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Strict Stability of Fuzzy Differential Equations with Impulse Effect

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Abstract—In this paper, some results on strict stability have been extended for fuzzy differential equations with impulse effect using Lyapunov functions and Razumikhin technique.

Keywords—Fuzzy differential equations, Impulsive differential equations, Strict stability, Lyapunov function, Razumikhin technique.

I. Introduction

Differential equations have been used to model the evolutionary mechanisms of various dynamical processes in diverse field of application. But, when a dynamical system is modeled by deterministic ordinary differential equations, we cannot usually be sure that the model is perfect because, in general, the knowledge of dynamical system is often incomplete and vague. Imprecision due to uncertainty or vagueness requires the introduction of fuzzy differential equations. The study of fuzzy differential equations has been initiated in 1978 by Kandel and Byatt [6], [7] and after that it has become an independent subject in conjection with fuzzy valued analysis and set valued differential equations. Buckley and Feuring [1] have given a very general formulation for fuzzy first order initial value problem and Kaleva [5] established an existence and uniqueness result for a solution of fuzzy differential equation. There exists sufficient literature on fuzzy differential equations (see monograph [8] and references therein).

Some authors have also worked on fuzzy functional differential equations (see [2], [11]), but so far a small amount of work is done for the fuzzy differential equations with impulse effect. Srivastava and Gupta [12] have established the local and global existence and uniqueness results for fuzzy differential equation with impulse effect using Hukuhara derivative and contraction principle. Guo et al. [4] established some existence results for the impulsive functional differential equations utilizing the Hullermeier approach.

Stability is one of the important qualitative property and has been studied by many authors in last few years for fuzzy differential equations [3], [13]. Strict stability is a kind of stability that can give us some information about the rate of decay of the solutions and has been studied by many authors for differential equations with or without impulse effect (see [9], [10], [14]).

The aim of this paper is to establish some results on strict stability for fuzzy differential equation with impulse effect. Various sufficient conditions are derived using Lyapunov functions and Razumikhin technique.

The paper is organized as follows. Section 2 introduces some preliminary definitions and notations which will be used throughout the paper. In Section 3, some sufficient conditions have been established for strict stability of trivial solution of fuzzy differential equation with impulse effect.

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II. Preliminaries

The idea of fuzzy set was proposed by Lofti Zadeh in the 1960s, as a tool to handle uncertainty that is due to vagueness or imprecision rather than to randomness.

The concept of fuzzy set is based on the idea that each element x in the base set X is assigned a membership grade $u(x)$ taking values in $[0,1]$. According to Zadeh, a fuzzy subset of X is a non-empty subset $\{(x,u(x)) : x \in X\}$ of $X \times [0,1]$ for some function $u : X \to [0,1]$. In particular, a fuzzy subset of $\Re^n$ is defined in terms of a membership function which assigns to each point $x \in \Re^n$ a grade of membership in the fuzzy set. Such a membership function $u : \Re^n \to [0,1]$ is used to denote the corresponding fuzzy set.

For instance, the function $u : \Re^1 \to [0,1]$ with

$$u(x) = \begin{cases} 0, x \leq 2 \\ \frac{1}{99}(x-2), 2 < x \leq 100 \\ 1, 100 < x \end{cases} \tag{1}$$

is an example of a fuzzy set of real numbers $>> 2$. It is necessary to mention here that there may be many other reasonable choices of membership grade function different from as defined above.

Let u be any fuzzy set in X. Then for each $\alpha \in (0,1]$, the set $[u]^\alpha = \{x \in X : u(x) \geq \alpha\}$ is called α-level set of a fuzzy set.

Clearly, α-level sets of a fuzzy set are crisp sets.

