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Blow up in Polynomial Differential Equations

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Abstract**—** Methods to detect and localize time singularities of polynomial and quasi-polynomial ordinary differential equations are systematically presented and developed. They are applied to examples taken form different fields of applications and they are also compared to better known methods such as those based on the existence of linear first integrals or Lyapunov functions.

Keywords**—** blow up, finite escape time, polynomial ODE, singularity, Lotka–Volterra equation, Painlevé analysis, Ψ-series, global existence

I. Introduction

A. Introductory example

The physical meaning of a singularity might be blowing up which either represents a realistic problem or an important disadvantage of a given model.

In the case of linear systems the problem is trivial, but in full generality nothing can be said about nonlinear equations. However, the case of differential equations with polynomial right hand side (or slightly more generally: with quasi-polynomial right hand side) is relatively easy to treat, still very important in many fields of applications such as chemical kinetics, electrical engineering, population biology etc.

While it is known that the domain of the solutions to linear differential equations is the whole domain of the coefficient functions, therefore it is the whole real line in the constant coefficient case, it is by far not true for nonlinear equations as the following simple example shows.

The domain of the right hand side of the differential equation of the initial value problem $\dot{x}(t) = x^2(t), \quad x(0) = x_0 \in \mathbb{R}^+$ can be taken to be $\Omega := \mathbb{R} \times \mathbb{R}^+$, still the maximal solution $]-\infty, 1/x_0[\ \ni t \mapsto x(t) := \frac{1}{1/x_0 - t}$ can only be defined on a proper subset of $\mathbb{R}$, or to put it another way, $t_*(x_0) := \sup \mathcal{D}_x = 1/x_0 < +\infty$. Such a singularity is usually called **movable singularity**, because it can be moved by changing the initial value. From the practical point of view the phenomenon can be called a **blow up**. Think only of the case when the solution describes the concentration in an autocatalytic chemical reaction. One speaks of **global existence**, if the solution does not blow up.

The main questions obviously are the (necessary or sufficient) conditions of blow up, and the time when it occurs—without explicitly solving the differential equation.

We mention that the topics of blow up is thoroughly studied for the case of partial differential equations, and also for stochastic processes. From the literature of the first topics see the randomly selected papers [4], [13] with chemical applications and useful lists of references. In the case of the second topics the term **finite escape time** or **first infinity** [5, pp. 257–271] is more often used.

B. The definition of blowing up

Let the solution to the initial value problem be $\dot{\mathbf{x}} = \mathbf{f} \circ \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0$ (with $\mathbf{f} \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n), \mathbf{x}_0 \in \mathbb{R}^n$) be $\boldsymbol{\varphi}(t, \mathbf{x}_0)$ at time $t \in I$ with some interval $I \subset \mathbb{R}$ containing 0. This solution is said to blow up if there exist $t_* \in \mathbb{R}^+$ such that for all $M \in \mathbb{R}$ there exists $\varepsilon \in \mathbb{R}^+$ for which and for all $t < t_*$ such that $t_* - t < \varepsilon$ the inequality $||\boldsymbol{\varphi}(t; \mathbf{x}_0)|| \geq M$ holds.

II. An algebraic method by Getz and Jacobson

Getz and Jacobson [9] has provided a sufficient condition for blowing up in quadratic polynomial systems based on an estimate using the corresponding scalar equation. Here we describe the method

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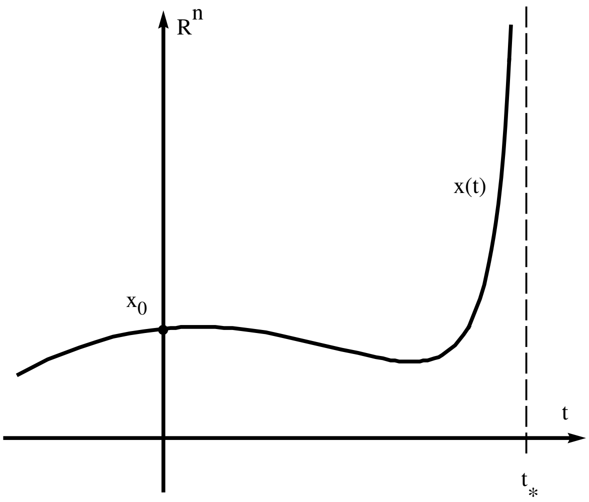

