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$\mathcal{H}_\infty$ approach to functional projective synchronization for chaotic systems with disturbances

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Abstract—This paper presents a method for functional projective $\mathcal{H}_\infty$ synchronization problem of chaotic systems with external disturbance. Based on Lyapunov theory and linear matrix inequality (LMI) formulation, the novel feedback controller is established to not only guarantee stable synchronization of both drive and response systems but also reduce the effect of external disturbance to an $\mathcal{H}_\infty$ norm constraint.

Keywords—Chaotic systems, functional projective $\mathcal{H}_\infty$ synchronization, LMI.

I. Introduction

Chaos is very interesting nonlinear phenomenon and has extensive applications in many areas. Since the first work of Pecora and Carrol in 1990 [1], chaos synchronization has received increasing attention due to its theoretical challenge and its great potential applications in secure communication, economics, signal generator design, chemical reaction, biological systems and so on [2].

The idea of synchronization is to use the output of the master system to control the slave system so that the output of the response system follows the output of the master system asymptotically. Up to date, a number of synchronization schemes by using various control theories such as variable structure control, observer-based control, time-delay feedback approach, back-stepping design technique, active control, parameters adaptive control, nonlinear control have been proposed in the literature [3]-[8]. Recently, fractional-order chaotic systems has been attracted lots of attention since it has been found that many systems in interdisciplinary fields can be described by fractional differential equations. Thus, the synchronization schemes above are extended to fractional-order chaotic systems [9]-[10] due to its potential applications in secure communication and control processing.

On the other hand, some noises or disturbances always exist in real systems that may cause instability and poor performance. Therefore, the effect of the noises or disturbances must be also reduced in synchronization process for chaotic systems. Motivated by this, Y.Y. Hou et. al. [11] firstly adopted the $\mathcal{H}_\infty$ control concept to reduce the effect of the disturbance for chaotic synchronization problem of a general class of chaotic systems. Recently, dynamic control method for designing $\mathcal{H}_\infty$ synchronization of chaotic systems has been proposed in [12]. More recently, a generalized synchronization method, called functional projective synchronization, has been developed [13]-[14]. In the scheme, the responses of the synchronized dynamical states synchronize up to not a constant but a scaling function.

In this paper, the problem of functional projective $\mathcal{H}_\infty$ chaos synchronization to general chaotic system with disturbance is considered. The functional projective synchronization scheme is the general synchronization concept including complete synchronization, anti-synchronization, phase synchronization, and generalized synchronization [3]-[8]. A new stabilizing controller for functional projective $\mathcal{H}_\infty$ synchronization between drive and response chaotic systems is proposed. The resulting closed-loop error system is asymptotically stable and the $\mathcal{H}_\infty$-norm from the disturbance to controlled output is reduced to a prescribed level. Based on the Lyapunov method and LMI framework, an existence criterion for such controller is represented in terms of LMI.

Notation: $\|\cdot\|$ refers to the Euclidean vector norm and the induced matrix norm. For symmetric matrices X and Y, the notation $X > Y$ (respectively, $X \geq Y$) means that the matrix $X - Y$ is positive definite, (respectively, nonnegative). $diag\{\cdots\}$ denotes the block diagonal matrix. $\star$ represents the symmetric part of a matrix. $\lambda_{min}(A)$ denotes the smallest eigenvalue of A.

II. Problem statement and main results

Consider a class of chaotic systems described by the nonlinear differential equation as follows:

$$\dot{x}(t) = Ax(t) + f(x(t)) \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the state variable, the matrices $A \in \mathcal{R}^{n\times n}$ is a constant matrix, and $f(x(t)) \in \mathcal{R}^n$ is a nonlinear function. Note that all the chaotic systems can be written of the form (1). The synchronization problem of system (1) is considered using the drive-response configuration. This is to say, if the system (1) regarded as the drive system, a suitable response system with control input should be constructed to synchronize the drive system. According to the above drive-response concept, slave chaotic systems can be described by the following

