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More on Gaussian quadratures for fuzzy functions

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Abstract—In this paper, the Gaussian type quadrature rules for fuzzy functions are discussed. The errors representation and convergence theorems are given. Moreover, four kinds of Gaussian type quadrature rules with error terms for approximate of fuzzy integrals are presented. The present paper complements the theoretical results of the paper by T. Allahviranloo and M. Otadi [T. Allahviranloo, M. Otadi, Gaussian quadratures for approximate of fuzzy integrals, Applied Mathematics and Computation 170 (2005) 874-885]. The obtained results are illustrated by solving some numerical examples.

Keywords—Guassian quadrature rules, Fuzzy number, Fuzzy integral, Fuzzy solution.

I. INTRODUCTION

WITH consideration of approximation theory, the integration problem plays major role in various areas, and in many applications we usually can not be sure that this problems are perfect, i.e., some of the system's parameters and measurements are represented by fuzzy numbers rather than crisp one. So it is important to develop fuzzy integration and solve them.

Since the fuzzy sets was originally introduced by Zadeh [2] in 1965, the concept of fuzzy sets and fuzzy numbers were studied by many scholars, such as fuzzy number and arithmetic operations with these numbers investigated by Zadeh in [3] and followed by others. Goetschel and Voxman [7] suggested a new approach. They represented the fuzzy number as a parameterized triple and then embedded the set of fuzzy numbers into a topological vector space. This enabled them to design the basics of a fuzzy calculus.

The concept of integration of fuzzy functions was first introduced by Dubois and Prade [12]. Alternative approaches were later suggested by Goetschel and Voxman [7], Kaleva [11], and others. While Goetschel and Voxman [7] preferred a Riemann integral type approach, Kaleva [11] chose to define the integral of fuzzy function, using the Lebesgue type concept for integration.

As one of the applications of fuzzy integration, Wu and Ma investigated the Fuzzy Fredholm integral equation of the second kind (FF-2) in [16], and established the existence of a unique solution to FF-2. Several applications of fuzzy integral and fuzzy differential equation in fuzzy control were presented in [17], [18], [19].

The numerical methods for fuzzy integral or differential equation was also considered by many scholars (see for example, [8], [9], [10], [13], [14], [15]). In [8], the author presented the Newton Cot's methods for the integration of fuzzy functions, however the numerical examples shown that numerical results of this methods are not good approximate to the exact solution, and the methods became useless for numerical processes when the integral points more than eight, see [1]. In [9], authors considered Gaussian quadratures for approximate of fuzzy integrals, the convergence and three Gaussian type quadrature rules are given. In this paper, we presented four kinds of Gaussian quadrature rules for fuzzy functions. The convergence and uniform errors representation theorems are given as well. The present paper complements the theoretical results of paper [9]. For the fuzzy integration yields fuzzy number in parametric form, we used the parametric form of methods. The structure of this paper is organized as follows:

In Section 2, we bring some basic definitions and results on fuzzy numbers and fuzzy integrations. In Section 3, we introduce the integration formulas of Gaussian type rules for fuzzy functions. The uniform errors representation and convergence theorems are given. Four kinds of Gaussian quadratures are presented in Section 4. The proposed algorithms are illustrated by solving some examples in Section 5. The conclusions are drawn in Section 6.

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II. PRELIMINARIES

For convenience of the readers, in this section, we give some of the notations, definitions and lemmas on fuzzy numbers and fuzzy integrations, which will be used in the sequel.

Definition 2.1. [2] A fuzzy number is a fuzzy set like $u : R \to I = [0, 1]$ satisfies:

1. u is an upper semicontinuous,
2. $u(x) = 0$ outside some interval $[c, d]$,
3. there are real numbers a, b such that $c \leq a \leq b \leq d$ and
   (3.1) $u(x)$ is monotonic increasing on $[c, a]$,
   (3.2) $u(x)$ is monotonic decreasing on $[b, d]$,
   (3.3) $u(x) = 1$, $a \leq x \leq b$.

The set of all these fuzzy numbers is denoted by E^1. An equivalent parametric form is also given as follows:

Definition 2.2. [4] An arbitrary fuzzy number is represented, in parametric form, by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded left continuous nondecreasing function over $[0, 1]$,
2. $\overline{u}(r)$ is a bounded left continuous nonincreasing function over $[0, 1]$,
3. $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

A crisp number α can be simply expressed as $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \leq r \leq 1$.

