# A Note on Negative Hypergeometric Distribution and Its Approximation 

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#### Abstract

In this paper, at first we explain about negative hypergeometric distribution and its properties. Then we use the wfunction and the Stein identity to give a result on the poisson approximation to the negative hypergeometric distribution in terms of the total variation distance between the negative hypergeometric and poisson distributions and its upper bound.


Keywords-Negative hypergeometric distribution, Poisson distribution, Poisson approximation, Stein-Chen identity, w-function.

## I. INTRODUCTION

IET a box contain $S$ items of which are defective and $R$ are non defective. Items are inspected at random (one at a time) without replacement, from box until the number of non defective items reaches a fixed numberr.

Let $X$ be the number of defective items the sample, then $X$ has a negative hypergeometric distribution and denoted by $N H(R, S, r)$. Its probability function can be expressed as;

$$
\begin{equation*}
p_{X}(k)=\frac{\binom{r+k-1}{r}\binom{R-r+S-k}{S-k}}{\binom{R+S}{S}} \quad k=0,1, \ldots, S \tag{1}
\end{equation*}
$$

Where $R, S \in \mathbb{N}$ and $r \in\{1,2, \ldots, R\}$.
Now, we show the mean and variance of $X$ are $\mu=\frac{r S}{R+1}$ and $\sigma^{2}=\frac{r S(R+S+1)(R-r+1)}{(R+1)^{2}(R+2)}$, respectively.

Proof
$\mu=E(X)=\sum_{k=0}^{S} k \frac{\binom{r+k-1}{r}\binom{R-r+S-k}{S-k}}{\binom{R+S}{S}}$
$=\frac{1}{\binom{R+S}{S}} \sum_{k=0}^{S} \frac{(r+k-1)!}{(k-1)!(r-1)!}\binom{R-r+S-k}{S-k}$
$=\frac{1}{\binom{R+S}{S}} \sum_{k=0}^{S-1} \frac{(r+k)!}{k!(r-1)!}\binom{R-r+S-k-1}{S-k-1}$
$=\frac{r}{\binom{R+S}{S}} \sum_{k=0}^{S-1}\binom{r+k}{k}\binom{R-r+S-k-1}{S-k-1}$
Note that, we have
$\sum_{j=0}^{k}\binom{a+k-j-1}{k-j}\binom{b+j-1}{j}=\binom{a+b+k-1}{k}$
Then
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$\mu=\frac{r}{\binom{R+S}{S}}\binom{R+S}{S-1}=\frac{r S}{R+1}$
For variance we obtain $E(X(X-1))$ then,

$$
\begin{aligned}
& E(X(X-1))=\sum_{k=0}^{S} k(k-1) \frac{\binom{r+k-1}{r}\binom{R-r+S-k}{S-k}}{\binom{R+S}{S}} \\
& =\frac{r(r+1)}{\binom{R+S}{S}} \sum_{k=0}^{S-2}\binom{r+k+1}{k}\binom{R-r+S-k-2}{S-k-2} \\
& =\frac{r(r+1)}{\binom{R+S}{S}}\binom{R+S}{S-2}=\frac{r S(r+1)(S-1)}{(R+1)(R+2)} \\
& \quad \text { Then }
\end{aligned}
$$

$$
\begin{equation*}
\sigma^{2}=\frac{\mathrm{rS}(\mathrm{R}+\mathrm{S}+1)(\mathrm{R}-\mathrm{r}+1)}{(\mathrm{R}+1)^{2}(\mathrm{R}+2)} \tag{2}
\end{equation*}
$$

Suppose that $S$ and $R$ tend to $\infty$ in such a way that $\frac{S}{R+1} \rightarrow \theta$ $(0<\theta<1)$, then the negative hypergeometric distribution converges to the negative binomial distribution with parameters $r$ and $\frac{\theta}{1+\theta}$. Similarly this distribution may converge to the binomial or poisson or normal distribution if the conditions on their parameters are appropriate.
It should be noted that if $\frac{\mathrm{r}}{\mathrm{R}+1}$ is not be small and $S$ is sufficiently large, then $N H(R, S, r)$ can also approximated by the normal distribution with mean $\frac{\mathrm{Sr}}{\mathrm{R}+1}$ and variance $\frac{\mathrm{Sr}(\mathrm{R}-\mathrm{r}+1)}{(\mathrm{R}+1)^{2}}$. In this case, a bound on the normal approximation can be derived by using the same method in [3].

In this paper, we use the w-function associated with the random variable $X$ together with the Stein-Chen identity to give an upper bound for the total variation distance between the negative hypergeometric and poisson distributions.