The support $[u]^0$ of a fuzzy set is then defined as the closure of the union of all its level sets, that is

$$[u]^0 = \overline{\cup_{\alpha \in (0,1]} [u]^\alpha}.$$

The union, intersection and complement of fuzzy sets can be defined pointwise in terms of their membership grades. Let u and v be any two fuzzy sets, then the complement u^c, their union $u \vee v$ and the intersection $u \wedge v$ are defined respectively as

$$u^c(x) = 1 - u(x), \tag{2}$$

$$u \vee v(x) = u(x) \vee v(x) = \max\{u(x), v(x)\}, \tag{3}$$

$$u \wedge v(x) = u(x) \wedge v(x) = \min\{u(x), v(x)\}, \tag{4}$$

for each $x \in X$.
A fuzzy subset u is said to be convex if $u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)]$ for some $\lambda \in [0,1]$ and $x, y \in X$.
In case of fuzzy numbers, that is fuzzy sets $u : \Re \to [0,1]$, fuzzy convexity means that the level sets are intervals.
The space $E^n = \{u : \Re^n \to I = [0,1]\}$ denotes the space of all fuzzy subsets u of $\Re^n$ which satisfy the following assumptions:
(a) u is normal i.e. there exist an $x_0 \in \Re^n$ such that $u(x_0) = 1$;
(b) $[u]^0 = \overline{\{u \in \Re^n : u(x) > 0\}}$ is compact;
(c) u is upper semicontinuous;
(d) u is fuzzy convex, i.e. $u(\lambda x + (1-\lambda)y \geq \min\{u(x), u(y)\}, 0 \leq \lambda \leq 1$.

Remark: An advantage of only requiring the $u \in E^n$ to be upper semicontinuous and not continuous is that the nonempty compact convex subsets of $\Re^n$ can then also be included in E^n by means of their characteristic functions.
For later purposes, we define $\widehat{0} \in E^n$ as $\widehat{0}(x) = 1$ if $x = 0$ and $\widehat{0}(x) = 0$ if $x \neq 0$.
Consider the following fuzzy differential equation with impulse effect

$$\begin{aligned} u' &= f(t,u),\ t \neq t_k, \\ u(t_k^+) &= u(t_k) + I_k(u(t_k)),\ k = 1,2,..., \\ u(t_0^+) &= u_0, \end{aligned} \tag{5}$$

where $f : J \times E^n \to E^n, J = [t_0, \infty), I_k : E^n \longrightarrow E^n$ is continuous, $u_0 \in E^n$ and $t_k, k = 1,2,...$ are points of impulses such that $t_k < t_{k+1}$ and $t_k \to \infty$ as $k \to \infty$.
Now, we have following definitions:
Definition 2.1: The trivial solution of (5) is said to be strictly stable if, for any $\epsilon_1 > 0$ and $t_0 \in \Re_+$, there exists a $\delta_1 = \delta_1(t_0, \epsilon_1) > 0$ such that $d[u_0, \hat{0}] < \delta_1$ implies $d[u(t), \hat{0}] < \epsilon_1, t \geq t_0$ and for every $\delta_2 \leq \delta_1$, there exists an $0 < \epsilon_2 < \delta_2$ such that $d[u_0, \hat{0}] > \delta_2$ implies $d[u(t), \hat{0}] > \epsilon_2$.

Definition 2.2: The trivial solution is said to be strictly uniformly stable, if δ_1, δ_2 and ϵ_2 in above definition are independent of t_0.

Definition 2.3: The trivial solution is said to be strictly attractive, if given $t \geq t_0$ and $\alpha_1 > 0, \epsilon_1 > 0$, for every $\alpha_2 \leq \alpha_1$, there exists $\epsilon_2 < \epsilon_1$ and $T_1(t_0, \epsilon_1), T_2(t_0, \epsilon_2)$ such that $\alpha_2 < d[u_0, \hat{0}] < \alpha_1$ implies $\epsilon_2 < d[u(t), \hat{0}] < \epsilon_1$ for $t_0 + T_1 \leq t \leq t_0 + T_2$.

Definition 2.4: The trivial solution is said to be strictly uniformly attractive if T_1, T_2 in above definition are independent of t_0.

Definition 2.5: A function $V : \Re_+ \times E^n \to \Re_+$ is said to belong to class V_0 if
(i) $V(t,u)$ is continuous in $(t_{k-1}, t_k] \times E^n$ and for each $u \in E^n,\ k = 1,2,..., \lim_{(t,v)\to(t_k^+,u)} V(t,v) = V(t_k^+, u)$ exists;
(ii) $V(t,u)$ is locally Lipschitzian in u and $V(t,0) \equiv 0$.