Lemma 2.1. [5] *Let $v, w \in E^1$ and s be real number. Then for $0 \leq r \leq 1$*

(a) $u = v$ *if and only if* $\underline{u}(r) = \underline{v}(r)$ *and* $\overline{u}(r) = \overline{v}(r)$;

(b) $v + w = (\underline{v}(r) + \underline{w}(r), \overline{v}(r) + \overline{w}(r))$;
(c) $v - w = (\underline{v}(r) - \overline{w}(r), \overline{v}(r) - \underline{w}(r))$;
(d) $v \cdot w$
$= (\min\{\underline{v}(r) \cdot \underline{w}(r), \underline{v}(r) \cdot \overline{w}(r), \overline{v}(r) \cdot \underline{w}(r), \overline{v}(r) \cdot \overline{w}(r)\},$
$\max\{\underline{v}(r) \cdot \underline{w}(r), \underline{v}(r) \cdot \overline{w}(r), \overline{v}(r) \cdot \underline{w}(r), \overline{v}(r) \cdot \overline{w}(r)\})$;
(e) $sv = \begin{cases} (s\underline{v}(r), s\overline{v}(r)), & s \geq 0, \\ (s\overline{v}(r), s\underline{v}(r)), & s < 0. \end{cases}$

E^1 with addition and multiplication as defined by Lemma 2.1 is a convex cone which is then embedded isomorphically and isometrically into a Banach space.

Definition 2.3. For arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$ the quantity

$$D(u, v) = \sup_{0 \leq r \leq 1} \{\max[|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|]\}$$

is the distance between u and v.

It is shown in [6] that E^1 with D is a complete metric space.

Definition 2.4. A function $f : R \to E^1$ is called a fuzzy function. If for arbitrary fixed $t_0 \in R$ and $\varepsilon > 0$, there exist a $\delta > 0$ such that

$$|t - t_0| < \delta \Longrightarrow D[f(t), f(t_0)] < \varepsilon,$$

then f is said to be continuous in the metric D.

Throughout this work, we consider fuzzy functions on a finite interval $[a, b] \subset R$. The integral of a fuzzy function using the Riemann integral concept can be defined as follows,

Definition 2.5. [7] Let $f : [a, b] \to E^1$. For each partition $P = \{t_0, t_1, ..., t_n\}$ of $[a, b]$ and for arbitrary $\xi_i : t_{i-1} \leq \xi_i \leq t_i$, $1 \leq i \leq n$, let

$$R_p = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}).$$

The definite integral of $f(t)$ over $[a, b]$ is

$$\int_a^b f(t)\mathrm{d}t = \lim R_p, \quad \max_{1 \leq i \leq n} |t_i - t_{i-1}| \to 0$$

provided that this limit exists in the metric D.

If the fuzzy function $f(t)$ is continuous in the metric D, its definite integral exists [7]. Further more,

$$\begin{aligned} \underline{\int_a^b f(t;r)\mathrm{d}t} &= \int_a^b \underline{f}(t;r)\mathrm{d}t, \\ \overline{\int_a^b f(t;r)\mathrm{d}t} &= \int_a^b \overline{f}(t;r)\mathrm{d}t. \end{aligned} \tag{1}$$

By virtue of [11], the integral of (1) possesses a unique solution $(\underline{x}, \overline{x})$ which is a fuzzy function, i.e. for each t, the pair $(\underline{x}(t; r), \overline{x}(t; r))$ is a fuzzy number.

III. Guassian quadrature rules

Now we consider the general integrals of the form

$$I(f) := \int_a^b \omega(x) f(x)\mathrm{d}x,$$

where $\omega(x)$ is a given crisp nonnegative weight function on the interval $[a, b]$. Also, the interval $[a, b]$ may be infinite, e.g., $(0, +\infty)$ or $(-\infty, +\infty)$. The weight function must meet the following requirements:

(a) $\omega(x) \geq 0$ is measurable on the finite or infinite interval $[a, b]$.

(b) All moments $\mu_k := \int_a^b x^k \omega(x)$, $k = 0, 1, \cdots$, exist and finite.

