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Solving Inhomogeneous Wave Equation Cauchy Problems using Homotopy Perturbation Method

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Abstract**—**In this paper, He's homotopy perturbation method (HPM) is applied to spatial one and three spatial dimensional inhomogeneous wave equation Cauchy problems for obtaining exact solutions. HPM is used for analytic handling of these equations. The results reveal that the HPM is a very effective, convenient and quite accurate to such types of partial differential equations (PDEs).

Keywords**—**Homotopy perturbation method; Exact solution; Cauchy problem; inhomogeneous wave equation.

I. Introduction

THE homotopy perturbation method (HPM) was firstly proposed by He [1-4]. The HPM deforms a difficult problem into simple problems which can be easily solved. In [3], a comparison of HPM and homotopy analysis method was made, revealing that the former is more powerful than the latter. The method gives rapidly convergent series of the exact solution if such a solution exists. Recently, many authors applied the HPM to various problems and demonstrated the efficiency of the HPM to handle nonlinear structure and solve various physics and engineering problems [5-8].

The study of various types of waves – elastic, acoustic, and electromagnetic – and other oscillational phenomena leads to the wave equation. It arises in the study of many important physical problems involving wave propagation, such as the transverse vibrations of an elastic string and the longitudinal or torsional oscillations of a rod [9].

The main purpose of this paper is to apply the HPM to Cauchy problem of some inhomogeneous wave equations for establishing closed form exact solutions and proving that the HPM is a very efficient, suitable, quite accurate and simple to such types of hyperbolic differential equations Cauchy problem.

II. Basic idea of He's homotopy perturbation method

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation [1]:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{1}$$

with the boundary conditions

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma, \tag{2}$$

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where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω.

Generally speaking, the operator A can be divided into two parts which are L and N, where L is linear, but N is nonlinear. Eq. (1) can therefore be rewritten as follows:

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

By the homotopy technique, we construct a homotopy $V(r,p): \Omega \times [0,1] \to \mathbb{R}$ which satisfies:

$$H(V,p) = (1-p)\left[L(V) - L(u_0)\right] + p\left[A(V) - f(r)\right] = 0, \quad p \in [0,1], \; r \in \Omega, \tag{4}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of Eq. (1), which satisfies the boundary conditions.

Obviously, from Eq. (4) we will have:

$$H(V,0) = L(V) - L(u_0) = 0, \tag{5}$$

$$H(V,1) = A(V) - f(r) = 0, \tag{6}$$

the changing process of p from zero to unity is just that of $V(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(V) - L(u_0)$ and $A(V) - f(r)$ are called homotopic.

According to the HPM, we can first use the embedding parameter p as a "small parameter", and assume that the solution of Eq. (4) can be written as a power series in p,

$$V = V_0 + p\,V_1 + p^2\,V_2 + \ldots. \tag{7}$$

Setting $p = 1$ results in the approximate solution of Eq. (1):

$$u = \lim_{p \to 1} V = V_0 + V_1 + V_2 + \ldots. \tag{8}$$

The series in Eq. (8) is convergent for most cases, and also the rate of convergent depends on the nonlinear operator $A(V)$ [1].

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method (HPM), which has eliminated the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantage of the traditional perturbation techniques.

Remark: We noticed that use of the following modified equation of the homotopy $V(r,p)$,

$$\begin{aligned} H(V,p) &= (1-p)\left[L(V)-L(u_0)\right] \\ &+ p\left[A(V)\right]-f(r)=0, \quad p\in[0,1], \ r\in\Omega, \end{aligned} \tag{9}$$

increases the convergence of the one-term approximation ($u_1 = V_0 + V_1$) to the exact solution rather than use of the homotopy equation described in (4).

