

# Solving of the Fourth Order Differential Equations with the Neumann Problem

Marziyeh Halimi, Roushanak Lotfekar, Simin Mansouri Borojeni

**Abstract**—In this paper we considered the Neumann problem for the fourth order differential equation. First we define the weighted Sobolev space  $W_\alpha^2$  and generalized solution for this equation. Then we consider the existence and uniqueness of the generalized solution, as well as give the description of the spectrum and of the domain of definition of the corresponding operator.

**Keywords**—Neumann problem, weighted Sobolev spaces, generalized solution, spectrum of linear operators. 2000 mathematic subject classification: 34A05, 34A30.

## I. INTRODUCTION

IN the paper we consider the Neumann problem for the following ordinary differential equation of the fourth order

$$Lu \equiv (t^\alpha u''')' + au = f, \quad (1)$$

Where  $0 \leq \alpha \leq 4$ ,  $t \in [0, b]$ ,  $f \in L_2(0, b)$ ,  $a = \text{const}$ .

We define the weighted Sobolev space  $W_\alpha^2$  and consider the behavior of the functions from  $W_\alpha^2$  near to  $t = 0$ . Then we define the generalized solution for the Neumann problem for the equation (1) and under some conditions on  $a$  prove that it exists and is unique, as well as give the description of the spectrum of the operator  $L$  and of the domain of definition  $D(L)$ . The problem (1) for the degenerate operator  $L$  was considered by A. A. Dezin [1,2].

Note that the Dirichlet problem for the degenerate Ordinary differential equations of the second and fourth order have been considered in [3] and for the higher order in [4].

## II. THE SPACE $W_\alpha^2$

Let  $\alpha \geq 0$  and  $t$  belong to the finite interval  $(0, b)$ .

Consider the set  $W_\alpha^2$  of the functions  $u(t)$ , which have the generalized derivative of the second order, such that the following semi-norm ;

$$\|u\|_1 = \int_0^b t^\alpha |u''(t)|^2 dt$$

Marziyeh Halimi is a Faculty Member of Ilam Branch, Islamic Azad University, Iran.

Roushanak Lotfekar is a Faculty Member of Islamic Azad University, Ilam Branch, Iran. Corresponding author of this Paper ( Phone: 00989188410036; email: rlotfekar@yahoo.com)

Simin Mansouri Borojeni, Payam Noor Iran

Is finite. First note, that for the functions  $u \in W_\alpha^2$  for every  $t_0 \in (0, b]$  exist finite values  $u(t_0)$  and  $u'(t_0)$  (see [4], [8]). Now we clear the behavior of the functions  $u(t)$  and  $u'(t)$  near to  $t = 0$ .

*Proposition 1.* For  $u \in W_\alpha^2$  we have the following estimates

$$|u(t)|^2 \leq (c_1 + c_2 t^{3-\alpha}) \|u\|_1^2, \quad \alpha \neq 1, \alpha \neq 3, \quad (2)$$

$$|u'(t)|^2 \leq (c_1 + c_2 t^{1-\alpha}) \|u\|_1^2, \quad \alpha \neq 1. \quad (3)$$

For  $\alpha = 3$  we replace in (2)  $t^{3-\alpha}$  by  $|\ln t|$  and for  $\alpha = 1$  we replace  $t^{3-\alpha}$  by  $t^2 |\ln t|$  in (2) and  $t^{1-\alpha}$  by  $|\ln t|$  in (3). For the proof note that we use the same way as in [3] and [5]. Namely we use the following representation

$$u(t) = u(t_0) + \int_{t_0}^t u'(\tau) d\tau$$

And apply the Cauchy inequality.

From Proposition 1 it follows that for  $0 \leq \alpha < 1$  (weak degeneration) the values  $u(0)$  and  $u'(0)$  are finite, for  $1 \leq \alpha < 3$  only  $u(0)$  is finite while for  $\alpha \geq 3$  both  $u(0)$  and  $u'(0)$  can be infinite.