(c) For nonnegative polynomials $s(x)$ on $[a, b]$, $\int_a^b \omega(x)s(x)dx = 0$ implies $s(x) \equiv 0$.

We will examine the integration rules of the type

$$\widetilde{Q}(f) := \sum_{i=1}^{n} \omega_i f(x_i).$$

The Newton-Cot's formulas are of this form, but the abscissas x_i were required to form a uniform partition of the interval $[a, b]$. Relaxing this restriction and choosing the x_i as well as the ω_i so as to maximize the order of the integration method will leads to a class of well-defined so-called Gaussian quadrature rules [1]. The exact form of Gaussian quadrature rules, with weight coefficient $\omega_i > 0$ and $a < x_i < b$ for $i = 1, \cdots, n$, can be determined by choosing some orthogonal polynomials. x_i is called the Gaussian point.

Let f be a fuzzy function with parametric form $f(x) = \left(\underline{f}(x), \overline{f}(x)\right)$, $0 \leq r \leq 1$, the Gaussian quadrature rules

$$\int_a^b \omega(x) f(x)\mathrm{d}x \approx \sum_{i=1}^{n} \omega_i f(x_i) \tag{2}$$

provide approximate value for $\int_a^b \omega(x) f(x)\mathrm{d}x$. The parametric form (2) is as follows:

$$\begin{aligned} \int_a^b \omega(x)\underline{f}(x;r)\mathrm{d}x &= \sum_{i=1}^{n} \omega_i \underline{f}(x_i;r) + E(\underline{f};r), \\ \int_a^b \omega(x)\overline{f}(x;r)\mathrm{d}x &= \sum_{i=1}^{n} \omega_i \overline{f}(x_i;r) + E(\overline{f};r), \end{aligned} \tag{3}$$

where $E(\underline{f}; r)$ and $E(\overline{f}; r)$ are the errors of this approximation.

If we let

$$Q(\underline{f}; r) = \sum_{i=1}^{n} \omega_i \underline{f}(x_i; r)$$

and

$$Q(\overline{f}; r) = \sum_{i=1}^{n} \omega_i \overline{f}(x_i; r),$$

then it follows from (3) that

$$\begin{aligned} I(\underline{f}; r) &= \int_a^b \omega(x)\underline{f}(x;r)\mathrm{d}x \\ &= Q(\underline{f}; r) + E(\underline{f}; r), \\ I(\overline{f}; r) &= \int_a^b \omega(x)\overline{f}(x;r)\mathrm{d}x \\ &= Q(\overline{f}; r) + E(\overline{f}; r). \end{aligned} \tag{4}$$

In the following, we use notation

$$Q(f; r) = (Q(\underline{f}; r), Q(\overline{f}; r))$$

$$I(f; r) = (I(\underline{f}; r), I(\overline{f}; r))$$

$$f = (\underline{f}, \overline{f})$$

and

$$p = (\underline{p}, \overline{p})$$

for convenient, where p is fuzzy polynomial.

Definition 3.1. $Q(f;r)$ of the integral formulas is called convergent if

$$Q(f;r) \to I(f;r), \quad n \to \infty,$$

for all continuous function f on $[a,\ b]$ in the metric D.

The following theorem is proving that $Q(f;r)$ converge to $I(f;r)$ uniformly.

Theorem 3.1. *If f is continuous (in the metric D). Assume that $Q(f;r) = \sum_{i=1}^{n} \omega_i f(x_i;r)$ convergence for all polynomials and that is uniformly bounded. Then $Q(f;r)$ converges to $I(f;r)$ is uniform for r.*

Proof. Let $\varepsilon > 0$ be arbitrary and for the continuity of f, by the Weierstrass approximation theorem there exists a polynomial p such that

$$D(f-p) \leq \frac{\varepsilon}{2(C+b-a)},$$

where $C > 0$ is a constant. Then by the assumption we have $Q(p) \to I(p)$ as $n \to \infty$, there exists $N(\varepsilon) \in N$ such that

$$D(Q(p) - I(p)) \leq \frac{\varepsilon}{2}$$

for all $n \geq N(\varepsilon)$. Now with the aid of the triangle inequality in the metric D and using the uniformly bounded of $Q(p)$ we can estimate

