

On the Numbers of Various Young Tableaux

Hsuan-Chu Li

Abstract—We demonstrate a way to count the number of Young tableau of shape $\lambda = (k, k, \dots, k)$ with $|\lambda| = lk$ by expanding Schur function. This result gives an answer to the question that was put out by Jenny Buontempo and Brian Hopkins.

Keywords—Young tableau, Schur function.

I. INTRODUCTION

Given a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of p , a semi-standard Young tableau of shape λ $SSYT^\lambda$ is an arrangement of p boxes, λ_1 in the first row, λ_2 in the second row sharing left border with the first row, etc., with each boxes having a label from $\{1, 2, \dots, p\}$ such that labels weakly increasing across rows and strictly increasing down columns. If the labels increase across rows and down columns, then it becomes a Young tableau of shape λ . The following two examples show a $SSYT^{5221}$ and a Young tableau of shape $(5, 2, 2, 1)$ respectively.

1	2	2	2	5
2	4			
3	6			
5				

1	3	5	8	10
2	4			
6	9			
7				

Let

$$G_{\{n; a_1, a_2, \dots, a_n\}} = \begin{bmatrix} x_1^{a_1} & x_1^{a_2} & \cdots & x_1^{a_n} \\ x_2^{a_1} & x_2^{a_2} & \cdots & x_2^{a_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{a_1} & x_n^{a_2} & \cdots & x_n^{a_n} \end{bmatrix}$$

are integers and $0 < x_1 < x_2 < \dots < x_n$ [1, p.142]. In connection with Schur functions we define the partition λ associated with G as the nonincreasing sequence of nonnegative integers

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) = (a_n - (n-1), a_{n-1} - (n-2), \dots, a_1),$$

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and get

$$G_{\{n; a_1, a_2, \dots, a_n\}} = [x_i^{j-1+\lambda_{n-j+1}}]_{1 \leq i, j \leq n},$$

and Schur function associated to λ is defined as

$$s_\lambda(x_1, x_2, \dots, x_n) = \frac{\det G_{\{n; a_1, a_2, \dots, a_n\}}}{\det G_{\{n; 0, 1, \dots, n-1\}}}.$$

In [2], Jenny Buontempo and Brian Hopkins considered Young tableaux consisting of two rows of equal length counted by the Catalan numbers and gave two combinatorial proofs.

In a recent paper [3], we succeeded to provide the unique LU factorization of $G_{\{n; a_1, a_2, \dots, a_n\}}$ avoiding Schur function and expressed any Schur function in an explicit form. Now we state the results as follows:

1. G_n can be factorized as $G_n = L_n U_n$, where $L_n = [L_n(i, j)]$ is a lower triangular matrix with unit main diagonal and $U_n = [U_n(i, j)]$ is an upper triangular matrix, whose entries are defined as follows:

$$L_n(i, j) = \begin{cases} 1, i = j; \\ 0, i < j; \\ (\frac{x_i}{x_1})^{a_1}, j = 1, i \geq 2; \\ (\frac{x_i}{x_j})^{a_1} \frac{S_{\{x_j \rightarrow x_i\}} A_j}{A_j}, i \geq j+1, j \geq 2. \end{cases}$$

and

$$U_n(i, j) = \begin{cases} x_1^{a_j}, i = 1; \\ 0, i > j; \\ x_i^{a_1} B_i, i = j \geq 2; \\ x_i^{a_1} S^{\{a_1 \rightarrow a_j\}}(B_i), j \geq i+1, i \geq 2. \end{cases}$$

where

$$A_2 = B_2 = x_2^{a_2-a_1} - x_1^{a_2-a_1};$$

$$A_k = \{S_{\{x_{k-1} \rightarrow x_k\}}(A_{k-1})\} A_{k-1} - S_{\{x_{k-1} \rightarrow x_k\}}(A_{k-1}) S^{\{a_{k-1} \rightarrow x_k\}}(A_{k-1}), k \geq 3, (1.1)$$

$$A_k = S_{\{x_{k-1} \rightarrow x_k\}}(B_{k-1}) - \{S^{\{a_{k-1} \rightarrow x_k\}}(B_{k-1})\} (\frac{S_{\{x_{k-1} \rightarrow x_k\}}(A_{k-1})}{A_{k-1}}), k \geq 3, (1.2)$$

$$S_{\{x_{k-1} \rightarrow x_k\}}(A_{k-1}) := x_k \text{ substitutes for } x_{k-1} \text{ in } A_{k-1};$$

$$S^{\{a_{k-1} \rightarrow x_k\}}(A_{k-1}) := a_k \text{ substitutes for } a_{k-1} \text{ in } A_{k-1}.$$