Fig. 1. What blow up means

and formulate a series of problems and possible generalizations. Let $n \in \mathbb{N}; \mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n \in \mathbb{R}^{n\times n}; \mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n, \in \mathbb{R}^n; c_1, c_2, \dots, c_n \in \mathbb{R}$, and consider the initial value problem

$$\dot{x}_i = \mathbf{x}^\top \mathbf{A}_i \mathbf{x} + \mathbf{b}_i^\top \mathbf{x} + c_i \ (i = 1, 2, \dots, n),\ \mathbf{x}(0) = \mathbf{x}_0 \tag{1}$$

(where the matrices $\mathbf{A}_i$ may be assumed to be symmetric). With $\boldsymbol{\omega} := (\omega_1, \omega_2, \dots, \omega_n)$ let us introduce $\mathbf{A} := \sum_{i=1}^n \omega_i \mathbf{A}_i, \quad \mathbf{b} := \sum_{i=1}^n \omega_i \mathbf{b}_i, \quad c := \sum_{i=1}^n \omega_i c_i, \quad \Delta := \frac{\lambda}{\boldsymbol{\omega}^\top \boldsymbol{\omega}}(\mathbf{b}^\top \mathbf{A}^{-1}\mathbf{b} - 4c)$, where λ is the smallest eigenvalue of the matrix $\mathbf{A}$.

The main result of Getz and Jacobson is: If there exists $\boldsymbol{\omega} \in \mathbb{R}^n$ such that $\mathbf{A}$ is positive definite, then the solution to (1) blows up

1) if $\Delta > 0$, then for all $\mathbf{x}_0 \in \mathbb{R}^n$;
2) if $\Delta \le 0$, then for all $\mathbf{x}_0 \in \mathbb{R}^n$ fulfilling $\boldsymbol{\omega}^\top \mathbf{x}_0 > -\frac{1}{2}\boldsymbol{\omega}^\top \mathbf{A}^{-1}\mathbf{b} + \frac{\sqrt{\Delta}\boldsymbol{\omega}^\top \boldsymbol{\omega}}{2\lambda}$.

They also give an upper estimate for the time of blowing up in both cases.

A series of problems can be formulated which would help using the theorem. We also give partial solutions or remarks.

1) How to give a lower estimate for the blow up time?
2) Which are the conditions on the matrices $\mathbf{A}_i$ to ensure the existence of $\boldsymbol{\omega} \in \mathbb{R}^n$ such that $\mathbf{A}$ is positive definite? (It is trivially true that if all the matrices $\mathbf{A}_i$ are semidefinite and at least one of them is definite, then there exist $\boldsymbol{\omega}$ for which $\mathbf{A}$ is positive definite. As a special case we get that solutions to $\dot{x} = -x^2$ blow up if started from a negative initial state.
3) If there exists more than one $\boldsymbol{\omega}$, how to chose among them in order to receive the best estimate for the blow up time?
4) If the equation $\dot{x}_i = \sum_{j,k=1}^n a_i^{jk} x_j x_k + \sum_{j=1}^n b_i^j x_j + c_i$ is the induced kinetic differential equation of a mass action type kinetic differential equation [8, page 35], then $a_i^{jk} \ge 0 \quad (i \ne j, i \ne k), b_i^j \ge 0 \quad (i \ne j), c_i \ge 0.$ How can these conditions utilize to apply the above theorem? In the special case, if $a_i^{ik} = a_i^{ji} = 0, b_i^i = 0, c_i = 0$, and if there exists i, j, k for which $a_i^{jk} > 0$, then the solutions with positive initial states blow up. It may still happen (see Sec. III-C) that no linear combination of the coefficient matrices is positive definite.

III. Transformation of Differential Equations

Polynomial differential equations can be transformed into a simpler form; a fact useful in itself, but especially useful to study blowing up.