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equations:

$$\dot{y}(t) = Ay(t) + f(y(t)) + Dw(t) + Bu_1(t) + u_2(t) \quad (2)$$

where $y(t) \in \mathcal{R}^n$ is the state vector of slave system, B and D are constant matrices with appropriate dimensions, $w(t) \in \mathcal{R}^l$ is the disturbance, and $u_1(t)$ and $u_2(t)$ are the linear and nonlinear control inputs, respectively.
Define the synchronization error as

$$e(t) = y(t) - \alpha(t)x(t). \quad (3)$$

where $\alpha(t) = diag\{\alpha_1(t), \alpha_2(t), \cdots, \alpha_n(t)\}, (\alpha_i(t) \neq 0)$ is the scaling function factor and $\alpha_i(t)$ is assumed to be differentiable in time.
Then, the dynamics of synchronization error between the drive and response systems given in Eqs. (1)-(2) is given by

$$\dot{e} = Ae + g - \dot{\alpha}(t)x + Dw + Bu_1 + u_2. \quad (4)$$

where $g = f(y(t)) - \alpha(t)f(x(t))$.
Next, in order to make the $\mathcal{H}_\infty$ synchronization between drive system (1) and response one (2), we propose the following control laws:

$$u_1(t) = -\beta B^T Pe(t), \quad (5)$$

$$u_2(t) = -\frac{g(t)g^T(t)Pe(t)}{\|g^T(t)Pe(t)\| + 0.5\epsilon\|e(t)\|^2} - \frac{\dot{\alpha}(t)x(t)x^T(t)\dot{\alpha}(t)Pe(t)}{\|x^T(t)\dot{\alpha}(t)Pe(t)\| + 0.5\epsilon\|e(t)\|^2} \quad (6)$$

where β and ϵ are the positive scalars, and the feedback gain P is the control parameter which is determined later.

Definition 1. The drive (1) and response (2) chaotic systems with functional projective error (3) achieve the functional projective $\mathcal{H}_\infty$ synchronization with the disturbance attenuation γ if the following conditions are satisfied [15]:

- With zero disturbance, the synchronization error systems (3) with certain controller is exponentially stable.
- With zero initial condition and a given constant $\gamma > 0$, the following condition holds:

$$J = \int_0^\infty [e^T(t)e(t) - \gamma^2 w^T(t)w(t)]dt \leq 0, \quad \left(\text{i.e} \sup_{w\neq 0,\ w\in L_2[0,\infty]} \frac{\|e(t)\|_2}{\|w(t)\|_2} \leq \gamma\right). \quad (7)$$

Then, the controller is said to be the functional projective $\mathcal{H}_\infty$ synchronization controller with the disturbance attenuation γ. The parameter γ is called the $\mathcal{H}_\infty$-norm bound of this controller.

Theorem 1. For given positive scalars γ, η, and ϵ, there exist a stabilizing controller given by Eqs. (5)-(6) for the error system (4) if there exist a positive-definite matrix X and a positive scalar β satisfying the following LMI:

$$\begin{bmatrix} XA^T + AX - 2\beta BB^T + \eta X & D & \delta X \\ \star & -\gamma^2 I & 0 \\ \star & \star & -\delta I \end{bmatrix} < 0 \quad (8)$$

where $\delta = \sqrt{4\epsilon + 1}$. Then, the $\mathcal{H}_\infty$ synchronization with the disturbance attenuation γ is obtain by the controller.
Proof. Let us consider the following Lyapunov function:

$$V = e^T(t)Pe(t). \quad (9)$$

Taking the time derivative of V along the solution of (4), we have

$$\begin{aligned} \dot{V} &= e^T\big(A^T P + PA - 2\beta PBB^T P\big)e \\ &\quad + 2e^T P(g + Dw - \dot{\alpha}(t)x) + 2e^T Pu_2. \end{aligned} \quad (10)$$