2. The Schur function $s_\lambda(x_1, x_2, \dots, x_n)$ can be expressed as

$$s_\lambda(x_1, x_2, \dots, x_n) = \frac{x_1^{a_1} \times \prod_{2 \leq i \leq n} x_i^{a_i} B_i}{\prod_{1 \leq i < j \leq n} (x_j - x_i)}$$

where B_i are defined as above.

Basing on the above facts, we establish a method to count the number of various Young tableaux, including those with more than 2 rows, while remaining within a tableaux context.

II. MAIN RESULT AND AN EXAMPLE

Now we are in a position to state the main theorem.

Theorem II.1 The number of Young tableau of shape $\lambda = (k, k, \dots, k)$ with $|\lambda| = lk$ is the coefficient of the term $x_1 x_2 \dots x_{lk}$ in the expansion of $s_{(k, k, \dots, k)}(x_1, x_2, \dots, x_{lk})$.

Proof. By definition, the Schur function s_λ is the symmetric function defined as:

$$s_\lambda(x_1, x_2, \dots, x_{lk}) = \sum_{T \in SSYT^\lambda} X^T = \sum_{T \in SSYT^\lambda} x_1^{m_1} x_2^{m_2} \dots x_{lk}^{m_{lk}}$$

where $|\lambda| = lk$, m_i is the number of entries i in $SSYT^\lambda$ T for $i = 1, 2, \dots, lk$. In particular, when $(m_1, m_2, \dots, m_{lk}) = (1, 1, \dots, 1)$, then the coefficient of the term $x_1 x_2 \dots x_{lk}$ is exactly the number of Young tableau of shape $\lambda = (k, k, \dots, k)$ with $|\lambda| = lk$.

Now let us take a look at the following example to see how to count the number of Young tableau of shape λ .

Example II.2 By using the recurrence formula (1.1) and (1.2), we can get

$$A_3 = (x_3^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1});$$

$$A_4 = [(x_4^{a_4-a_1} - x_1^{a_4-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_4^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_4-a_1} - x_1^{a_4-a_1})]$$

$$\times [(x_3^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})]$$

$$- [(x_4^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_4^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})]$$

$$\times [(x_3^{a_4-a_1} - x_1^{a_4-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1}) - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_4-a_1} - x_1^{a_4-a_1})];$$

$$A_5 = \{[(x_5^{a_5-a_1} - x_1^{a_5-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_5^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_5-a_1} - x_1^{a_5-a_1})]$$

$$\begin{aligned} & \times [(x_3^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})] \\ & - [(x_5^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_5^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})] \\ & \times [(x_3^{a_5-a_1} - x_1^{a_5-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_5-a_1} - x_1^{a_5-a_1})] \\ & \times \{[(x_4^{a_4-a_1} - x_1^{a_4-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_4^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_4-a_1} - x_1^{a_4-a_1})] \\ & - (x_4^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})\} \\ & \times [(x_3^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})] \\ & - [(x_4^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_4^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})] \\ & \times [(x_3^{a_4-a_1} - x_1^{a_4-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1}) - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_4-a_1} - x_1^{a_4-a_1})] \\ & - [(x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_4-a_1} - x_1^{a_4-a_1})] \\ & - \{[(x_5^{a_4-a_1} - x_1^{a_4-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_5^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_4-a_1} - x_1^{a_4-a_1})] \\ & \times [(x_3^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})] \\ & - [(x_5^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_5^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})] \\ & \times [(x_3^{a_4-a_1} - x_1^{a_4-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1}) - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_4-a_1} - x_1^{a_4-a_1})] \\ & - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_4-a_1} - x_1^{a_4-a_1})\} \\ & \times \{[(x_4^{a_5-a_1} - x_1^{a_5-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_4^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_5-a_1} - x_1^{a_5-a_1})] \\ & \times [(x_3^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})] \\ & - [(x_4^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_4^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})] \\ & \times [(x_3^{a_5-a_1} - x_1^{a_5-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1}) - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_5-a_1} - x_1^{a_5-a_1})] \\ & - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_5-a_1} - x_1^{a_5-a_1})\}]; \end{aligned}$$

$$\begin{aligned} A_6 = \{ \{ & [(x_6^{a_6-a_1} - x_1^{a_6-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_6^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_6-a_1} - x_1^{a_6-a_1})] \\ & \times [(x_3^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})] \\ & - [(x_6^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_6^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})] \\ & - (x_6^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1}) \} \} \end{aligned}$$