Definition 2.6: Let $V \in V_0$, for each $(t,u) \in (t_{k-1}, t_k] \times E^n$, D^+V is defined as

$$D^+V(t,u) = \lim_{h \to 0^+} \sup \frac{1}{h}[V(t+h, u + hf(t,u)) - V(t,u)].$$

We define,
(i) $\mathcal{K}_0 = \{a \in C(\Re_+, \Re_+) : a(0) = 0,\ a(s) > 0 \text{ for } s > 0\}$.
(ii) $\mathcal{K} = \{a \in \mathcal{K}_0 : a \text{ is strictly increasing in } \Re_+\}$.
(iii) $\mathcal{K}_1 = \{\phi \in C(\Re_+, \Re_+) : \phi \text{ is increasing and } \phi(s) < s \text{ for } s > 0\}$.

III. Main results

In this section, we shall investigate some sufficient conditions for strict stability of trivial solution of the fuzzy impulsive differential equation (5) by using Lyapunov function with Razumikhin technique.

Theorem 3.1 Assume that
(i) There exists $V_1 \in V_0$ such that $b_1(d[u, \hat{0}]) \leq V_1(t,u) \leq a_1(d[u, \hat{0}]),\ a_1, b_1 \in \mathcal{K}$;
(ii) $D^+V_1(t,u) \leq g(t)w(V_1(t,u)),\ g, w : \Re_+ \to \Re_+$ are locally integrable;
(iii) For all $k \in \mathbf{N},\ V_1(t_k^+, u(t_k) + I_k(u(t_k))) \leq \psi_1(V_1(t_k, u(t_k)))$, where $\psi_1 \in \mathcal{K}_1$;
(iv) There exist a constant $A > 0$ such that $\int_{t_{k-1}}^{t_k} g(s)ds < A,\ k \in \mathbf{N}$ and for any $v > 0$ such that $\int_v^{\psi_1^{-1}(v)} \frac{ds}{w(s)} \geq A$;
(v) There exists $V_2 \in V_0$ such that $b_2(d[u, \hat{0}]) \leq V_2(t,u) \leq a_2(d[u, \hat{0}]),\ a_2, b_2 \in \mathcal{K}$;
(vi) $D^+V_2(t,u) \leq h(t)p(V_2(t,u)),\ h, p : \Re_+ \to \Re_+$ are locally integrable;
(vii) For all $k \in \mathbf{N},\ V_2(t_k^+, u(t_k) + I_k(u(t_k))) \geq \psi_2^{-1}(V_2(t_k, u(t_k)))$, where $\psi_2 \in \mathcal{K}_1$;
(viii) There exist a constant $B > 0$ such that $\int_{t_{k-1}}^{t_k} h(s)ds < B,\ k \in \mathbf{N}$ and for any $v > 0$ such that $\int_v^{\psi_2(v)} \frac{ds}{p(s)} \geq B$;
Then the trivial solution of (5) is strictly uniform stable.
Proof: Let $0 < \epsilon_1 < \rho$. Choose $\delta_1 = \delta_1(\epsilon_1) > 0$ such that $\psi_1^{-1}(a_1(\delta_1)) < b_1(\epsilon_1)$.
Firstly we prove that for $u_0 \in E^n,\ d([u_0, \hat{0}]) < \delta_1$ implies $d([u(t), \hat{0}]) < \epsilon_1,\ t \geq t_0$.
Clearly, $V_1(t_0, u_0) \leq a_1(d[u_0, \hat{0}]) < a_1(\delta_1) < \psi_1^{-1}(a_1(\delta_1))$.
We claim that

$$V_1(t, u(t)) \leq \psi_1^{-1}(a_1(\delta_1)),\ t_0 < t \leq t_1. \tag{6}$$

If (6) does not hold, then there is an $\hat{t} \in (t_0, t_1]$ such that

$$V_1(\hat{t}, u(\hat{t})) > \psi_1^{-1}(a_1(\delta_1)) > a_1(\delta_1) > V_1(t_0, u_0).$$