## II. UsEFUl DEFINITION AND Propositions

A. Let X be a non-negative integer-valued random variable with distribution $F$ and let $P_{\lambda}$ denote the poisson distribution with mean $\lambda$. The total variation distance between two distribution defined by:

$$
\begin{equation*}
\mathrm{d}_{\mathrm{TV}}\left(\mathrm{~F}, \mathrm{P}_{\lambda}\right)=\sup _{\mathrm{A}}\left|\mathrm{~F}(\mathrm{~A})-\mathrm{P}_{\lambda}(\mathrm{A})\right| \tag{3}
\end{equation*}
$$

Where A runs over subset of non-negative integers. To obtain an upper bound for the total variation distance in terms of the w-function, we apply the Stein-chen identity (see [2]) according to which for every positive constant, every subset A of non-negative integers and some function $g=g_{\lambda, A}$,

$$
\begin{equation*}
F(A)-P_{\lambda}(A)=E(\lambda g(X+1)-X g(X)) \tag{4}
\end{equation*}
$$

The explicit formula for the function g can be found e.g in [2], but what we really need are the following estimates valid uniformly for all A:

$$
\begin{gather*}
\sup _{\mathrm{k}}|g(\mathrm{k})| \ll \min \left(1, \lambda^{-1 / 2}\right) \\
|\Delta \mathrm{g}|=\sup _{\mathrm{k}}|\Delta \mathrm{~g}(\mathrm{k})| \ll \lambda^{-1}\left(1-\mathrm{e}^{-\lambda}\right) \tag{5}
\end{gather*}
$$

where $\Delta g(k)=g(k+1)-g(k)($ see [1]).
B. Let a non-negative integer-valued random variable X with distribution $F=\{p(k), k=0,1,2, \ldots\}$ have mean $\mu$ and variance $\sigma^{2}$. Define a function w associated with the random variable X by the relation

$$
\begin{equation*}
\text { C. } \quad \sigma^{2} w(k) p(k)=\sum_{i=0}^{k}(\mu-i) p(i), k=0,1,2, \ldots \tag{6}
\end{equation*}
$$

Immediately from the above we have

$$
\begin{equation*}
\mathrm{w}(0)=\frac{\mu}{\sigma^{2}} \tag{7}
\end{equation*}
$$

$\mathrm{w}(\mathrm{k}+1)=\frac{\mathrm{p}(\mathrm{k})}{\mathrm{p}(\mathrm{k}+1)} \mathrm{w}(\mathrm{k})+\frac{\mu-(\mathrm{k}+1)}{\sigma^{2}} \quad \mathrm{k}=0,1,2, \ldots$
And

$$
\begin{equation*}
w(k) \gg 0, k=0,1,2, \ldots \tag{8}
\end{equation*}
$$

Proposition1. If a non-negative integer-valued random variable X with distribution $\mathrm{p}(\mathrm{k})>0$, for all k in support of X and $0<\sigma^{2}=\operatorname{Var}(X)<\infty$, then;

$$
\begin{equation*}
\operatorname{Cov}(\mathrm{X}, \mathrm{~g}(\mathrm{X}))=\sigma^{2} \mathrm{E}(\mathrm{w}(\mathrm{X}) \Delta \mathrm{g}(\mathrm{X})) \tag{9}
\end{equation*}
$$

For any function $\mathrm{g}: \mathbb{N} \cup\{0\} \rightarrow \mathbb{R}$ for which $\mathrm{E}(\mathrm{w}(\mathrm{X}) \Delta \mathrm{g}(\mathrm{X}))<\infty \quad$.By taking $g(\mathrm{x})=\mathrm{x}$, we have $\mathrm{E}(\mathrm{w}(\mathrm{X}))=1$ (see [4]).

Proposition2. (Reference [6])Let $\mathrm{w}(\mathrm{X})$ be the w -function associated with the negative hypergeometric random variable, then;

$$
\begin{equation*}
\mathrm{w}(\mathrm{k})=\frac{(\mathrm{r}+\mathrm{k})(\mathrm{S}-\mathrm{k})}{(\mathrm{R}+1) \sigma^{2}} \tag{10}
\end{equation*}
$$

Where $\sigma^{2}=\frac{\mathrm{rS}(\mathrm{R}+\mathrm{S}+1)(\mathrm{R}-\mathrm{r}+1)}{(\mathrm{R}+1)^{2}(\mathrm{R}+2)}$.

## Proof

Following (7), we have

$$
\begin{align*}
w(k) & =\frac{p(k-1)}{p(k)} w(k-1)+\frac{\mu-k}{\sigma^{2}} \\
& =\frac{\mu}{\sigma^{2}}+\frac{p(k-1)}{p(k)} w(k-1)-\frac{k}{\sigma^{2}} \tag{11}
\end{align*}
$$

With replacing (1) in (11) we have
$w(k)=\frac{r S}{(R+1) \sigma^{2}}+\frac{k(R-r+S-k+1)}{(r+k-1)(S-k+1)} w(k-1)-\frac{k}{\sigma^{2}}$

$$
\mathrm{k}=1,2, \ldots, \mathrm{~S}
$$

And as we told before $w(0)=\frac{\mathrm{rS}}{(\mathrm{R}+1) \mathrm{\sigma}^{2}}$.
We will show that (10) holds for every $k \in\{1,2, \ldots, S\}$.
Equation (11) holds for $\mathrm{k}=1$ i.e

$$
\mathrm{w}(1)=\frac{(\mathrm{r}+1)(\mathrm{S}-1)}{(\mathrm{R}+1) \sigma^{2}}
$$

We assume that (11) holds for $k=i-1$, then we will prove that holds for $k=i$.