A polynomial equation can be rewritten into the form

$$\dot{x}_i = x_i \left(\lambda_i + \sum_{j=1}^m A_{ij} \prod_{k=1}^n x_k^{B_{jk}} \right) \qquad (i = 1, 2, \dots, n) \tag{2}$$

with $\mathbf{A} = (A_{ij}) \in \mathbb{R}^{n\times m}, \mathbf{B} = (B_{jk}) \in \mathbb{R}^{m\times n}$, and $\boldsymbol{\lambda} = (\lambda_i) \in \mathbb{R}$. Let us also suppose that the equations are simplified in such a way that all the monomials occur once, i.e. the matrix $\mathbf{B}$ has no two equal rows. A large class of dynamical models in physics, biology and chemistry can be formulated in this form, especially the induced kinetic differential equations of all mass action type reactions. On the other hand, this form allows us to explicitly write down the Taylor series form of the solution [2]. (In what follows below we get rid of the restriction that the elements of $\mathbf{B}$ are all integers larger than -1, they should not even be integers at all.)

Let $\mathbf{C} \in \mathbb{R}^{n\times n}$ be an invertible matrix, and let us introduce the new variables through $x_i := \prod_{j=1}^n y_j^{C_{ij}}$. Then (2) will be transformed into an equation of the same form with the matrices as follows: $\boldsymbol{\lambda}' = \mathbf{C}^{-1}\lambda, \quad \mathbf{A}' = \mathbf{C}^{-1}\mathbf{A}, \quad \mathbf{B}' = \mathbf{BC}.$

From this it can also be seen that this transformation has two invariants, $\mathbf{B}\boldsymbol{\lambda}$ and $\mathbf{BA}$. (The question is if it may or may not have other invariants, as well.)

Now let us consider two special cases when this transformation gives a simpler form than the original one. The simpler form may help reveal if blow up occurs or not.

A. Decoupling

Suppose that the rank of the matrix $\mathbf{B}$ is $r < n$, then there exists $\boldsymbol{\varphi}_{r+1}, \boldsymbol{\varphi}_{r+2}, \dots, \boldsymbol{\varphi}_n \in \mathbb{R}^n$ for which one has $\mathbf{B}\boldsymbol{\varphi}_k = \mathbf{0} \quad (k = r+1, r+2, \dots, n)$, therefore it is appropriate to choose $\mathbf{C}$ as $\mathbf{C} := \begin{pmatrix} \mathbf{I}^{r\times r} & & & \\ & \boldsymbol{\varphi}_{r+1} & \cdots & \boldsymbol{\varphi}_n \\ \mathbf{0}^{(n-r)\times r} & & & \end{pmatrix}$, where $\mathbf{I}^{r\times r}$ is an $r \times r$ unit matrix, and $\mathbf{0}^{(n-r)\times r}$ is a zero matrix. In this case we have $\mathbf{B}' = (\mathbf{B}^{m\times r}\ \mathbf{0}^{m\times(n-r)})$, what means that the first r equations only contain the first $n-r$ variables, and solving these we can substitute these variables into the last $n-r$ equations to get a linear equation (with nonconstant coefficients). This procedure is thus a special case of lumping [17] and has been applied in the special case of second order reactions [16, pp. 290–295] earlier. From the point of our main concern we have that under the above conditions the system blows up if and only if the nonlinear part blows up.

B. Lotka–Volterra form

In the special case when $m = n$ and $\mathbf{B}$ is invertible, an especially simple form is obtained: a second order equation, what is more an equation of the Lotka–Volterra form results. Let $\mathbf{C} := \mathbf{B}^{-1}$, then $\boldsymbol{\lambda}' = \mathbf{B}\boldsymbol{\lambda}, \mathbf{A}' = \mathbf{BA}, \mathbf{B}' = \mathbf{I}$, which shows that these parameters are invariant for any further quasi-monomial transformation, meaning that we have arrived at the simplest form in a certain sense. The example below has a further merit: it shows that we may be lucky enough to get rid off from non-integer exponents. This may also mean that the original equation has not fulfilled the **Lipschitz condition** (global or local) whereas the transform surely obeys it. Another important point is that the transformed equation is always a kinetic differential equation (in the sense that there exist a reaction inducing it through the kinetic mass action law), although the original equation may possibly not have this property. These questions also merit a deeper investigation.