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$$\begin{aligned} & \times [(x_3^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) \\ & - (x_3^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})] \\ & - [(x_5^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) \\ & - (x_5^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})] \\ & \times [(x_3^{a_4-a_1} - x_1^{a_4-a_1})(x_2^{a_3-a_1} - x_1^{a_2-a_1}) \\ & - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_5-a_1} - x_1^{a_4-a_1})] \} \\ & \times \{ [(x_4^{a_6-a_1} - x_1^{a_6-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) \\ & - (x_4^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_6-a_1} - x_1^{a_6-a_1})] \\ & \times [(x_3^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) \\ & - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})] \\ & - [(x_5^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) \\ & - (x_4^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})] \\ & \times [(x_3^{a_6-a_1} - x_1^{a_6-a_1})(x_2^{a_3-a_1} - x_1^{a_2-a_1}) \\ & - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_6-a_1} - x_1^{a_6-a_1})] \} \}; \end{aligned}$$

$$B_4 = \{ (x_4^{a_4-a_1} - x_1^{a_4-a_1}) - (x_2^{a_4-a_1} - x_1^{a_4-a_1}) \times \left[\frac{(x_4^{a_2-a_1} - x_1^{a_2-a_1})}{(x_2^{a_2-a_1} - x_1^{a_2-a_1})} \right] \} \\ - \{ (x_3^{a_4-a_1} - x_1^{a_4-a_1}) - (x_2^{a_4-a_1} - x_1^{a_4-a_1}) \times \left[\frac{(x_3^{a_2-a_1} - x_1^{a_2-a_1})}{(x_2^{a_2-a_1} - x_1^{a_2-a_1})} \right] \} \\ \times \left\{ \frac{(x_4^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1})}{(x_3^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1})} \right. \\ \left. - \frac{(x_4^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})}{(x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})} \right\};$$

$$\begin{aligned}
B_5 = & \{ (x_5^{a_5-a_1} - x_1^{a_5-a_1}) - (x_2^{a_5-a_1} - x_1^{a_5-a_1}) \times \left[\frac{(x_5^{a_2-a_1} - x_1^{a_2-a_1})}{(x_2^{a_2-a_1} - x_1^{a_2-a_1})} \right] \} \\
& - \{ (x_3^{a_5-a_1} - x_1^{a_5-a_1}) - (x_2^{a_5-a_1} - x_1^{a_5-a_1}) \times \left[\frac{(x_3^{a_2-a_1} - x_1^{a_2-a_1})}{(x_2^{a_2-a_1} - x_1^{a_2-a_1})} \right] \} \\
& \times \left\{ \frac{(x_5^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1})}{(x_3^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1})} \right. \\
& \left. - \frac{(x_5^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})}{(x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})} \right\} \\
& - \{ (x_4^{a_5-a_1} - x_1^{a_5-a_1}) - (x_2^{a_5-a_1} - x_1^{a_5-a_1}) \times \left[\frac{(x_4^{a_2-a_1} - x_1^{a_2-a_1})}{(x_2^{a_2-a_1} - x_1^{a_2-a_1})} \right] \}
\end{aligned}$$

575

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$$\begin{aligned}
& - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})] \\
& - [(x_4^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) \\
& - (x_4^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1})] \\
& \times [(x_3^{a_5-a_1} - x_1^{a_5-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) \\
& - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_5-a_1} - x_1^{a_5-a_1})] \} \} \}.
\end{aligned}$$