III. Applications

The Cauchy problem of the inhomogeneous wave equation reads [9],

$$\frac{\partial^2 u(r,t)}{\partial t^2} - a^2\nabla^2 u(r,t) = f(r,t), \tag{10}$$

$$u(r,0)=g_1(r), \qquad \frac{\partial u(r,0)}{\partial t}=g_2(r), \tag{11}$$

where a is a real constant, $\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is the Laplace's operator.

According to Eq. (9), a homotopy $V(r,t,p):\Omega\times[0,1]\to\mathbb{R}$ can be constructed as follows:

$$\begin{aligned} &(1-p)\left(V_{tt}-u_{0,tt}\right)+p\left(V_{tt}-a^2\nabla^2 V\right) \\ &-f(r,t)=0, \quad p\in[0,1], \quad (r,t)\in\Omega, \end{aligned} \tag{12}$$

where $u_0 = V_0(r,0)=u(r,0)$ and $u_{0,tt}=\frac{\partial^2 u_0}{\partial t^2}$.

Suppose the solution of Eq. (12) in the form of:

$$V(r,t)=V_0(r,t)+p\,V_1(r,t)+p^2\,V_2(r,t)+\ldots. \tag{13}$$

Substituting Eq. (13) into Eq. (12), and equating the terms with the identical powers of p, yields,

$$\begin{aligned} p^0:&\quad V_{0,tt}-f(r,t)=0, \\ p^1:&\quad V_{1,tt}-a^2\nabla^2 V_{0,tt}=0, \\ p^2:&\quad V_{2,tt}-a^2\nabla^2 V_{1,tt}=0, \\ &\vdots \\ p^n:&\quad V_{n,tt}-a^2\nabla^2 V_{n-1,tt}=0, \quad n=3,4,5,..., \end{aligned} \tag{14}$$

with the following initial conditions:

$$\begin{aligned} V_i(r,0)&=\begin{cases} g_1(\mathrm{r}), & i=0, \\ 0, & i=1,2,..., \end{cases} \\ V_{i,t}(r,0)&=\begin{cases} g_2(\mathrm{r}), & i=0, \\ 0, & i=1,2,.... \end{cases} \end{aligned} \tag{15}$$

A) One spatial dimensional wave equation

Example 1. Firstly, we consider the following wave equation,

$$\begin{aligned} &u_{tt}-a^2u_{xx}=(\gamma+\delta x)h(t), \\ &u(x,0)=\alpha+\beta x, \quad u_t(x,0)=m\sin(nx), \end{aligned} \tag{16}$$

where α, β, γ, δ, m and n are real constants and $h(t)$ is any integrable function of t.

Solving the system (14), with the initial conditions (15) for $f=(\gamma+\delta x)h(t)$, $g_1=\alpha+\beta x$ and $g_2=m\sin(nx)$, yields:

$$\begin{aligned} V_0(x,t)&=\alpha+\beta x+m\sin(nx)t \\ &+(\gamma+\delta x)\left\{\iint h(t)\mathrm{d}t\mathrm{d}t-\left[\iint h(t)\mathrm{d}t\mathrm{d}t\right]_{t=0}-t\left[\int h(t)\mathrm{d}t\right]_{t=0}\right\}, \\ V_1(x,t)&=-\frac{1}{6}a^2n^2m\sin(nx)t^3, \\ V_2(x,t)&=\frac{1}{120}a^4n^4m\sin(nx)t^5, \\ V_3(x,t)&=-\frac{1}{5040}a^6n^6m\sin(nx)t^7, \\ V_k(x,t)&=(-1)^k\frac{1}{(2k+1)!}a^{2k}n^{2k}m\sin(nx)t^{2k+1}, \\ &k=4,\ 5,\ 6.... \end{aligned} \tag{17}$$