From the inequality (2) for  $0 \leq \alpha < 4$  we get the inequality

$$\|u\|_{L_2(0,b)} \leq c \|u\|_1, \quad (4)$$

i.e. we have the following embedding

$$W_\alpha^2 \subset L_2(0, b). \quad (5)$$

The embedding (5) is true also for  $\alpha = 4$ . Indeed, using Hardy's inequality (see [6]) we get

$$\int_0^b |u(t)|^2 dt = \int_0^b \left| u(b) + \int_t^b u'(\tau) d\tau \right|^2 dt \leq c \int_0^b t^2 |u'(t)|^2 dt,$$

$$\int_0^b t^2 |u'(t)|^2 dt = \int_0^b \left| u'(b) + \int_t^b u''(\tau) d\tau \right|^2 dt \leq (c_1 + c_2) \int_0^b t^4 |u''(t)|^2 dt.$$

The embedding (5) for  $\alpha > 4$  fails. Indeed, the function  $u(t) = t^{-1/2}$  belong to  $W_\alpha^2$  for  $\alpha > 4$ , but  $u \notin L_2(0, b)$ .

Therefore, to remain in  $L_2(0, b)$ , in the following we assume that  $0 \leq \alpha \leq 4$ .

Now we can define the following norm in  $W_\alpha^2$

$$\|u\|_{W_\alpha^2}^2 = \int_0^b \left( t^\alpha |u''(t)|^2 + |u(t)|^2 \right) dt. \tag{6}$$

The space  $W_\alpha^2$  is Hilbert space with the scalar product

$$(u, v)_\alpha = (t^\alpha u'', v'') + (u, v),$$

where  $(\cdot, \cdot)$  stands for the scalar product in  $L_2(0, b)$ .

It is evident that for  $0 \leq \alpha \leq 4$  we have the following inequality

$$\|u\|_{L_2(0,b)} \leq c \|u\|_{W_\alpha^2}. \tag{7}$$

*Proposition 2.* The embedding (5) for  $0 \leq \alpha < 4$  is compact. Indeed, using inequality (3) we get

$$\begin{aligned} \|u(t+h) - u(t)\|_{L_2(0,b)}^2 &= \int_0^b |u(t+h) - u(t)|^2 dt = \int_0^b \left| \int_t^{t+h} u'(\tau) d\tau \right|^2 dt \leq \\ &\leq \int_0^b \int_t^{t+h} \left( \sqrt{c_1} + c_3 t^{\frac{1-\alpha}{2}} \right) d\tau dt \cdot \|u\|_1^2 = \int_0^b \left( \sqrt{c_1} h + c_4 \left( (t+h)^{\frac{3-\alpha}{2}} - t^{\frac{3-\alpha}{2}} \right) \right)^2 dt \cdot \|u\|_1^2 \leq \\ &\leq \left( c_5 |h| + 2c_4 \int_0^b \left( (t+h)^{\frac{3-\alpha}{2}} - t^{\frac{3-\alpha}{2}} \right)^2 dt \right) \cdot \|u\|_1^2 \leq c |h|^\beta \|u\|_1^2, \end{aligned}$$

i.e. we have the following inequality

$$\|u(t+h) - u(t)\|_{L_2(0,b)} \leq c \|u\|_{W_\alpha^2}.$$

The result now follows from the precompactness criterion in  $L_2(0, b)$ .

Note also that for  $\alpha = 4$  the continuous embedding (5) is not compact (see [1]).

### III. THE NEUMANN PROBLEM

*Definition 1.* The function  $u \in W_\alpha^2$  is called the generalized solution of the Neumann problem for the equation (1), if for every  $v \in W_\alpha^2$  we have the equality

$$(t^\alpha u'', v'') + a(u, v) = (f, v). \tag{8}$$

Note that, if the generalized solution  $u \in W_\alpha^2$  is classical,

then we get for  $\alpha = 0$  the following conditions (see [7])

$$u''(0) = u'''(0) = u''(b) = u'''(b) = 0.$$

Consider the particular case of the equation (1) when  $a = 1$ .

$$Bu \equiv (t^\alpha u'')'' + u = f. \tag{9}$$

*Proposition 3.* For every  $f \in L_2(0, b)$  the generalized solution of the Neumann problem for the equation (9) exists and is unique.