By mathematical induction, (11) holds for every $k \in$ $\{1,2, \ldots, S\}$.

## III. Poisson Approximation

We will prove our main result by using the w-function associated with the negative hypergeometric random variable $X$ and the Stein-Chen identity.
For the Stein-Chen identity, using definition 1, its applied for every positive constant $\lambda$, and every subset A of $g=$ $g_{A}: \mathbb{N} \cup\{0\} \rightarrow \mathbb{R}$, yield
$N H(R, S, r)\{A\}-P o(\lambda)\{A\}=E(\lambda g(X+1)-X g(X))$
For any subset A of $\mathbb{N} \cup\{0\}$, Barbour et al in [2] proved that:

$$
\begin{equation*}
\sup _{A, k}|\Delta \mathrm{~g}(\mathrm{k})|=\sup _{A, k}|\mathrm{~g}(\mathrm{k}+1)-\mathrm{g}(\mathrm{k})| \ll \lambda^{-1}\left(1-\mathrm{e}^{-\lambda}\right) \tag{13}
\end{equation*}
$$

The following theorem gives a result of the poisson approximation to the negative hypergeometric distribution.

Theorem. Let $X$ be negative hypergeometric random variable, $\lambda=\frac{\mathrm{rS}}{\mathrm{R}+1}$ and $r \gg S-1$, then for $A \subseteq \mathbb{N} \cup\{0\}$

$$
\begin{equation*}
d_{T V}(N H(R, S, r), P o(\lambda)) \leq\left(1-e^{-\lambda}\right) \frac{(R+1)(r+1)-S(R-r+1)}{(R+1)(R+2)} \tag{14}
\end{equation*}
$$

Proof
From (12) it follows that
$|N H(R, S, r)\{A\}-P o(\lambda)\{A\}|=|E(\lambda g(X+1)-X g(X))|$
$=|E(\lambda g(X+1))-\operatorname{Cov}(X, g(X))-\mu E(g(X))|$
$=|\lambda E(\Delta g(X))-\operatorname{Cov}(X, g(X))|$
$=\left|\lambda E(\Delta g(X))-\sigma^{2} E(w(X) \Delta g(X))\right| \quad$ by (9)
$\ll E\left|\left(\lambda-\sigma^{2} w(X)\right) \Delta g(X)\right|$
$\ll \sup |\Delta \mathrm{g}(\mathrm{x})| E\left|\lambda-\sigma^{2} w(X)\right|$
$\ll \lambda^{x>1}\left(1-\mathrm{e}^{-\lambda}\right) E\left|\lambda-\sigma^{2} w(X)\right| \quad$ by (13)
Then
$|N H(R, S, r)\{A\}-P o(\lambda)\{A\}| \ll \lambda^{-1}\left(1-\mathrm{e}^{-\lambda}\right) E\left|\lambda-\sigma^{2} w(X)\right|$
Now we show that $\lambda-\sigma^{2} w(X) \gg 0$. As by proposition 2,

$$
\begin{align*}
\lambda-\sigma^{2} w(X) & =\frac{r S}{R+1}-\sigma^{2} \frac{(r+k)(S-k)}{(R+1) \sigma^{2}}  \tag{15}\\
& =\frac{\mathrm{rS}}{\mathrm{R}+1}-\frac{(\mathrm{r}+\mathrm{k})(\mathrm{S}-\mathrm{k})}{(\mathrm{R}+1)} \\
& =\frac{\mathrm{k}(\mathrm{k}-\mathrm{S}+\mathrm{r})}{\mathrm{R}+1} \gg 0
\end{align*}
$$

Thus

$$
\begin{aligned}
E\left|\lambda-\sigma^{2} w(X)\right| & =E\left(\lambda-\sigma^{2} w(X)\right) \\
& =\lambda-\sigma^{2} E(w(X)) \\
& =\lambda-\sigma^{2} \\
& =\lambda \frac{(\mathrm{R}+1)(\mathrm{r}+1)-\mathrm{S}(\mathrm{R}-\mathrm{r}+1)}{(\mathrm{R}+1)(\mathrm{R}+2)}
\end{aligned}
$$

Then we have

$$
\begin{aligned}
d_{T V}(N H(R, S, r) & , P o(\lambda)) \\
& \leq(1 \\
& -e^{-\lambda)} \frac{(R+1)(r+1)-S(R-r+1)}{(R+1)(R+2)}
\end{aligned}
$$

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If $r=S-1$ then

$$
\begin{aligned}
& d_{T V}(N H(R, S, r), P o(\lambda)) \\
& \quad \leq\left(1-\mathrm{e}^{-\lambda}\right) \frac{(\mathrm{R}+1)(\mathrm{r}+1)-(\mathrm{r}+1)(\mathrm{R}-\mathrm{r}+1)}{(\mathrm{R}+1)(\mathrm{R}+2)} \\
& \text { Thus } \\
& d_{T V}(N H(R, S, r), P o(\lambda)) \leq\left(1-e^{-\lambda}\right) \frac{r(r+1)}{(R+1)(R+2)}<\frac{r}{R}
\end{aligned}
$$

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