

# Certain Conditions for Strongly Starlike and Strongly Convex Functions

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**Abstract**—In the present paper, we investigate a differential subordination involving multiplier transformation related to a sector in the open unit disk  $\mathbb{E} = \{z : |z| < 1\}$ . As special cases to our main result, certain sufficient conditions for strongly starlike and strongly convex functions are obtained.

**Keywords**—Analytic function, Multiplier transformation, Strongly starlike function, Strongly convex function.

## I. INTRODUCTION

Let  $\mathcal{H}$  be the class of functions analytic in the open unit disk  $\mathbb{E} = \{z : |z| < 1\}$  and for  $a \in \mathbb{C}$  (set of complex numbers) and  $n \in \mathbb{N} = \{1, 2, \dots\}$ , let  $\mathcal{H}[a, n]$  be the subclass of  $\mathcal{H}$  consisting of functions  $f$  of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$$

Let  $\mathcal{A}$  be the class of functions  $f$ , analytic in  $\mathbb{E}$  and normalized by the conditions  $f(0) = f'(0) - 1 = 0$ .

A function  $f \in \mathcal{A}$  is said to be strongly starlike of order  $\alpha$ ,  $0 < \alpha \leq 1$ , if

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\alpha\pi}{2},$$

equivalently

$$\frac{zf'(z)}{f(z)} \prec \left( \frac{1+z}{1-z} \right)^\alpha.$$

A function  $f \in \mathcal{A}$  is said to be strongly convex of order  $\alpha$ ,  $0 < \alpha \leq 1$ , if

$$\left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\alpha\pi}{2},$$

equivalently

$$1 + \frac{zf''(z)}{f'(z)} \prec \left( \frac{1+z}{1-z} \right)^\alpha.$$

For two analytic functions  $f$  and  $g$  in the open unit disk  $\mathbb{E}$ , we say that  $f$  is subordinate to  $g$  in  $\mathbb{E}$  and write as  $f \prec g$  if there exists a Schwarz function  $w$  analytic in  $\mathbb{E}$  with  $w(0) = 0$  and  $|w(z)| < 1$ ,  $z \in \mathbb{E}$  such that  $f(z) = g(w(z))$ .

In case the function  $g$  is univalent, the above subordination is equivalent to  $f(0) = g(0)$  and  $f(\mathbb{E}) \subset g(\mathbb{E})$ .

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Let  $\mathcal{A}_p$  denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad p \in \mathbb{N},$$

which are analytic in the open unit disk  $\mathbb{E} = \{z : |z| < 1\}$ . We note that  $\mathcal{A}_1 = \mathcal{A}$ .

For  $f \in \mathcal{A}_p$ , we define the multiplier transformation  $I_p(n, \lambda)$  as

$$I_p(n, \lambda)[f](z) = z^p + \sum_{k=p+1}^{\infty} \left( \frac{k+\lambda}{p+\lambda} \right)^n a_k z^k, \quad (\lambda \geq 0, n \in \mathbb{Z}).$$

The operator  $I_p(n, \lambda)$  has been recently studied by Aghalary et al. [1]. Earlier, the operator  $I_1(n, \lambda)$  was investigated by Cho and Kim [2] and Cho and Srivastava [3], whereas the operator  $I_1(n, 1)$  was studied by Uralegaddi and Somanatha [9].  $I_1(n, 0)$  is the well-known Sălăgean [8] derivative operator  $D^n$ , defined as:

$$D^n[f](z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

where  $f \in \mathcal{A}$ .

In 1989, the operator  $I_1(n, 0)$  has been studied by Owa, Shen and Obradović [7]. Recently, Li and Owa [4] studied the operator  $I_1(n, 0)$ . Many significant results regarding the operator  $I_p(n, \lambda)$  have been obtained by different authors.

In the present paper, we study a differential subordination involving multiplier transformation in a sector. As special cases to our main result, we derive some simple sufficient conditions for members of the class  $\mathcal{A}$  to be strongly starlike and strongly convex functions.

## II. PRELIMINARIES

We shall need the following lemma to prove the main result.