Let us start form the equation

$$\begin{aligned} \dot{x}_1 &= x_1 - \tfrac{x_1 x_2^{2/3}}{x_3^2} && = x_1\left(1 - \tfrac{x_2^{2/3}}{x_3^2}\right) \\ \dot{x}_2 &= -x_2 + x_2^2 - \tfrac{3x_2^{5/3}}{x_3^2} && = x_2\left(-1 + x_2 - \tfrac{3x_2^{2/3}}{x_3^2}\right) \\ \dot{x}_3 &= 2x_3 + 5x_1x_2^3 + x_2x_3 && = x_3\left(2 + \tfrac{5x_1x_2^3}{x_3^1} + x_2\right) \end{aligned}$$

then

$$\mathbf{B} = \begin{pmatrix} 0 & \frac{2}{3} & -2 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} -1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 1 & 5 \end{pmatrix}$$

$$\boldsymbol{\lambda} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Now we get the Lotka–Volterra form of the equations

$$\begin{aligned} \dot{y}_1 &= y_1\left(-\frac{14}{3} - 2y_1 - \frac{4}{3}y_2 - 10y_3\right) \\ \dot{y}_2 &= y_2\left(-1 - 3y_1 + y_2\right) \\ \dot{y}_3 &= y_3\left(-4 - 10y_1 + 2y_2 - 5y_3\right). \end{aligned}$$

We also mention that Brenig and Goriely [3] has carried out a systematic investigation how to find an appropriate $\mathbf{C}$ matrix, of which we also mention one below. Beklemisheva [1] treated the case when $\boldsymbol{\lambda} = \mathbf{0}$, and she always supposes that $\mathbf{B}$ is invertible.

C. The algebraic method applied to the Lotka–Volterra form

If our equation is of the Lotka–Volterra form, i.e. it has the form $\dot{x}_i = x_i \sum_{j=1}^n \alpha_{ij} + x_i\beta_i \quad (i = 1, 2, \dots, n)$, then we get an even simpler and more explicit form for the parameters to use the Getz–Jacobson result. However, the situation may still not be so simple as the following example shows. Beklemisheva [1] transformed an equation into the following Lotka–Volterra form:

$$\begin{aligned} \dot{y}_1 &= y_1(3 + y_1) \\ \dot{y}_2 &= y_2(9 + 2y_1) \\ \dot{y}_3 &= y_3(4 + y_1 - y_2 + 2y_3) \end{aligned}$$

Now

$$\mathbf{A} = \sum_{i=1}^{3} \omega_i \mathbf{A}_i = \frac{1}{2}\begin{pmatrix} 2\omega_1 & 2\omega_2 & 2\omega_3 \\ 2\omega_2 & 0 & -\omega_3 \\ \omega_3 & -\omega_3 & 4\omega_3 \end{pmatrix},$$

and the matrix is surely not positive definite, as it has a zero element on its main diagonal, whereas the system obviously blows up, see the first variable.

IV. Local analysis around time singularities

A. Decomposition of the right hand side into an appropriate sum

As a first step, we decompose the right hand side of the investigated differential equation $\dot{\mathbf{x}} = \mathbf{f} \circ \mathbf{x} \quad (\mathbf{f} \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n))$ into the sum $\mathbf{f} = \mathbf{g} + \mathbf{h}$ in such a way that for some $\boldsymbol{\alpha}, \mathbf{p} \in \mathbb{R}^n, \boldsymbol{\alpha} \neq \mathbf{0}$ the function $\tau \mapsto \boldsymbol{\alpha} \odot \tau^{\mathbf{p}}$ is the solution to the equation $\dot{\mathbf{x}} = \mathbf{g} \circ \mathbf{x}$ (what is equivalent to saying that $\mathbf{p} \odot \boldsymbol{\alpha} \odot \tau^{\mathbf{p-1}} = \mathbf{g}(\boldsymbol{\alpha} \odot \tau^{\mathbf{p}})$, and $\mathbf{h}$ is small in the sense that $\mathbf{h}(\tau^{\mathbf{p}}) = \varepsilon(\tau)\tau^{\mathbf{p-1}}$ with $\lim_{\tau \to 0} \varepsilon(\tau) = 0$. (Here $\odot$ denotes componentwise multiplication.)

The condition on $\mathbf{h}$ can also be expressed using componentwise division of vectors. The meaning of this condition is that $\mathbf{h}$ is not dominant in the neighborhood of the singularity belonging to/characterized by the pair of vectors $\boldsymbol{\alpha}, \mathbf{p}$.