Applying the nonlinear control input $u_2(t)$ to Eq. (10) gives that

$$\begin{aligned} \dot{V} &\leq e^T Qe + \frac{\|g^T Pe\| \cdot \epsilon\|e\|^2}{\|g^T Pe\| + 0.5\epsilon\|e\|^2} \\ &\quad + \frac{\|x^T\dot{\alpha}(t)Pe\ \| \cdot \epsilon\|e\|^2}{\|x^T\dot{\alpha}(t)Pe\ \| + 0.5\epsilon\|e\|^2} + 2e^T PDw, \\ &\leq e^T Qe + 4\epsilon\|e\|^2 + 2e^T PDw, \end{aligned} \quad (11)$$

where $Q = A^T P + PA - 2\beta PBB^T P$ and the well-known inequality $0 \leq ab/(a+b) \leq a\ \forall\ a, b > 0$ is used.
Thus, if the inequality, $e^T(Q + 4\epsilon I)e + 2e^T PDw \leq 0$, holds, i.e.,

$$\begin{bmatrix} e(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} Q + 4\epsilon I & PD \\ D^T P & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ w(t) \end{bmatrix} \leq 0, \quad (12)$$

we have $\dot{V} \leq 0$.
Now, in order to establish the $\mathcal{H}_\infty$ performance for error system, consider the following performance index $J(e(t), w(t))$:

$$J(e(t), w(t)) = \dot{V} + e^T(t)e(t) - \gamma^2 w^T(t)w(t). \quad (13)$$

Substituting (11) into (13) yields

$$J \leq \begin{bmatrix} e(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} Q + 4\epsilon I + I & PD \\ D^T P & -\gamma^2 I \end{bmatrix} \begin{bmatrix} e(t) \\ w(t) \end{bmatrix}. \quad (14)$$

If there exist a constant $\eta > 0$ such that

$$\bar{\Theta} = \begin{bmatrix} Q + (4\epsilon + 1)I + \eta P & PD \\ D^T P & -\gamma^2 I \end{bmatrix} < 0, \quad (15)$$

then, we have

$$J < -\begin{bmatrix} e(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} \eta P & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ w(t) \end{bmatrix}. \quad (16)$$

From Eq. (16), we can easily obtain that

$$\dot{V}|_{w(t)=0} < -\eta\lambda_{min}(P)\|e(t)\|^2 < 0 \text{ for all } e(t) \neq 0. \quad (17)$$

Based on Lyapunov stability theory, the synchronization error system (4) with the linear controller $u_1(t)$ and nonlinear controller $u_2(t)$ is exponentially stable for $w(t) = 0$.
Integrating the function in Eq. (16) from 0 to ∞, we have

$$V(\infty) - V(0) + \int_0^\infty \big(\|e(t)\|_2^2 - \gamma^2\|w(t)\|_2^2\big)dt \leq 0. \quad (18)$$

With zero initial condition, we have

$$\int_0^\infty \big(\|e(t)\|_2^2 - \gamma^2\|w(t)\|_2^2\big)dt \leq 0. \quad (19)$$

By Definition 1, functional projective $\mathcal{H}_\infty$ synchronization with the disturbance attenuation γ is obtained by the control (5)-(6). Finally, here we give an equivalent condition of stability criterion (15) which can be solved by various efficient convex algorithms. The fact that $\bar{\Theta} < 0$ given in (15) by postmultiplying and premultiplying the matrix $diag\{P^{-1}, I\}$ and by its transpose, respectively, is equivalent to

$$\begin{bmatrix} (1,1) & D \\ D^T & -\gamma^2 I \end{bmatrix} < 0 \tag{20}$$

where $X = P^{-1}, (1,1) = XA^T + AX - 2\beta BB^T + \eta X + (4\epsilon + 1)XX$. By Schur Complement, the inequality (20) is equivalent to the LMI (8). This completes the proof. ■

Remark 1. When the scaling function $\alpha_i(t)$ equals to any constant such as $\alpha_i(t) = 1$, $\alpha_i(t) = -1$, and $\alpha_i(t) =$ nonzero constant, respectively, the synchronization problem becomes complete synchronization, anti-synchronization, and projective synchronization, respectively.