From the continuity of $V_1(t, u(t))$ in $(t_0, t_1]$, there exists $s_1 \in (t_0, \hat{t})$ such that

$$\begin{aligned} V_1(s_1, u(s_1)) &= \psi_1^{-1}(a_1(\delta_1)), \\ V_1(t, u(t)) &\leq \psi_1^{-1}(a_1(\delta_1)),\ t_0 \leq t < s_1, \end{aligned} \tag{7}$$

and also there exist an $s_2 \in (t_0, s_1)$ such that

$$\begin{aligned} V_1(s_2, u(s_2)) &= a_1(\delta_1), \\ V_1(t, u(t)) &\geq a_1(\delta_1),\ t \in [s_2, s_1]. \end{aligned} \tag{8}$$

Therefore, we integrate $D^+V(t,u) \leq g(t)w(V(t,u))$ in $[s_2, s_1]$ and by condition (iv), we have

$$\int_{V_1(s_2,u(s_2))}^{V_1(s_1,u(s_1))} \frac{ds}{w(s)} \leq \int_{s_2}^{s_1} g(t)dt \leq \int_{t_{k-1}}^{t_k} g(t)dt < A. \tag{9}$$

On the other hand (7), (8) and (iv) implies that

$$\int_{V_1(s_2,u(s_2))}^{V_1(s_1,u(s_1))} \frac{ds}{w(s)} = \int_{a_1(\delta_1)}^{\psi_1^{-1}(a_1(\delta_1))} \frac{ds}{w(s)} \geq A.$$

This is contradiction to inequality (9). So the inequality (6) holds.
From condition (iii), we have

$$\begin{aligned} V_1(t_1^+, u(t_1^+)) &= V_1(t_1^+, u(t_1) + I_1(u(t_1))) \\ &\leq \psi_1(V_1(t_1, u(t_1)) \leq a_1(\delta_1). \end{aligned} \quad (10)$$

Next, we claim that

$$V_1(t, u(t)) \leq \psi_1^{-1}(a_1(\delta_1)),\ t_1 < t \leq t_2. \quad (11)$$

Since $a_1(\delta_1) < \psi_1^{-1}(a_1(\delta_1))$, if inequality (11) does not hold, $\exists\ \hat{r} \in (t_1, t_2]$ such that

$$V_1(\hat{r}, u(\hat{r})) > \psi_1^{-1}(a_1(\delta_1)) > a_1(\delta_1) > V_1(t_1, u(t_1)).$$

From the continuity of $V_1(t, u(t))$ in $(t_1, t_2]$, there exist an $r_1 \in (t_1, \hat{r})$ such that

$$\begin{aligned} V_1(r_1, u(r_1)) &= \psi_1^{-1}(a_1(\delta_1)), \\ V_1(t, u(t)) &\leq \psi_1^{-1}(a_1(\delta_1)),\ t_1 \leq t < r_1 \end{aligned} \quad (12)$$

and also there exist an $r_2 \in (t_1, r_1)$ such that

$$\begin{aligned} V_1(r_2, u(r_2)) &= a_1(\delta_1), \\ V_1(t, u(t)) &\geq a_1(\delta_1),\ t \in [r_2, r_1]. \end{aligned} \quad (13)$$

Therefore, after integrating the inequality $D^+V(t,u) \leq g(t)w(V(t,u))$ in $[r_2, r_1]$, similar as above and we get a contradiction. So inequality (11) holds.
From condition (iii), we have

$$\begin{aligned} V_1(t_2^+, u(t_2^+)) &= V_1(t_2^+, u(t_2) + I_2(u(t_2))) \\ &\leq \psi_1(V_1(t_2, u(t_2)) \\ &\leq \psi_1(\psi_1^{-1}(a_1(\delta_1))) \\ &\leq a_1(\delta_1). \end{aligned} \quad (14)$$

By similar argument as before, we can prove that for $k = 1, 2, ...$

$$\begin{aligned} V_1(t, u(t)) \leq \psi_1^{-1}(a_1(\delta_1)),\ t_k < t \leq t_{k+1}, \\ V_1(t_{k+1}^+, u(t_{k+1}^+)) \leq a_1(\delta_1). \end{aligned} \quad (15)$$