Substituting Eq. (17) into Eq. (8) yields,

$$\begin{aligned} u(x,t)&=\alpha+\beta x+m\sin(nx)\{t-\frac{1}{6}a^2n^2t^3+\frac{1}{120}a^4n^4t^5 \\ &-\frac{1}{5040}a^6n^6t^7+\sum_{k=4}^{\infty}(-1)^k\frac{1}{(2k+1)!}a^{2k}n^{2k}t^{2k+1}\} \\ &+(\gamma+\delta x)\{\iint h(t)\mathrm{d}t\mathrm{d}t-\left[\iint h(t)\mathrm{d}t\mathrm{d}t\right]_{t=0}-t\left[\int h(t)\mathrm{d}t\right]_{t=0}\}. \end{aligned} \tag{18}$$

Consequently, the exact solution of Eq. (16)

$$\begin{aligned} u(x,t)&=\alpha+\beta x+\frac{m}{an}\sin(nx)\sin(ant) \\ &+(\gamma+\delta x)\{\iint h(t)\mathrm{d}t\mathrm{d}t-\left[\iint h(t)\mathrm{d}t\mathrm{d}t\right]_{t=0}-t\left[\int h(t)\mathrm{d}t\right]_{t=0}\}, \end{aligned} \tag{19}$$

is readily obtained upon using the Taylor series expansion of $\sin(ant)$.

Example 2. Secondly, consider the following wave equation,

$$\begin{aligned} &u_{tt}-u_{xx}=p\,e^{qt}, \quad u(x,0)=\alpha\sin(mx), \\ &u_t(x,0)=\beta\cos(nx), \end{aligned} \tag{20}$$

where α, β, m, n, p and q are real constants.

The solution of the system (14), with the initial conditions (15) for $a=1$, $f=pe^{qt}$, $g_1=\alpha\sin(mx)$ and $g_2=\beta\cos(nx)$, gives:

$$V_0(x,t)=\frac{p}{q^2}\left(e^{qt}-qt-1\right)+\beta\cos(nx)t+\alpha\sin(mx),$$

$$V_1(x,t)=-\frac{1}{6}\beta n^2\cos(nx)t^3-\frac{1}{2}\alpha m^2\sin(mx)t^2,$$

$$V_2(x,t)=\frac{1}{120}\beta n^4\cos(nx)t^5+\frac{1}{24}\alpha m^4\sin(mx)t^4,$$

$$V_3(x,t)=-\frac{1}{5040}\beta n^6\cos(nx)t^7-\frac{1}{720}\alpha m^6\sin(mx)t^6,$$

$$V_k(x,t)=\frac{(-1)^k\beta}{(2k+1)!}n^{2k}\cos(nx)t^{2k+1}+\frac{(-1)^k\alpha}{(2k)!}m^{2k}\sin(mx)t^{2k},\quad k=4,\ 5,\ 6.... \tag{21}$$

Substituting Eq. (21) into Eq. (8) yields,

$$u(x,t)=\frac{p}{q^2}\left(e^{qt}-qt-1\right)+\beta\cos(nx)\{t-\frac{1}{6}n^2t^3+\frac{1}{120}n^4t^5-\frac{1}{5040}n^6t^7+\sum_{k=4}^{\infty}(-1)^k\frac{1}{(2k+1)!}n^{2k}t^{2k+1}\}+\alpha\sin(mx)\{1-\frac{1}{2}m^2t^2+\frac{1}{24}m^4t^4-\frac{1}{720}m^6t^6+\sum_{k=4}^{\infty}(-1)^k\frac{1}{(2k)!}m^{2k}t^{2k}\}. \tag{22}$$

Consequently, the exact solution of Eq. (20)

$$u(x,t)=\frac{p}{q^2}\left(e^{qt}-qt-1\right)+\frac{\beta}{n}\cos(nx)\sin(nt)+\alpha\sin(mx)\cos(mt), \tag{23}$$

follows immediately upon using the Taylor series expansions of $\sin(nt)$ and $\cos(mt)$.