**Lemma 2.1:** ([5]). Let  $\mu > 0$  be a real number and let  $\beta_0 = \beta_0(\mu, n)$ ,  $n \in \mathbb{N}$  be the root of the equation  $\beta\pi = \frac{3\pi}{2} - \arctan(n\mu\beta)$ .

Let

$$\alpha = \alpha(\beta, \mu, n) = \beta + \frac{2}{\pi} \arctan(n\mu\beta), \quad 0 < \beta \leq \beta_0. \quad (1)$$

If  $P \in \mathcal{H}[1, n]$ , then  $P(z) + \mu z P'(z) \prec \left( \frac{1+z}{1-z} \right)^\alpha$  implies

$$P(z) \prec \left( \frac{1+z}{1-z} \right)^\beta.$$

## III. MAIN RESULT

**Theorem 3.1:** If  $f \in \mathcal{A}_p$  satisfies

$$(1 - \gamma) \frac{I_p(n, \lambda)[f](z)}{z^p} + \gamma \frac{I_p(n+1, \lambda)[f](z)}{z^p} \prec \left( \frac{1+z}{1-z} \right)^\alpha, \quad (2)$$

then

$$\frac{I_p(n+1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} \prec \left( \frac{1+z}{1-z} \right)^\delta,$$

where  $\alpha = \alpha\left(\frac{\gamma}{p+\lambda}, \delta\right)$  satisfies the equation

$$2 \arctan \left[ \frac{\gamma}{p+\lambda} (\delta - \alpha) \right] + \pi(\delta - 2\alpha) = 0, \quad (3)$$

and  $\gamma$  and  $\delta$  are real numbers such that  $\gamma \geq 1$ ,  $0 < \delta \leq 1$ .

*Proof:* Let us define

$$\frac{I_p(n, \lambda)[f](z)}{z^p} = u(z). \quad (4)$$

Differentiate (4) logarithmically, we obtain

$$\frac{z I_p'(n, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} - p = \frac{z u'(z)}{u(z)}. \quad (5)$$

In view of the equality

$$z I_p'(n, \lambda)[f](z) = (p + \lambda) I_p(n+1, \lambda)[f](z) - \lambda I_p(n, \lambda)[f](z),$$

(5) reduces to

$$\frac{I_p(n+1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} = 1 + \frac{z u'(z)}{(p + \lambda) u(z)}.$$

A little calculation yields

$$\begin{aligned} u(z) + \frac{\gamma}{p + \lambda} z u'(z) \\ = (1 - \gamma) \frac{I_p(n, \lambda)[f](z)}{z^p} + \gamma \frac{I_p(n+1, \lambda)[f](z)}{z^p}. \end{aligned}$$

Therefore, in view of (2), we have

$$u(z) + \frac{\gamma}{p + \lambda} z u'(z) \prec \left( \frac{1+z}{1-z} \right)^\alpha. \quad (6)$$

We note that for  $\alpha + \beta = \delta$  and  $\mu = \frac{\gamma}{p + \lambda}$ , the condition (3) corresponds to the condition (1) of Lemma 2.1. Therefore, in view of Lemma 2.1, we have

$$u(z) \prec \left( \frac{1+z}{1-z} \right)^\beta \quad (7)$$

where  $\beta$  satisfies the condition (1) of Lemma 2.1.

Let us, now, write  $u(z) + \frac{\gamma}{p + \lambda} z u'(z) = v(z)$  and therefore, we have

$$\frac{I_p(n+1, \lambda)[f](z)}{z^p} = \left( 1 - \frac{1}{\gamma} \right) u(z) + \frac{1}{\gamma} v(z).$$

Obviously,  $\frac{I_p(n+1, \lambda)[f](z)}{z^p}$  is a convex combination of  $u(z)$  and  $v(z)$ .