General conditions to ensure neither the existence nor the uniqueness of such $\boldsymbol{\alpha}$ and $\mathbf{p}$ are known. However, in the case of polynomial equations the situation is clearer. In the one-dimensional case we have the following table.

p	α	$g(x)$	$h(x)$
1	g_0	g_0	$h_1 x + h_2 x^2 + \ldots$
-1	$-\frac{1}{g_2}$	$g_2 x^2$	$h_0 + h_1 x$
$-\frac{1}{2}$	$\pm\sqrt{-\frac{1}{2g_3}}$	$g_3 x^3$	$h_0 + h_1 x + h_2 x^2$
$\ldots$			
$-\frac{1}{k-1}$	$\left(-\frac{1}{g_k(k-1)}\right)^{\frac{1}{k-1}}$	$g_k x^k$	$\sum_{i=0}^{k-1} h_i x^i$

The meaning of the table is that there exists a unique decomposition with the given p and nonzero α if some natural conditions are fulfilled.

B. Companion system

Now using one of the vectors $\mathbf{p}$ above and the supposed value t^* of first infinity let us introduce a transformation changing the singularity of the original system into the stationary point of the companion system. Let us write down the equation for the new function

$$s \mapsto \mathbf{X}(s) := \begin{pmatrix} \mathbf{x}(t) \odot e^{-\mathbf{p}s} \\ e^{qs} \end{pmatrix} \in \mathbb{R}^{n+1},$$

the definition of which comes from $\mathbf{x}(t) = \tau^{\mathbf{p}} \odot \tilde{\mathbf{X}}(\log(\tau)) \quad \tau := t - t_*$, where $\tilde{\mathbf{X}}$ are the first n coordinate functions of $\mathbf{X}$, and the $1/q$ is the smallest integer such that $\mathbf{h}(t^{\mathbf{p}}\mathbf{x}) = t^{\mathbf{p}-1} \sum_{i=1}^{+\infty} t^{iq} \mathbf{f}^{(i)}(\mathbf{x})$.

For example, consider the following system:

$$\begin{aligned} \dot{x}_1 &= x_1^2 - 3x_1 \\ \dot{x}_2 &= 2x_1x_2 + 9x_1. \end{aligned}$$

Desomposition of the right hand side is

$$\mathbf{g}(x_1, x_2) := \begin{pmatrix} x_1^2 \\ 2x_1x_2 \end{pmatrix}, \quad \mathbf{h}(x_1, x_2) := \begin{pmatrix} -3x_1 \\ 9x_1 \end{pmatrix}.$$

The equations for the pair $(\boldsymbol{\alpha}, \mathbf{p})$ are

$$\begin{aligned} p_1 - 1 &= 2p_1 \\ p_2 - 1 &= p_1 + p_2 \\ p_1\alpha_1 &= \alpha_1^2 \\ p_2\alpha_2 &= 2\alpha_1\alpha_2. \end{aligned}$$

One possible solution is

$$\boldsymbol{\alpha} = (-1, 1), \quad \mathbf{p} = (-1, -2).$$

Furthermore, let us check if the condition holds for the $\mathbf{h}$ function:

$$\lim_{\tau \to 0} \frac{\mathbf{h}(\tau^{\mathbf{p}})}{\tau^{\mathbf{p}-1}} = \lim_{\tau \to 0} \begin{pmatrix} \dfrac{h(\tau^{p_1}, \tau^{p_2})}{\tau^{p_1-1}} \\ \dfrac{h(\tau^{p_1}, \tau^{p_2})}{\tau^{p_2-1}} \end{pmatrix} = \lim_{\tau \to 0} \begin{pmatrix} -3\tau \\ 9\tau^2 \end{pmatrix} = 0,$$

that is, the $\mathbf{h}$ function is the non-dominant part of $\mathbf{f}$ around the singularity determined by the $(\boldsymbol{\alpha}, \mathbf{p})$ pair.