Now considering $\lambda = (2,2,2) = (2,2,2,0,0,0)$, then

$(a_1, a_2, a_3, a_4, a_5, a_6) = (0,1,2,5,6,7)$ and

$$s_{(2,2,2)}(x_1, x_2, x_3, x_4, x_5, x_6)$$

$$\begin{aligned}
& = \frac{\det G_{\{6;0,1,2,5,6,7\}}}{\prod_{1 \leq i, j \leq 6} (x_j - x_i)} = \frac{\prod_{2 \leq i \leq 6} B_i}{\prod_{1 \leq i, j \leq 6} (x_j - x_i)} \\
& = x_1^2 x_2^2 x_3^2 + x_1^2 x_2^2 x_3 x_4 + x_1^2 x_2^2 x_3^2 x_4 + x_1 x_2^2 x_3^2 x_4 \\
& + x_1^2 x_2^2 x_4^2 + x_1^2 x_2 x_3 x_4^2 + x_1 x_2^2 x_3 x_4^2 + x_1^2 x_3^2 x_4^2 \\
& + x_1^2 x_2 x_3^2 x_4^2 + x x_2^2 x_3^2 x_4^2 + x_1^2 x_2^2 x_3 x_5 + x_1^2 x_2 x_3^2 x_5 \\
& + x_1 x_2^2 x_3^2 x_5 + x_1^2 x_2^2 x_4 x_5 + 2x_1^2 x_2 x_3 x_4 x_5 + 2x_1 x_2^2 x_3 x_4 x_5 \\
& + x_1^2 x_3^2 x_4 x_5 + 2x_1 x_2 x_3^2 x_4 x_5 + x_2^2 x_3^2 x_4 x_5 + x_1^2 x_2 x_4^2 x_5 \\
& + x_1 x_2^2 x_4^2 x_5 + x_1^2 x_3 x_4^2 x_5 + 2x_1 x_2 x_3 x_4^2 x_5 + x_2^2 x_3 x_4^2 x_5 \\
& + x_1 x_3^2 x_4^2 x_5 + x_2 x_3^2 x_4^2 x_5 + x_1^2 x_2^2 x_5^2 + x_1^2 x_2 x_3 x_5^2 \\
& + x_1 x_2^2 x_3 x_5^2 + x_1^2 x_3^2 x_5^2 + x_1 x_2 x_3^2 x_5^2 + x_2^2 x_3^2 x_5^2 \\
& + x_1^2 x_2 x_4 x_5^2 + x_1 x_2^2 x_4 x_5^2 + x_1^2 x_3 x_4 x_5^2 + 2x_1 x_2 x_3 x_4 x_5^2 \\
& + x_2^2 x_3 x_4 x_5^2 + x_1 x_3^2 x_4 x_5^2 + x_2 x_3^2 x_4 x_5^2 + x_1^2 x_4^2 x_5^2 \\
& + x_1 x_2 x_4^2 x_5^2 + x_2^2 x_4^2 x_5^2 + x_1 x_3 x_4^2 x_5^2 + x_2 x_3 x_4^2 x_5^2 \\
& + x_3^2 x_4^2 x_5^2 + x_1^2 x_2^2 x_3 x_6 + x_1^2 x_2 x_3^2 x_6 + x_1 x_2^2 x_3^2 x_6 \\
& + x_1^2 x_2^2 x_4 x_6 + 2x_1^2 x_2 x_3 x_4 x_6 + 2x_1 x_2^2 x_3 x_4 x_6 + x_1^2 x_3^2 x_4 x_6 \\
& + 2x_1 x_2 x_3^2 x_4 x_6 + x_2^2 x_3^2 x_4 x_6 + x_1^2 x_2 x_4^2 x_6 + x_1 x_2^2 x_4^2 x_6 \\
& + x_1^2 x_3 x_4^2 x_6 + 2x_1 x_2 x_3 x_4^2 x_6 + x_2^2 x_3 x_4^2 x_6 + x_1 x_3^2 x_4^2 x_6 \\
& + x_2 x_3^2 x_4^2 x_6 + x_1^2 x_2^2 x_5 x_6 + 2x_1^2 x_2 x_3 x_5 x_6 + 2x_1 x_2^2 x_3 x_5 x_6 \\
& + x_1^2 x_3^2 x_5 x_6 + 2x_1 x_2 x_3^2 x_5 x_6 + x_2^2 x_3^2 x_5 x_6 + 2x_1^2 x_2 x_4 x_5 x_6 \\
& + 2x_1 x_2^2 x_4 x_5 x_6 + 2x_1^2 x_3 x_4 x_5 x_6 + 5x_1 x_2 x_3 x_4 x_5 x_6 + 2x_2^2 x_3 x_4 x_5 x_6 \\
& + 2x_1 x_3^2 x_4 x_5 x_6 + 2x_2 x_3^2 x_4 x_5 x_6 + x_1^2 x_4^2 x_5 x_6 + 2x_1 x_2 x_4^2 x_5 x_6 \\
& + x_2^2 x_4^2 x_5 x_6 + 2x_1 x_3 x_4^2 x_5 x_6 + 2x_2 x_3 x_4^2 x_5 x_6 + x_3^2 x_4^2 x_5 x_6 \\
& + x_1^2 x_2 x_5^2 x_6 + x_1 x_2^2 x_5^2 x_6 + x_1^2 x_3 x_5^2 x_6 + 2x_1 x_2 x_3 x_5^2 x_6 \\
& + x_2^2 x_3 x_5^2 x_6 + x_1 x_3^2 x_5^2 x_6 + x_2 x_3^2 x_5^2 x_6 + x_1^2 x_4 x_5^2 x_6 \\
& + 2x_1 x_2 x_4 x_5^2 x_6 + x_2^2 x_4 x_5^2 x_6 + 2x_1 x_3 x_4 x_5^2 x_6 + 2x_2 x_3 x_4 x_5^2 x_6 \\
& + x_3^2 x_4 x_5^2 x_6 + x_1 x_4^2 x_5^2 x_6 + x_2 x_4^2 x_5^2 x_6 + x_3 x_4^2 x_5^2 x_6 \\
& + x_1^2 x_2^2 x_6^2 + x_1^2 x_2 x_3 x_6^2 + x_1 x_2^2 x_3 x_6^2 + x_1^2 x_3^2 x_6^2
\end{aligned}$$