Since $a_1(\delta_1) < \psi_1^{-1}(a_1(\delta_1))$, by inequalities (6), we have

$$V_1(t, u(t)) \leq \psi_1^{-1}(a_1(\delta_1)) < b_1(\epsilon_1), t \geq t_0.$$

By condition (i), we have

$$b_1(d[u(t), \hat{0}]) \leq V_1(t, u(t)) \leq \psi_1^{-1}(a_1(\delta_1)) < b_1(\epsilon_1),\ t \geq t_0.$$

Thus

$$d[u(t), \hat{0}] < \epsilon_1,\ t \geq t_0.$$

Now, let $0 < \delta_2 \leq \delta_1$ and choose $0 < \epsilon_2 \leq \delta_2$ such that $a_2(\epsilon_2) < \psi_2(b_2(\delta_2))$.
Next, we prove that $d([u_0, \hat{0}]) > \delta_2$ implies $d([u(t), \hat{0}]) > \epsilon_2,\ t \geq t_0$.
Clearly, $V_2(t_0, u_0) \geq b_2(d[u_0, \hat{0}]) > b_2(\delta_2) > \psi_2(b_2(\delta_2))$.
We claim that

$$V_2(t, u(t)) \geq \psi_2(b_2(\delta_2)),\ t_0 < t \leq t_1. \quad (16)$$

Suppose inequality (16) does not hold, then there is a $\bar{t} \in (t_0, t_1]$ such that

$$V_2(\bar{t}, u(\bar{t})) < \psi_2(b_2(\delta_2)) < b_2(\delta_2) \leq V_2(t_0, u_0).$$

From the continuity of $V_2(t, u(t))$ in $(t_0, t_1]$, there exist a $t^{'} \in (t_0, \bar{t})$ such that

$$\begin{aligned} V_2(t^{'}, u(t^{'})) &= \psi_2(b_2(\delta_2)), \\ V_2(t, u(t)) &\geq \psi_2(b_2(\delta_2)),\ t_0 \leq t < t^{'} \end{aligned} \quad (17)$$

and also there exist an $t^{''} \in (t_0, t^{'})$ such that

$$\begin{aligned} V_2(t^{''}, u(t^{''})) &= b_2(\delta_2), \\ V_2(t, u(t)) &\leq b_2(\delta_2),\ t \in [t^{''}, t^{'}]. \end{aligned} \quad (18)$$

After integrating $D^+V_2(t,u) \leq h(t)p(V_2(t,u))$ in $[t^{''}, t^{'}]$ and by condition (viii), we have

$$\int_{V_2(t'', u(t''))}^{V_2(t', u(t'))} \frac{ds}{p(s)} \leq \int_{t''}^{t'} h(t)dt \leq \int_{t_{k-1}}^{t_k} h(t)dt < B. \quad (19)$$

On the other hand (17), (18) and (viii) implies that

$$\int_{V_2(t'', u(t''))}^{V_2(t', u(t'))} \frac{ds}{p(s)} = \int_{b_2(\delta_2)}^{\psi_2(b_2(\delta_2))} \frac{ds}{p(s)} \geq B,$$

which is contradiction to inequality (19). So the inequality (16) holds.
From condition (vii) and inequality (16), we have

$$V_2(t_1^+, u(t_1^+)) \geq \psi_2^{-1}(V_2(t_1, u(t_1))) \geq b_2(\delta_2).$$

Next, we prove that

$$V_2(t, u(t)) \geq \psi_2(b_2(\delta_2)),\ t_1 < t \leq t_2. \quad (20)$$

If inequality (20) does not hold, then there is a $\bar{q} \in (t_1, t_2]$ such that

$$V_2(\bar{q}, u(\bar{q})) < \psi_2(b_2(\delta_2)) < b_2(\delta_2) \leq V_2(t_1, u(t_1)).$$