$$\begin{aligned} & D(Q(f) - I(f)) \\ \leq \ & \sum_{i=1}^{n} |\omega_i| D(f(x_i) - p(x_i)) + D(Q(p) - I(p)) \\ & + \int_a^b D(p(x) - f(x)) dt, \end{aligned}$$

since the weight coefficient ω_i is nonnegative, we can let $\sum_{i=1}^{n} \omega_i = C > 0$, then we have

$$\begin{aligned} & D(Q(f) - I(f)) \\ \leq \ & \frac{C\varepsilon}{2(C+b-a)} + \frac{\varepsilon}{2} + \frac{(b-a)\varepsilon}{2(C+b-a)} \\ = \ & \varepsilon \end{aligned}$$

for all $n \geq N(\varepsilon)$; that is $Q(f;r)$ converges to $I(f;r)$, i.e., $Q(\underline{f};r)$, $Q(\overline{f};r)$ converge uniformly to $I(\underline{f};r)$, $I(\overline{f};r)$ as well.

We complete the proof.

Suppose that there exist an orthogonal polynomial $p_n(x)$ associated with the weight function $\omega(x)$ (note that here the $p_n(x)$ is a crisp one), then the following error representation theorem is obtained. First we given two lemmas, which will be used in the sequel.

Lemma 3.1. [1] *Let the real function $a(x)$ be $n+1$ times differentiable on the interval $[a,b]$, for $m+1$ support abscissa $\xi_i \in [a,b]$,*

$$\xi_0 < \xi_1 < \cdots < \xi_m.$$

If $b(x)$ is the Hermite interpolation polynomials of $a(x)$ with degree at most n, then for any $\bar{x} \in [a,b]$, there exists $\bar{\xi} \in I[\xi_0, \cdots, \xi_m, \bar{x}]$ such that

$$a(\bar{x}) - b(\bar{x}) = \frac{c(\bar{x}) f^{(n+1)}(\bar{\xi})}{(n+1)!},$$

where $c(x) := (x-\xi_0)^{n_0}(x-\xi_1)^{n_1} \cdots (x-\xi)^{n_m}$, and $\sum_{i=0}^{m} n_i = 1$.

Lemma 3.2. [1] *Let $x_1, \cdots, x_n$ be the roots of the nth orthogonal polynomial $p_n(x)$, and let $\omega_1, \cdots, \omega_n$ be the solution of the (nonsingular) system of equations*

$$\sum_{i=1}^{n} p_k(x_i)\omega_i = \begin{cases} (p_0, p_0), & \text{if } k = 0, \\ 0, & \text{if } k = 1, 2, \cdots, n-1, \end{cases}$$

where the scalar product $(p_0, p_0) = \int_a^b \omega(x)(p_0(x))^2 dx$, and $\omega(x)$ be the weight function.

Then $\omega_i > 0$ for $i = 1, 2, \cdots, n$, and

$$\int_a^b \omega(x) p(x) \mathrm{d}x = \sum_{i=1}^{n} \omega_i p(x_i)$$

holds for all polynomials p with $\text{degree}(p) \leq 2n-1$.

Theorem 3.2. *If for fuzzy function $f = (\underline{f}(r), \overline{f}(r)) \in C^{2n}[a,b]$ (in the metric D), $0 \leq r \leq 1$. Then the error for the Guassian quadrature rules (4) of order n is given by*

$$E(\underline{f};r) = \frac{\underline{f}^{2n}(\xi;r)}{(2n)!}(p_n, p_n),$$

$$E(\overline{f};r) = \frac{\overline{f}^{2n}(\xi;r)}{(2n)!}(p_n, p_n),$$

for some $\xi \in (a,b)$.