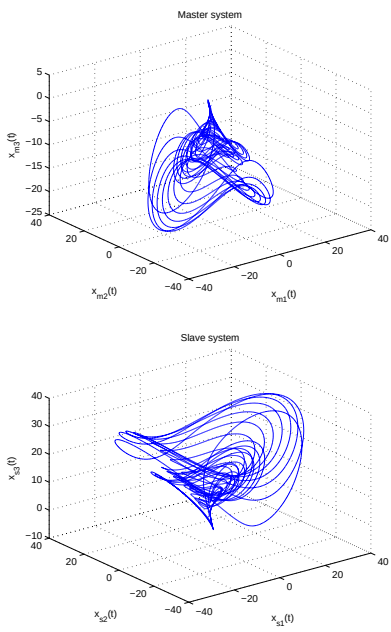

Fig. 1. The state trajectories of synchronization of 4-dimensional chaotic system without disturbance signal $w(t)$ in Case 1

Example: Let us consider the following four-dimensional (4D) chaotic system [16] described by

$$\begin{cases} \dot{x}_1 &= a(x_2 - x_1) + x_2x_3x_4, \\ \dot{x}_2 &= b(x_1 + x_2) - x_1x_3x_4, \\ \dot{x}_3 &= -cx_3 + x_1x_2x_4, \\ \dot{x}_4 &= -dx_4 + x_1x_2x_3, \end{cases} \tag{21}$$

where x_1, x_2, x_3 and x_4 are state variables, and a, b, c and d are all positive real constant parameters.
For drive-response concept for synchronization, the following

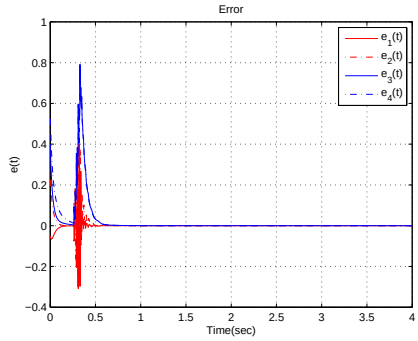

Fig. 2. The synchronization error of 4-dimensional chaotic system without disturbance signal $w(t)$ in Case 1

system parameters are considered:

$$A = \begin{bmatrix} -30 & 30 & 0 & 0 \\ 10 & 10 & 0 & 0 \\ 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & -10 \end{bmatrix}, \quad B = D = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$f(x) = \begin{bmatrix} x_2x_3x_4 \\ -x_1x_3x_4 \\ x_1x_2x_4 \\ x_1x_2x_3 \end{bmatrix}, \quad f(y) = \begin{bmatrix} y_2y_3y_4 \\ -y_1y_3y_4 \\ y_1y_2y_4 \\ y_1y_2y_3 \end{bmatrix}.$$

Now, in order to make functional projective $\mathcal{H}_\infty$ synchronization of the systems (1) and (2) via control laws (5) and (6), let us solve the problem given in Theorem 1 with the constants $\eta = 0.1$ and $\epsilon = 0.1$ and the disturbance attenuation $\gamma = 0.5$. By MATLAB's LMI Control Toolbox, one can see that the LMI given in Eq. (8) is feasible and get a possible solution set: $\beta = 9.9931$ and

$$X = \begin{bmatrix} 1.1704 & -0.1754 & 0.2100 & 0.2100 \\ -0.1754 & 0.4812 & 0.2083 & 0.2083 \\ 0.2100 & 0.2083 & 4.1670 & -0.0537 \\ 0.2100 & 0.2083 & -0.0537 & 4.1670 \end{bmatrix}.$$

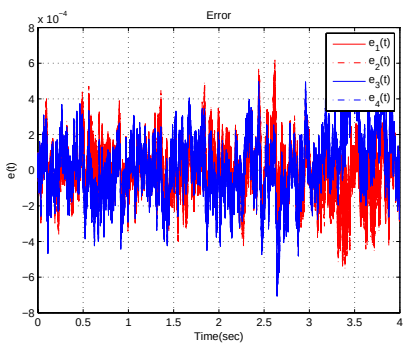

Fig. 3. The error state $e(t)$ for 4-dimensional chaotic system with disturbance signal $w(t)$ in Case 1

In the numerical simulations, the fourth-order Rung-Kutta method is used to solve the systems with time step size 0.0001. For the simulation, we assume that the following initial conditions, $(x_{m1}(0), x_{m2}(0), x_{m3}(0), x_{m4}(0)) =$