$$\begin{aligned}
& + x_1 x_2 x_3^2 x_6^2 + x_2^2 x_3^2 x_6^2 + x_1^2 x_2 x_4 x_6^2 + x_1 x_2^2 x_4 x_6^2 \\
& + x_1^2 x_3 x_4 x_6^2 + 2x_1 x_2 x_3 x_4 x_6^2 + x_2^2 x_3 x_4 x_6^2 + x_1 x_3^2 x_4 x_6^2 \\
& + x_2 x_3^2 x_4 x_6^2 + x_1^2 x_4^2 x_6^2 + x_1 x_2 x_4^2 x_6^2 + x_2^2 x_4^2 x_6^2 \\
& + x_1 x_3 x_4^2 x_6^2 + x_2 x_3 x_4^2 x_6^2 + x_3^2 x_4^2 x_6^2 + x_1^2 x_2 x_5 x_6^2 \\
& + x_1 x_2^2 x_5 x_6^2 + x_1^2 x_3 x_5 x_6^2 + 2x_1 x_2 x_3 x_5 x_6^2 + x_2^2 x_3 x_5 x_6^2 \\
& + x_1 x_3^2 x_5 x_6^2 + x_2 x_3^2 x_5 x_6^2 + x_1^2 x_4 x_5 x_6^2 + 2x_1 x_2 x_4 x_5 x_6^2 \\
& + x_2^2 x_4 x_5 x_6^2 + 2x_1 x_3 x_4 x_5 x_6^2 + 2x_2 x_3 x_4 x_5 x_6^2 + x_3^2 x_4 x_5 x_6^2 \\
& + x_1 x_4^2 x_5 x_6^2 + x_2 x_4^2 x_5 x_6^2 + x_3 x_4^2 x_5 x_6^2 + x_1^2 x_5^2 x_6^2 \\
& + x_1 x_2 x_5^2 x_6^2 + x_2^2 x_5^2 x_6^2 + x_1 x_3 x_5^2 x_6^2 + x_2 x_3 x_5^2 x_6^2 \\
& + x_3^2 x_5^2 x_6^2 + x_1 x_4 x_5^2 x_6^2 + x_2 x_4 x_5^2 x_6^2 + x_3 x_4 x_5^2 x_6^2 \\
& + x_4^2 x_5^2 x_6^2.
\end{aligned}$$

Observing that the coefficient of the term $x_1 x_2 x_3 x_4 x_5 x_6$ is 5, which is exactly the number of Young tableau of shape $\lambda = (2,2,2)$.

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