Now transforming the equations, we obtain

$$\begin{aligned} x_1(t) &:= \tau^{-1} X_1(\log(\tau)) \\ x_2(t) &:= \tau^{-2} X_2(\log(\tau)), \end{aligned}$$

so thus

$$\begin{aligned} X_1' &= X_1 + X_1^2 - 3X_3X_1 \\ X_2' &= 2X_2 + 2X_1X_2 + 9X_1X_3^2 \\ X_3' &= X_3. \end{aligned}$$

Let us return to studying the campanion system. Now the companion system has at least two stationary points: $\mathbf{X}_0 := \mathbf{0}$, and $\mathbf{X}_* := \begin{pmatrix} \boldsymbol{\alpha} \\ 0 \end{pmatrix}$. The eigenvalues of the linearized system are $\begin{pmatrix} -\mathbf{p} \\ q \end{pmatrix}$ for $\mathbf{X}_0$, and $\begin{pmatrix} \boldsymbol{\varrho} \\ q \end{pmatrix}$ for $\mathbf{X}_*$. The components $\varrho_1, \varrho_2, \ldots, \varrho_n$ of $\boldsymbol{\varrho}$ are said to be the **Kovalevskaya exponents** [11] of the singularity.

The $\mathbf{X}_*$ fixed point of the companion system defines an unstable manifold $W_u(\mathbf{X}_*)$, which is the solution of the original system around a singularity, that is

$$\lim_{s\to-\infty} \mathbf{X}_u(s) = \mathbf{X}_* \Rightarrow \lim_{t\to t_*} \|x_u(t)\| = +\infty.$$

Now let us consider the unstable manifold $W_u(\mathbf{X}_*)$ of the fixed point $\mathbf{X}_*$ of the companion system

$$\mathbf{X}_u(s) = \mathbf{X}_* + \sum_{|i|=1}^{+\infty} \mathbf{c}_i(s) e^{\langle \boldsymbol{\varrho}^u, i\rangle s},$$

where $\mathrm{Re}(\varrho_j^u) > 0$, $j = k, \ldots, n+1$. Returning to the original variable:

$$\mathbf{x}_u(t) = \tau^{\mathbf{p}} \left(\boldsymbol{\alpha} + \sum_{|i|=1}^{+\infty} \mathbf{c}_i(\log(\tau)) \tau^{\langle \boldsymbol{\varrho}^u, i\rangle} \right).$$

This is the local series of the solution around the singularity determined by $(\boldsymbol{\alpha}, \mathbf{p})$, called **Psi-series**.

C. Linear right hand side

As a simple application of the methods above the well known fact that the solutions to linear differential equations do not blow up can be given a new proof. Let us consider the equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \quad (\mathbf{A} \in \mathbb{R}^{n\times n})$, and let us look for solutions in the form $\boldsymbol{\alpha} \odot \tau^{\mathbf{p}} := (\alpha_1 \tau^{p_1}, \alpha_1 \tau^{p_2}, \ldots, \alpha_1 \tau^{p_n})$ with $\tau := t - t_*$. A simple substitution $\mathbf{p} \odot \boldsymbol{\alpha} \odot \tau^{\mathbf{p}-\mathbf{1}} = \boldsymbol{\alpha} \odot \mathbf{A} \cdot \tau^{\mathbf{p}}$ shows that $\mathbf{p} = \mathbf{p} - \mathbf{1}$ should hold, an impossibility. Therefore no such solutions and no singularities exist, the solutions remain bounded on any finite interval.

This result can also be obtained using the classical Gronwall inequality, if one assumes that the right hand side $\mathbf{f}$ is linear, or even if one assumes that there exists a continuous function $k : I \to \mathbb{R}$ such that for all $(t, \mathbf{p}) \in \mathcal{D}_{\mathbf{f}} := I \times \mathbb{R}^n \quad \mathbf{p}^\top \mathbf{f}(t, \mathbf{p}) \leq k(t)|\mathbf{p}|^2$ holds.

V. Classical approaches

A. Global Lipschitz property

The simplest criterion is that the right hand has the **global** or uniform Lipschitz property: this—through the Gronwall lemma—obviously implies global existence. However, only polynomials of the not more than first degree fulfil the global Lipschitz property. Thus, again we have the trivial result for linear differential equations.

B. The existence of linear first integrals: mass conservation

The existence of first integrals with nonnegative components can easily be used to exclude blow up in kinetic differential equations.