B) Three spatial dimensional wave equation

Example 3. Consider the following wave equation,

$$u_{tt}-a^2\nabla^2u(x,y,z,t)=\frac{byzt^3}{1+t^2},\quad u(x,y,z,0)=\alpha x\,e^{\beta y},\ u_t(x,y,z,0)=\gamma y\,e^{\delta z}, \tag{24}$$

where $\alpha,\ \beta,\ \gamma,\ \delta$ and b are real constants.

Solving the system (14), with the initial conditions (15) for $f=\frac{byzt^3}{1+t^2}$, $g_1=\alpha x\,e^{\beta y}$ and $g_2=\gamma y\,e^{\delta z}$, yields:

$$V_0(r,t)=byz\left(t+\frac{1}{6}t^3-\frac{1}{2}t\ln\left(1+t^2\right)-\tan^{-1}(t)\right)+\gamma y\,e^{\delta z}t+\alpha x\,e^{\beta y},$$

$$V_1(r,t)=\alpha\left(\frac{1}{2}a^2\beta^2x\,e^{\beta y}t^2\right)+\gamma\left(\frac{1}{6}a^2\delta^2y\,e^{\delta z}t^3\right),$$

$$V_2(r,t)=\alpha\left(\frac{1}{24}a^4\beta^4x\,e^{\beta y}t^4\right)+\gamma\left(\frac{1}{120}a^4\delta^4y\,e^{\delta z}t^5\right),$$

$$V_3(r,t)=\alpha\left(\frac{1}{720}a^6\beta^6x\,e^{\beta y}t^6\right)+\gamma\left(\frac{1}{5040}a^6\delta^6y\,e^{\delta z}t^7\right),$$

$$V_k(r,t)=\alpha\left(\frac{1}{(2k)!}a^{2k}\beta^{2k}x\,e^{\beta y}t^{2k}\right)+\gamma\left(\frac{1}{(2k+1)!}a^{2k}\delta^{2k}y\,e^{\delta z}t^{2k+1}\right),\quad k=4,\ 5,\ 6.... \tag{25}$$

Substituting Eq. (25) into Eq. (8) yields,

$$u(r,t)=byz\left(t+\frac{1}{6}t^3-\frac{1}{2}t\ln\left(1+t^2\right)-\tan^{-1}(t)\right)+\alpha x\,e^{\beta y}\left\{1+\frac{1}{2}a^2\beta^2t^2+\frac{1}{24}a^4\beta^4t^4+\sum_{k=3}^{\infty}\frac{1}{(2k)!}a^{2k}\beta^{2k}t^{2k}\right\}+\gamma y\,e^{\delta z}\left\{t+\frac{1}{6}a^2\delta^2t^3+\frac{1}{120}a^4\delta^4t^5+\sum_{k=3}^{\infty}\frac{1}{(2k+1)!}a^{2k}\delta^{2k}t^{2k+1}\right\}. \tag{26}$$

Consequently, the exact solution of Eq. (24)

$$u(r,t)=byz\left(t+\frac{1}{6}t^3-\frac{1}{2}t\ln\left(1+t^2\right)-\tan^{-1}(t)\right)+\alpha x\,e^{\beta y}\cosh(a\beta t)+\frac{\gamma}{a\delta}y\,e^{\delta z}\sinh(a\delta t), \tag{27}$$

is readily obtained upon using the Taylor series expansion of $\cosh(a\beta t)$ and $\sinh(a\delta t)$.

Example 4. Consider the following wave equation,

$$u_{tt}-a^2\nabla^2u(x,y,z,t)=\alpha xyz\sin(\beta t),\quad u(r,0)=\gamma x^2yz^2,\quad u_t(r,0)=\delta y\sin(mx)e^{mz}, \tag{28}$$

where $\alpha,\ \beta,\ \gamma,\ \delta$ and m are real constants.