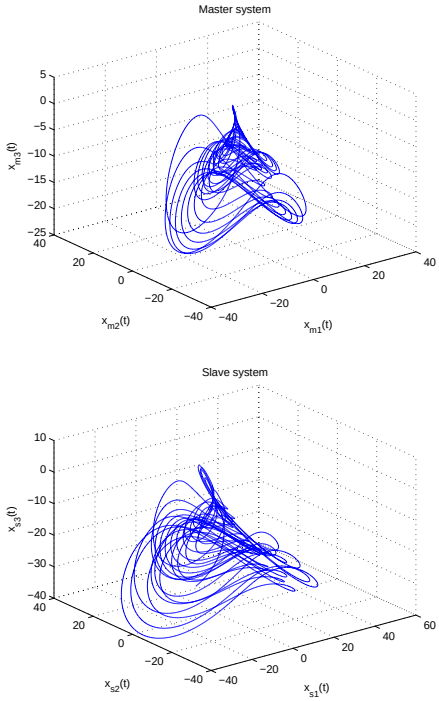

Fig. 4. The state trajectories of synchronization of 4-dimensional chaotic system without disturbance signal $w(t)$ in Case 2

$(0.1,\ -0.5,\ 0.2,\ -0.3)$, $(x_{s1}(0), x_{s2}(0), x_{s3}(0), x_{s4}(0)) = (-0.2,\ 1,\ 0.1,\ 1)$ are employed. The Gaussian noise with mean 0 and variance 1 is imposed on the response system. For functional projective synchronization, the scaling functions are chosen for two cases:

- **Case 1:** $\alpha_i(t) = -1.5, (i = 1, \cdots, 4)$: Projective synchronization
- **Case 2:** $\alpha_i(t) = 2 - \cos(t), (i = 1, \cdots, 4)$: Functional synchronization

First, without disturbance signal and by applying the controller (5)-(6) with the control parameters $P(= X^{-1})$ and β obtained above, each state trajectories of drive and response systems are illustrated in Fig. 1. In Fig. 2, The synchronization error between drive and response systems is illustrated. It shows that the synchronization error converges to zero exponentially. In order to observe the $\mathcal{H}_\infty$ performance with disturbance attenuation, the response of the controlled output error $e(t)$ is depicted in Fig. 3, which shows functional projective $\mathcal{H}_\infty$ synchronization controller reduces the effect of the disturbance input $w(t)$ on the controlled error state $e(t)$ to within a prescribed level $\gamma = 0.5$.

Next, simulation results for Case 2 are given in Figs. 4-6. When $\alpha_i(t) = 2 - \cos(t)$, the simulation results in Figs. 4-6 confirm the effectiveness of our proposed control scheme for master-slave functional synchronization scheme.

III. Conclusions

The functional projective $\mathcal{H}_\infty$ synchronization method has been investigated for chaotic systems with disturbances. Based on Lyapunov theory and LMI formulation, the controller for the problem has designed to guarantee synchronization for drive and response chaotic systems. The controller reduces the $\mathcal{H}_\infty$-norm from the disturbance to the output error within a prescribed level. Finally, our method is applied to a 4-dimensional chaotic system in order to illustrate the effectiveness of the control scheme.

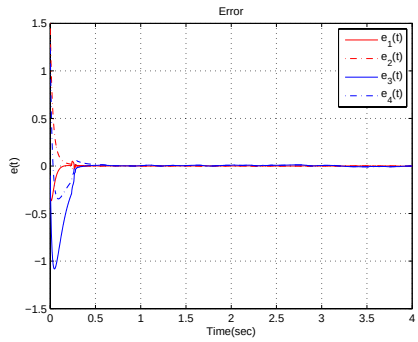

Fig. 5. The synchronization error of 4-dimensional chaotic system without disturbance signal $w(t)$ in Case 2

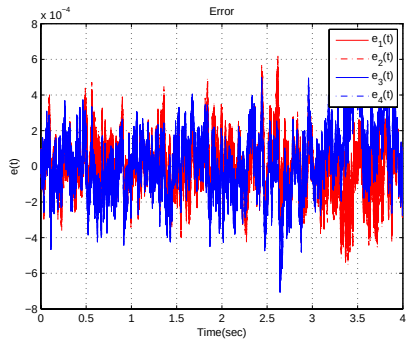

Fig. 6. The error state $e(t)$ for 4-dimensional chaotic system with disturbance signal $w(t)$ in Case 2

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