From the continuity of $V_2(t, u(t))$ in $(t_1, t_2]$, there exist a $q_1 \in (t_1, \bar{q})$ such that

$$\begin{aligned} V_2(q_1, u(q_1)) &= \psi_2(b_2(\delta_2)), \\ V_2(t, u(t)) &\geq \psi_2(b_2(\delta_2)),\ t_0 \leq t < q_1 \end{aligned} \quad (21)$$

and also there exist an $q_2 \in (t_1, q_1)$ such that

$$\begin{aligned} V_2(q_2, u(q_2)) &= b_2(\delta_2), \\ V_2(t, u(t)) &\leq b_2(\delta_2),\ t \in [q_2, q_1]. \end{aligned} \quad (22)$$

Therefore, after integrating $D^+V_2(t,u) \leq h(t)p(V_2(t,u))$ in $[q_2, q_1]$ and similar as before, we get a contradiction. So the inequality (20) holds.
From condition (vii), we have

$$\begin{aligned} V_2(t_2^+, u(t_2^+)) &= V_2(t_2^+, u(t_2) + I_2(u(t_2))) \\ &\geq \psi_2^{-1}(V_2(t_2, u(t_2)) \\ &\geq b_2(\delta_2). \end{aligned} \quad (23)$$

By similar arguement as before, we can prove that for $k = 1, 2, ...,$

$$\begin{aligned} V_2(t, u(t)) &\geq \psi_2(b_2(\delta_2)),\ t_k < t \leq t_{k+1}, \\ V_2(t_{k+1}^+, u(t_{k+1}^+)) &\geq b_2(\delta_2). \end{aligned} \quad (24)$$

Since $b_2(\delta_2) < \psi_2(b_2(\delta_2))$, by inequalities (16), (24), we have

$$V_2(t, u(t)) \geq \psi_2(b_2(\delta_2)) > a_2(\epsilon_2).$$

By condition (v), we have

$$a_2(d[u(t),\hat{0}]) \geq V_2(t,u(t)) \geq a_2(\epsilon_2),\ t \geq t_0.$$

Thus

$$d[u(t),\hat{0}] > \epsilon_2,\ t \geq t_0.$$

Thus the trivial solution of (5) is strictly uniformly stable.

Theorem 3.2 Assume that
(i) There exists $V_1 \in V_0$ such that $b_1(d[u,\hat{0}]) \leq V_1(t,u) \leq a_1(d[u,\hat{0}]),\ a_1,b_1 \in \mathcal{K}$;
(ii) For any solution $u(t)$ of (5), $D^+V_1(t,u) \leq 0$;
(iii) For all $k \in \mathbf{N},\ V_1(t_k^+,u(t_k)+I_k(u(t_k))) \leq (1+d_k)V_1(t_k,u(t_k))$, where $d_k \geq 0$ and $\sum_{k=1}^{\infty} d_k < \infty$;
(iv) There exists $V_2 \in V_0$ such that $b_2(d[u,\hat{0}]) \leq V_2(t,u) \leq a_2(d[u,\hat{0}]),\ a_2,b_2 \in \mathcal{K}$;
(v) For any solution $u(t)$ of (5), $D^+V_2(t,u) \geq 0$;
(vi) For all $k \in \mathbf{N},\ V_2(t_k^+,u(t_k)+I_k(u(t_k))) \geq (1-c_k)V_2(t_k,u(t_k))$, where $0 \leq c_k < 1$ and $\sum_{k=1}^{\infty} c_k < \infty$;
Then the trivial solution of (5) is strictly uniformly stable.
Proof: Since $\sum_{k=1}^{\infty} d_k < \infty$ and $\sum_{k=1}^{\infty} c_k < \infty$, it follows that $\prod_{k=1}^{\infty}(1+d_k) = M$ and $\prod_{k=1}^{\infty}(1-c_k) = N$.
Clearly, $1 \leq M < \infty, 0 < N \leq 1$.
Let $0 < \epsilon_1 < \rho$. Choose $\delta_1 = \delta_1(\epsilon_1) > 0$ such that $Ma_1(\delta_1) < b_1(\epsilon_1)$.
Firstly, we prove that for $u_0 \in E^n,\ d([u_0,\hat{0}]) < \delta_1$ implies $d([u(t),\hat{0}]) < \epsilon_1,\ t \geq t_0$.
Clearly, $V_1(t_0,u_0) \leq a_1(d[u_0,\hat{0}]) < a_1(\delta_1)$.
We claim that