Proof. Consider the solution $h \in \prod_{2n-1}$ of the Hermite interpolation problem

$$h(x_i) = f(x_i), \quad h'(x_i) = f'(x_i) \quad i = 1, \cdots, n.$$

Since $\text{degree}(h) < 2n$, and from the Lemma 3.2, we have

$$\begin{aligned} & \int_a^b \omega(x) \underline{h}(x) \mathrm{d}x \\ = \ & \sum_{i=1}^{n} \omega_i \underline{h}(x_i) \\ = \ & \sum_{i=1}^{n} \omega_i \underline{f}(x_i) \end{aligned}$$

and

$$\begin{aligned} & \int_a^b \omega(x) \overline{h}(x) \mathrm{d}x \\ = \ & \sum_{i=1}^{n} \omega_i \overline{h}(x_i) \\ = \ & \sum_{i=1}^{n} \omega_i \overline{f}(x_i). \end{aligned}$$

Therefore the error term has the integral representation

$$\begin{aligned} & \int_a^b \omega(x)\underline{f}(x)\mathrm{d}x - \sum_{i=1}^n \omega_i \underline{f}(x_i) \\ = & \int_a^b \omega(x)(\underline{f}(x) - \underline{h}(x))\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} & \int_a^b \omega(x)\overline{f}(x)\mathrm{d}x - \sum_{i=1}^n \omega_i \overline{f}(x_i) \\ = & \int_a^b \omega(x)(\overline{f}(x) - \overline{h}(x))\mathrm{d}x. \end{aligned}$$

By Lemma 3.1 and note that x_i are the roots of the orthogonal polynomial $p_n(x)$, then

$$\begin{aligned} & \underline{f}(x) - \underline{h}(x) \\ = & \frac{\underline{f}^{2n}(\zeta)}{(2n)!}(x-x_1)^2 \cdots (x-x_n)^2 \\ = & \frac{\underline{f}^{2n}(\zeta)}{(2n)!}p_n^2(x) \end{aligned}$$

and

$$\begin{aligned} & \overline{f}(x) - \overline{h}(x) \\ = & \frac{\overline{f}^{2n}(\zeta)}{(2n)!}(x-x_1)^2 \cdots (x-x_n)^2 \\ = & \frac{\overline{f}^{2n}(\zeta)}{(2n)!}p_n^2(x) \end{aligned}$$

for some $\zeta = \zeta(x)$ on the interval $I(x_1, \cdots, x_n, x)$. $\underline{f}(x)$, $\overline{f}(x)$, $h(x)$, $p(x)$ are continuous on $[a, b]$, then from the well-known mean-value theorem of integral calculus, we can obtain the desired result.

IV. Four kind integration formulas of Gauss method

Let $f(x)$ be a fuzzy function with parametric form $f(x) = (\underline{f}(x), \overline{f}(x))$. For the weight function $\omega(x)$, there exist different orthogonal polynomials $p_n(x)$ associated with $\omega(x)$, which lead to different Gaussian rules. The important cases are given as follows:

1. Gauss-Legendre rules:

$$\int_{-1}^1 f(t)\mathrm{d}t \approx \sum_{i=0}^n \omega_i f(x_i),$$

with $\omega(x) = 1$, where x_i are the roots of the Legendre polynomials

$$p_k(x) = \frac{k!}{(2k!)}\frac{\mathrm{d}^k}{\mathrm{d}x^k}(x^2-1)^k, \quad k = 0, 1, \cdots,$$

and

$$\omega_i = \frac{2}{n+1}p'_{n+1}(x_i)p_n(x_i).$$

The error terms are

$$E(\underline{f}) = \frac{2^{2n+3}[(n+1)!]^4}{(2n+3)[(2n+2)!]^3}\underline{f}^{2n+2}(\eta),$$

$$E(\overline{f}) = \frac{2^{2n+3}[(n+1)!]^4}{(2n+3)[(2n+2)!]^3}\overline{f}^{2n+2}(\eta),$$

where $\eta \in [-1, 1]$.

2. Gauss-Laguerre rules:

$$\int_0^{+\infty} e^{-x} f(t)\mathrm{d}t \approx \sum_{i=0}^n \omega_i f(x_i),$$

with $\omega(x) = e^{-x}$, where x_i are the roots of the Laguerre polynomials

$$l_k(x) = e^x \frac{\mathrm{d}^k}{\mathrm{d}x^k}(x^k e^{-x}), \quad k = 0, 1, \cdots,$$

and

$$\omega_i = \frac{(n!)^2}{l'_{n+1}(x_i)l_n(x_i)}.$$

The error terms are:

$$E(\underline{f}) = \frac{[(n+1)!]^2}{(2n+2)!}\underline{f}^{2n+2}(\eta),$$

$$E(\overline{f}) = \frac{[(n+1)!]^2}{(2n+2)!}\overline{f}^{2n+2}(\eta),$$

where $\eta \in (0, +\infty)$.