Proof. Uniqueness of the generalized solution for the equation (9) immediately follows from the equality (8) (with  $a = 1$ ), if we put  $f = 0$  and  $v = u$ . To prove the existence define the functional  $l_f(v) = (f, v)$ ,  $f \in L_2(0, b)$  over the space  $W_\alpha^2$ . Using the inequality (7) we get

$$|l_f(v)|^2 = \left| \int_0^b f(t) \overline{v(t)} dt \right|^2 \leq \|f\|_{L_2(0,b)}^2 \|v\|_{L_2(0,b)}^2 \leq c \|f\|_{L_2(0,b)}^2 \|v\|_{W_\alpha^2}^2$$

I.e.  $l_f(v)$  is a linear continuous functional over the space  $W_\alpha^2$ . Using Riesz lemma on representation we get  $l_f(v) = (u_0, v)_\alpha$ ,  $u_0 \in W_\alpha^2$ . Therefore the function  $u_0$  is the generalized solution for the equation (9) (see [1]).

Define the operator  $B : L_2(0, b) \rightarrow L_2(0, b)$  corresponding to the Definition 1.

*Definition 2.* We say that the function  $u \in W_\alpha^2$  belong to the domain of the definition  $D(B)$  of the operator  $B$ , if exists  $f \in L_2(0, b)$  such that is valid the equality (8) and then we write  $Bu = f$ .

*Theorem 1.* The operator  $B : L_2(0, b) \rightarrow L_2(0, b)$  is positive and selfadjoint. The bounded operator  $B^{-1} : L_2(0, b) \rightarrow L_2(0, b)$  for  $0 \leq \alpha < 4$  is compact.

Proof. The symmetry and positive ness of the operator  $B$  is a direct consequence of the Definition 2. The coincidence of  $D(B)$  and  $D(B^*)$  ( $B^*$  is the adjoints to the  $B$  operator) Follows from the existence of a generalized solution of (9) for every  $f \in L_2(0, b)$  (see Proposition 3). We note that Definition 2 implies the inequality

$$\|u\|_{W_\alpha^2} \leq c \|Bu\|_{L_2(0,b)}.$$

The compactness of the operator  $B^{-1}$  for  $0 \leq \alpha < 4$  now follows from the Proposition 2.

*Corollary.* For  $0 \leq \alpha < 4$  the operator  $B$  has discrete spectrum, and its eigenfunction system is complete in  $f \in L_2(0, b)$  (see [2]).

Note that now we can rewrite the equation (1) in the form  $Bu = (1-a)u + f$ ,

i.e. we can regard the number  $1-a$  as a spectral parameter.

*Theorem 2.* The domain of definition of the operator  $L$  consists of the functions  $u(t)$  for which  $u(0)$  is finite

when  $0 \leq \alpha < \frac{7}{2}$  and  $u'(0)$  is finite for  $0 \leq \alpha < 2$ . The

values  $u(0)$  and  $u'(0)$  cannot be specified arbitrarily, but are determined by the right-hand side of (1).

Proof. Since  $D(L) = D(L - aI)$ , it is sufficient to consider properties of the Neumann problem for the equation

$$(t^\alpha u'')'' = f. \quad (10)$$

Let  $1 \leq \alpha < 2$ . The derivative of the general solution of the equation (10) has the following form

$$u'(t) = c_1 + c_2 t^{2-\alpha} + \int_0^t \tau^{-\alpha} \int_0^\tau (\tau - \eta) f(\eta) d\eta d\tau.$$

We have

$$\left| \int_0^t \tau^{-\alpha} \int_0^\tau (\tau - \eta) f(\eta) d\eta d\tau \right| \leq c t^{\frac{5}{2}-\alpha} \|f\|_{L_2(0,b)}.$$

For  $\alpha \geq 2$  the value  $u'(0)$  can in general be infinite. Let now  $3 \leq \alpha < \frac{7}{2}$ . Then we can write the solution in the following form

$$u(t) = c_3 + c_4 t - \int_0^t \int_\tau^b \eta^{-\alpha} \int_0^\eta (\eta - \xi) f(\xi) d\xi d\eta d\tau.$$

Then we have

$$\left| \int_0^t \int_\tau^b \eta^{-\alpha} \int_0^\eta (\eta - \xi) f(\xi) d\xi d\eta d\tau \right| \leq c \left| t^{\frac{7}{2}-\alpha} - \left( \frac{7}{2} - \alpha \right) b^{\frac{5}{2}-\alpha} t \right| \|f\|_{L_2(0,b)},$$

Which completes the proof.

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