Suppose we have a mass conserving reaction endowed with mass action type kinetics. Then, no solution of the induced kinetic differential equation with nonnegative initial condition blows up.

The proof of this statement is based upon a theorem by Volpert stating that the solutions are always nonnegative, and will be given in the lecture.

Let us remark in passing that the existence of a positive linear first integral does not exclude blow up, if the equation is not kinetic, see the equation $\dot{x} = x^2 \quad \dot{y} = -x^2$.

It may also happen that the phase volume decreases while the solution blows up at it can be seen from the **kinetic** example $\dot{x} = x^2 \quad \dot{y} = -x^3 y$.

C. Zero deficiency theorem and Volpert's results

The exclusion of blow up is the by-product of some very general statements on the dynamics of chemical reactions. The proof of these statements is based upon a Lyapunov function of the entropy form, see the references by Feinberg, Horn, Jackson and Volpert in [8].

(Feinberg, Horn, Jackson) The solutions to the induced kinetic differential equations of a weakly reversible reaction of the zero deficiency endowed with mass action type kinetics do not blow up.

(Volpert) The solutions to the induced kinetic differential equations of a reaction with acyclic Volpert graph endowed with mass action type kinetics do not blow up.

VI. An application: The simplest Volterra–Lotka model

Let us consider the simplest Volterra–Lotka reaction $A + X \longrightarrow 2X, \quad X + Y \longrightarrow 2Y, \quad Y \longrightarrow B,$ with many chemical, biological and economic applications having the induced kinetic differential equation $\dot{x} = k_1 x - k_2 xy \quad \dot{y} = k_2 xy - k_3 y$. (Here we assume that the concentration of A can be supposed to be constant.) The algebraic method does not work here, because $\mathbf{A}$ can always seen to be indefinite. Therefore we try to find the formal solution in the form of a Ψ-series. The only decomposition to provide an exact solution to the truncated system is obtained from

$$g(x,y) := \begin{pmatrix} -k_2xy \\ k_2xy \end{pmatrix} \quad h(x,y) := \begin{pmatrix} k_1x \\ -k_3y \end{pmatrix},$$

for which one has $\boldsymbol{\alpha} = \begin{pmatrix} -\frac{1}{k_2} & \frac{1}{k_2} \end{pmatrix}$ $\mathbf{p} = \begin{pmatrix} -1 & -1 \end{pmatrix}$, and the Kovalevskaya exponents are $\varrho = -1, \varrho = 1$. As $n-1=1$ of them is positive, therefore there exists an open set of initial values from which the solutions blow up and as $\mathrm{sign}(\beta) = (1,-1)$, the solution blows up for solutions with $x_0 \in \mathbb{R}^+, y_0 \in \mathbb{R}^-$ [12]. Although this is not relevant from the point of view of applications, the solutions do not blow up starting form nonnegative initial values, as it can be seen using an appropriate Lyapunov function, the result is interesting property of the model in itself. The blow up time has been shown to be around $t_* = 0.9656$ if $k_1 = k_2 = k_3 = 1, x_0 = 1, y_0 = -1$ by numerical calculations.

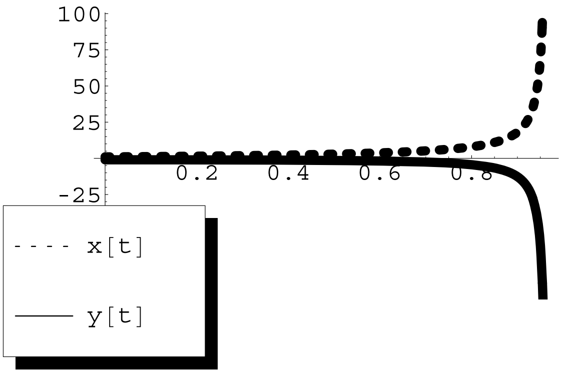

Fig. 2. Blow up in the Volterra–Lotka model

If we have $x_0 \in \mathbb{R}^-, y_0 \in \mathbb{R}^+$, then the solution blows up 'backwards': for negative times.

VII. Codes

The *Mathematica* code for the calculation of the the quasi-monomial transformation, for obtaining a solution in Taylor form, and for the calculation of the decompositions and the possible values of $\boldsymbol{\alpha}$ and $\mathbf{p}$ can be found on the home page [6].

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