\frac{\gamma}{2}, \delta\right)$ satisfies the equation

$$2\arctan\left[\frac{\gamma}{2}(\delta - \alpha)\right] + \pi(\delta - 2\alpha) = 0,$$

and γ and δ are real numbers such that $\gamma \geq 1,\ 0 < \delta \leq 1$.

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