3. Gauss-Hermite rules:

$$\int_{-\infty}^{+\infty} e^{-x^2} f(t)\mathrm{d}t \approx \sum_{i=0}^n \omega_i f(x_i),$$

with $\omega(x) = e^{-x^2}$, where x_i are the roots of the Hermite polynomials

$$h_k(x) = (-1)^k e^{x^2} \frac{\mathrm{d}^k}{\mathrm{d}x^k}(e^{-x^2}), \quad k = 0, 1, \cdots,$$

and

$$\omega_i = \frac{2^{n+1}n!\sqrt{\pi}}{h'_{n+1}(x_i)h_n(x_i)}.$$

The error terms are:

$$E(\underline{f}) = \frac{(n+1)!\sqrt{\pi}}{2^{n+1}(2n+2)!}\underline{f}^{2n+2}(\eta),$$

$$E(\overline{f}) = \frac{(n+1)!\sqrt{\pi}}{2^{n+1}(2n+2)!}\overline{f}^{2n+2}(\eta),$$

where $\eta \in (-\infty, +\infty)$.

4. Gauss-Chebyshev rules:

$$\int_{-1}^1 \frac{f(t)}{\sqrt{1-x^2}}\mathrm{d}t \approx \frac{\pi}{n+1}\sum_{i=0}^n f(\cos\frac{2i+1}{2n+1}\pi),$$

with $\omega(x) = \sqrt{1-x^2}$, where x_i are the roots of the Chebyshev polynomials

$$t_k(x) = \cos(k \arccos x), \quad k = 0, 1, \cdots,$$

and the error terms are:

$$E(\underline{f}) = \frac{\pi}{2^{n+1}(2n+2)!}\underline{f}^{2n+2}(\eta),$$

$$E(\overline{f}) = \frac{\pi}{2^{n+1}(2n+2)!}\overline{f}^{2n+2}(\eta),$$

where $\eta \in (-1, 1)$.

V. Numerical Examples

In this section we illustrate the methods in Section 3 by solving some numerical examples.

Example 4.1. Consider the fuzzy integral:

$$\int_{-1}^{1} \widetilde{k} x^4 \mathrm{d}x, \tag{5}$$

where $\widetilde{k} = (r, 2-r)$, the exact solution is $\frac{2}{5}(r, 2-r)$.

From Guass-Legendre rules (n=1):

$$Q(\underline{f}; r) = \frac{2}{9}r, \quad Q(\overline{f}; r) = \frac{2}{9}(2-r),$$

$$\underline{f}^{(4)} = 24, \quad \overline{f}^{(4)} = 24$$

and

$$E(\underline{f}; r) = \frac{8}{45}r, \quad E(\overline{f}; r) = \frac{8}{45}(2-r),$$

it is clear that (4) holds. If we take the *Simpson integration rule* [8] with step size $h = \frac{1}{2}$, then we have

$$\int_{-1}^{1} kx^4 \mathrm{d}x = \tfrac{3}{4}r - \tfrac{1}{60}r,$$

$$\int_{-1}^{1} (2-k)x^4 \mathrm{d}x = \tfrac{3}{4}(2-k) - \tfrac{1}{60}(2-k).$$

Example 4.2. Consider the following fuzzy integral:

$$\int_{-1}^{1} \frac{\widetilde{k}}{\sqrt{1-x^2}} \mathrm{d}x,$$

where $\widetilde{k} = (r-1, 1-r)$, the exact solution is $\pi(r-1, 1-r)$.

From Guass-Chebyshev rules :

$$Q(\underline{f}; r) = \pi(r-1), \quad Q(\overline{f}; r) = \pi(1-r),$$

$$\underline{f}^{(2)} = \overline{f}^{(2)} = 0$$

and

$$E(\underline{f}; r) = E(\overline{f}; r) = 0,$$

it is clear that (4) holds.

VI. Conclusion

We consider the Gaussian type quadratures for fuzzy functions in this paper. The uniform errors representation and convergence theorems are given. The present paper complements the theoretical results of the paper by T. Allahviranloo and M. Otadi [9]. The numerical examples show that the methods are effective and applicable for solving the fuzzy integrate equations.

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