In view of condition (1) of Lemma 2.1, we conclude that  $\alpha > \beta$ , thus, from (6) and (7), we have

$$\frac{I_p(n+1, \lambda)[f](z)}{z^p} \prec \left( \frac{1+z}{1-z} \right)^\alpha. \quad (8)$$

Write  $w(z) = \frac{I_p(n+1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)}$ , obviously  $w \in \mathcal{H}[1, 1]$  and we can rewrite  $w$  as

$$w(z) = \frac{I_p(n+1, \lambda)[f](z)/z^p}{u(z)}.$$

From (7) and (8), we obtain

$$|\arg w(z)| \leq \left| \arg \frac{I_p(n+1, \lambda)[f](z)}{z^p} \right| + |\arg u(z)|$$

$$< \alpha \frac{\pi}{2} + \beta \frac{\pi}{2} = (\alpha + \beta) \frac{\pi}{2} = \delta \frac{\pi}{2}.$$

Hence, we have

$$\frac{I_p(n+1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} \prec \left( \frac{1+z}{1-z} \right)^\delta, \quad z \in \mathbb{E}. \quad \blacksquare$$

## IV. APPLICATIONS TO UNIVALENT FUNCTIONS

In this section, using Theorem 3.1, we derive certain sufficient conditions for strongly starlike and strongly convex functions.

On writing  $p = 1$  and  $\lambda = 0$  in Theorem 3.1. We have the following result.

**Corollary 4.1:** If  $f \in \mathcal{A}$  satisfies

$$(1 - \gamma) \frac{D^n[f](z)}{z} + \gamma \frac{D^{n+1}[f](z)}{z} \prec \left( \frac{1+z}{1-z} \right)^\alpha,$$

then

$$\frac{D^{n+1}[f](z)}{D^n[f](z)} \prec \left( \frac{1+z}{1-z} \right)^\delta,$$

where  $\alpha = \alpha(\gamma, \delta)$  satisfies the equation

$$2 \arctan[\gamma(\delta - \alpha)] + \pi(\delta - 2\alpha) = 0,$$

and  $\gamma$  and  $\delta$  are real numbers with  $\gamma \geq 1$ ,  $0 < \delta \leq 1$ .

When we select  $p = 1$ ,  $n = 0$  and  $\lambda = 0$  in Theorem 3.1. We obtain the following result of Oros [6].

**Corollary 4.2:** If  $f \in \mathcal{A}$  satisfies

$$(1 - \gamma) \frac{f(z)}{z} + \gamma f'(z) \prec \left( \frac{1+z}{1-z} \right)^\alpha,$$

then

$$\frac{z f'(z)}{f(z)} \prec \left( \frac{1+z}{1-z} \right)^\delta,$$

where  $\alpha = \alpha(\gamma, \delta)$  satisfies the equation

$$2 \arctan[\gamma(\delta - \alpha)] + \pi(\delta - 2\alpha) = 0,$$

and  $\gamma$  and  $\delta$  are real numbers with  $\gamma \geq 1$ ,  $0 < \delta \leq 1$ .

By taking  $p = 1$ ,  $n = 1$  and  $\lambda = 0$  in Theorem 3.1. We obtain the following result.

**Corollary 4.3:** If  $f \in \mathcal{A}$  satisfies

$$f'(z) + \gamma z f''(z) \prec \left( \frac{1+z}{1-z} \right)^\alpha,$$

then

$$1 + \frac{z f''(z)}{f'(z)} \prec \left( \frac{1+z}{1-z} \right)^\delta,$$

where  $\alpha = \alpha(\gamma, \delta)$  satisfies the equation

$$2 \arctan[\gamma(\delta - \alpha)] + \pi(\delta - 2\alpha) = 0,$$

and  $\gamma$  and  $\delta$  are real numbers with  $\gamma \geq 1$ ,  $0 < \delta \leq 1$ .

By setting  $p = 1$ ,  $n = 0$  and  $\lambda = 1$  in Theorem 3.1. We have the following result.

*Corollary 4.4:* If  $f \in \mathcal{A}$  satisfies

$$\left(1 - \frac{\gamma}{2}\right) \frac{f(z)}{z} + \frac{\gamma}{2} f'(z) \prec \left(\frac{1+z}{1-z}\right)^\alpha,$$

then

$$\frac{1}{2} \left(1 + \frac{zf'(z)}{f(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^\delta,$$

where  $\alpha = \alpha\left(\frac{\gamma}{2}, \delta\right)$  satisfies the equation

$$2 \arctan \left[\frac{\gamma}{2}(\delta - \alpha)\right] + \pi(\delta - 2\alpha) = 0,$$

and  $\gamma$  and  $\delta$  are real numbers such that  $\gamma \geq 1$ ,  $0 < \delta \leq 1$ .

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