$$V_1(t,u(t)) \leq a_1(\delta_1),\ t_0 < t \leq t_1. \tag{25}$$

If the inequality (25) does not hold, then there is a $\hat{t} \in (t_0,t_1]$ such that

$$V_1(\hat{t},u(\hat{t})) > a_1(\delta_1) \geq V_1(t_0,u_0),$$

which implies that there is a $t^* \in (t_0,\hat{t}]$ such that $D^+V_1(t,u) > 0$, which is a contradiction to condition (ii).
Therefore, inequality (25) holds.
From condition (iii), we have

$$\begin{aligned} V_1(t_1^+,u(t_1^+)) &= V_1(t_1^+,u(t_1)+I_1(u(t_1))) \\ &\leq (1+d_1)V_1(t_1,u(t_1)) \\ &\leq (1+d_1)a_1(\delta_1). \end{aligned} \tag{26}$$

Now, we claim that

$$V_1(t,u(t)) \leq (1+d_1)a_1(\delta_1), t_1 < t \leq t_2. \tag{27}$$

If inequality (27) does not hold, then there is an $\hat{s} \in (t_1,t_2]$ such that

$$V_1(\hat{s},u(\hat{s})) > (1+d_1)a_1(\delta_1) \geq V_1(t_1,u(t_1)),$$

which implies that there is an $s_1 \in (t_1,\hat{s})$ such that $D^+V_1(t,u) > 0$, which is again a contradiction to condition (ii). Therefore, the inequality (27) holds.
From condition (iii), we have

$$\begin{aligned} V_1(t_2^+,u(t_2^+)) &= V_1(t_2,u(t_2)+I_2(u(t_2))) \\ &\leq (1+d_2)V_1(t_2,u(t_2)) \\ &\leq (1+d_1)(1+d_2)a_1(\delta_1). \end{aligned} \tag{28}$$

By similar arguement as before, we can prove that for $k = 1,2,...$

$$\begin{aligned} V_1(t,u(t)) \quad \leq \quad & (1+d_1)(1+d_2)...(1+d_k)a_1(\delta_1), \\ & t_k < t \leq t_{k+1}, \end{aligned}$$

which together with (25) implies

$$V_1(t,u(t)) \leq Ma_1(\delta_1), \forall\ t \geq t_0.$$

So from condition (i), we have

$$b_1(d[u(t),\hat{0}]) \leq V_1(t,u(t)) \leq Ma_1(\delta_1) < b_1(\epsilon_1),\ t \geq t_0.$$

Thus

$$d[u(t),\hat{0}] \leq \epsilon_1, t \geq t_0.$$

Now, let $0 < \delta_2 \leq \delta_1$ and choose $0 < \epsilon_2 < \delta_2$ such that $a_2(\delta_2) < Nb_2(\delta_2)$.
Next, we prove that for $u_0 \in E^n,\ d([u_0,\hat{0}]) > \delta_2$ implies $d([u(t),\hat{0}]) > \epsilon_2$.
Clearly, $V_2(t_0,u_0) \geq b_2(d[u_0,\hat{0}]) > b_2(\delta_2)$.
We claim that

$$V_2(t,u(t)) \geq b_2(\delta_2),\ t_0 < t \leq t_1. \tag{29}$$

If inequality (29) does not hold, then there is a $\bar{t} \in (t_0,t_1]$ such that

$$V_2(\bar{t},u(\bar{t})) < b_2(\delta_2) \leq V_2(t_0,u_0),$$

which implies that there is a $t^1 \in (t_0,\bar{t})$ such that $D^+V_2(t^1,u(t^1)) < 0$, which is a contradiction to the condition (v). Therefore, inequality (29) holds.
From condition (vi), we have

$$\begin{aligned} V_2(t_1^+,u(t_1^+)) &= V_2(t_1^+,u(t_1)+I_1(u(t_1))) \\ &\geq (1-c_1)V_2(t_1,u(t_1)) \\ &\geq (1-c_1)b_2(\delta_2). \end{aligned} \tag{30}$$