The solution of the system (14), with the initial conditions (15) for $f=\alpha xyz\sin(\beta t)$, $g_1=\gamma x^2yz^2$ and $g_2=\delta y\sin(mx)e^{mz}$, gives:

$$V_0(r,t) = \gamma x^2 y z^2 - \frac{\alpha}{\beta^2} xyz \sin(\beta t) + \frac{\alpha}{\beta} xyz\, t + \delta y \sin(mx) e^{mz}\, t,$$
$$V_1(r,t) = a^2 \gamma y (x^2 + z^2) t^2,$$
$$V_2(r,t) = \frac{1}{3} a^4 \gamma y\, t^4,$$
$$V_3(r,t) = 0,$$
$$V_k(r,t) = 0 \quad ,k = 4,\ 5,\ 6.... \tag{29}$$

Substituting Eq. (29) into Eq. (8) yields,

$$u(r,t) = \gamma x^2 y z^2 - \frac{\alpha}{\beta^2} xyz \sin(\beta t) + \frac{\alpha}{\beta} xyz\, t + \delta y \sin(mx) e^{mz}\, t + a^2 \gamma y (x^2 + z^2) t^2 + \frac{1}{3} a^4 \gamma y\, t^4, \tag{30}$$

which is the exact solution of Eq. (28).

Example 5. Consider the following wave equation,

$$u_{tt} - \nabla^2 u(x,y,z,t) = \alpha xyz\, e^{-\beta t},$$
$$u(r,0) = 2xy, \quad u_t(r,0) = x \sin(\sqrt{2}y) \cos(\sqrt{2}z), \tag{31}$$

where α, β, γ, δ and b are real constants.

Solving the system (14), with the initial conditions (15) for $a = 1$, $f = \alpha xyz\, e^{-\beta t}$, $g_1 = 2xy$ and $g_2 = x \sin(\sqrt{2}y)\cos(\sqrt{2}z)$, yields:

$$V_0(r,t) = 2xy + \frac{\alpha}{\beta} xyz\, t + x\, t \sin(\sqrt{2}y) \cos(\sqrt{2}z) + \frac{\alpha}{\beta^2} xyz \left(e^{-\beta t} - 1\right),$$
$$V_1(r,t) = -\frac{2}{3} x\, t^3 \sin(\sqrt{2}y) \cos(\sqrt{2}z),$$
$$V_2(r,t) = \frac{2}{15} x\, t^5 \sin(\sqrt{2}y) \cos(\sqrt{2}z),$$
$$V_3(r,t) = -\frac{4}{315} x\, t^7 \sin(\sqrt{2}y) \cos(\sqrt{2}z),$$
$$V_4(r,t) = \frac{2}{2835} x\, t^9 \sin(\sqrt{2}y) \cos(\sqrt{2}z),$$
$$\vdots \tag{32}$$

Substituting Eq. (32) into Eq. (8) yields,

$$u(r,t) = 2xy + \frac{\alpha}{\beta^2} xyz \left(e^{-\beta t} + \beta t - 1\right) + \frac{1}{2} x \sin(\sqrt{2}y) \cos(\sqrt{2}z) \left\{2t - \frac{4}{3} t^3 + \frac{4}{15} t^5 - \frac{8}{315} t^7 + \frac{4}{2835} t^9 + ...\right\}, \tag{33}$$

Consequently, the exact solution of Eq. (31)

$$u(r,t) = 2xy + \frac{\alpha}{\beta^2} xyz \left(e^{-\beta t} + \beta t - 1\right) + \frac{1}{2} x \sin(\sqrt{2}y) \cos(\sqrt{2}z) \sin(2t), \tag{34}$$

is readily obtained upon using the Taylor series expansion of $\sin(2t)$.

IV. Conclusions

A clear conclusion can be draw from our results that the homotopy perturbation method provides fast convergence series to exact solutions. It is also worth noting that the HPM is an effective, simple and quite accurate tool for handling and solving inhomogeneous wave equations and other hyperbolic-type PDEs. The various applications of He's homotopy perturbation method prove that it's an efficient method to handle various types of differential equations. It's predicted that the HPM will be found widely applications in science and engineering.

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