Now, we claim that

$$V_2(t,u(t)) \geq (1-c_1)b_2(\delta_2), t_1 < t \leq t_2. \tag{31}$$

If inequality (31) does not hold, then there is an $\hat{r} \in (t_1,t_2]$ such that

$$V_2(\hat{r},u(\hat{r})) < (1-c_1)b_2(\delta_2) \leq V_2(t_1,u(t_1)),$$

which implies that there is an $r_1 \in (t_1,\hat{r})$ such that $D^+V_2(r_1,u(r_1)) > 0$, which is a contradiction to condition (v). Therefore, inequality (31) holds.
Also, from condition (vi), we have

$$\begin{aligned} V_2(t_2^+,u(t_2^+)) &= V_1(t_2^+,u(t_2)+I_2(u(t_2))) \\ &\geq (1-c_2)V_2(t_2,u(t_2)) \\ &\geq (1-c_1)(1-c_2)b_2(\delta_2). \end{aligned} \tag{32}$$

By similar arguement as before, we can prove that for $k = 1,2,...$

$$V_2(t,u(t)) \geq (1-c_1)(1-c_2)...(1-c_k)b_2(\delta_2),\ t_k < t \leq t_{k+1},$$

which together with inequality (29) and condition (iv) imply

$$a_2(d[u(t),\hat{0}]) \geq V_2(t,u(t)) \geq Nb_2(\delta_2) > a_2(\epsilon_2), t \geq t_0.$$

Thus,

$$d[u(t), \hat{0}] > \epsilon_2, t \geq t_0.$$

Hence the zero solution of (5) is strictly uniformly stable.

REFERENCES

[1] J. J. Buckley, and T. H. Feuring, " Fuzzy Differential Equations", *Fuzzy sets and systems*, 110, 2000, pp. 43-54.
[2] P. Balasubramaniam and S. Muralisankar, *Existence and Uniqueness of a Fuzzy Solution for Nonlinear Neural Functional Differential Equations*, Comput. Math. Appl., 42 (2001), 961-967.
[3] P. Diamond, "Stability and Periodicity in Fuzzy Differential Equations", *IEEE Trans. Fuzzy Systems*, 8, 2000, pp. 583-590.
[4] M. Guo, X. Xue, and R. Li, " Impulsive Functional Differential Inclusions and Fuzzy Population Models", *Fuzzy Sets and Systems*, 138, 2003, pp. 601-615.
[5] O. Kaleva, " Fuzzy Differential Equations", *Fuzzy Sets and Systems*, 24, 1987, pp. 301-317.
[6] A. Kandel, and W. J. Byatt, *Fuzzy Differential Equations*, Proc. Internat. Conf. on Cybernetics and Society, Tokyo-Kyoto, Japan, November 3-7, 1978, pp. 1213-1216.
[7] A. Kandel, and W. J. Byatt, "Fuzzy Processes", *Fuzzy Sets and Systems*, 4, 1980, 117-152.
[8] V. Lakshmikantham, and R. N. Mohapatra, *Theory of Fuzzy Differential Equations and Inclusions*, Taylor and Francis Publishers, London, 2003.
[9] V. Lakshmikantham, and Y. Zhang, " Strict Practicle Stability of Delay Differential Equation", *Appl. Math.Comput.*, 118, 2001, pp. 275-285.
[10] V. Lakshmikantham, and R. N. Mohapatra, (2001), *Strict Stability of Differential Equation*, Nonlinear Analysis, 46, pp. 915-921.
[11] V. Lupulescu, "On a Class of Fuzzy Functional Differential Equations", *Fuzzy Sets and Systems*, 160, 2009, pp. 1547-1562.
[12] S. K. Srivastava and B. Gupta, "Existence of Nonlinear Fuzzy Differential Equations with Impulses", *JIMS* (to appear).
[13] A. S. Vatsala, " Impulsive Hybrid Fuzzy Differential Equations", *Facta Universitatis*, 3(14), 2003, pp. 851-859.
[14] Y. Zhang, and J. Sun, "Strict Stability of Impulsive Functional Differential Equations", *J. Math. Anal. Appl.*, 301, 2005, pp. 237-248.