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Lagrange and Multilevel Wavelet-Galerkin with Polynomial Time Basis for Heat Equation

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Abstract**—**The Wavelet-Galerkin finite element method for solving the one-dimensional heat equation is presented in this work. Two types of basis functions which are the Lagrange and multi-level wavelet bases are employed to derive the full form of matrix system. We consider both linear and quadratic bases in the Galerkin method. Time derivative is approximated by polynomial time basis that provides easily extend the order of approximation in time space. Our numerical results show that the rate of convergences for the linear Lagrange and the linear wavelet bases are the same and in order 2 while the rate of convergences for the quadratic Lagrange and the quadratic wavelet bases are approximately in order 4. It also reveals that the wavelet basis provides an easy treatment to improve numerical resolutions that can be done by increasing just its desired levels in the multilevel construction process.

Keywords**—**Galerkin finite element method, Heat equation , Lagrange basis function, Wavelet basis function.

I. Introduction

THE Galerkin approach is one of the very successful methods for finding approximate solutions from the partial differential equation. The main concept is using an appropriate basis function for the solution space of the governing equation, and then projecting the terms of approximate solution on the functional basis space. This process provides residual that needed to be minimized with respect to the functional basis. By this concept, the accuracy of numerical solutions depends directly on the type of basis function.

In this work, we apply the Galerkin method with wavelet bases called the Wavelet-Galerkin method to solve numerically the linear one-dimensional heat equation. Wavelets in our consideration are compactly supported wavelets introduced by Chen et al. [1]. They introduced the multilevel augmentation method related with some wavelet bases for solving certain boundary value problems. This method has then been applied for solving the sine-Gordon equation in [2] and some types of nonlinear boundary value problems in [3]. For solving the partial differential equations, the wavelet applications have been introduced by several authors, such as a wavelet-Galerkin method for solving parabolic equations [4], the singularly perturbed convection-dominated diffusion equation [5], non-homogeneous heat and wave equations [6], some types of elliptic problems [7], and diffusion equation [8].

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Instead of applying the Wavelet-Gelerkin method to solve the unsteady heat equation, we have compared the accuracy of numerical results when using the traditional Lagrange base.

Rates of convergence for two types of linear and quadratic bases are also presented. The rates of convergence for linear and quadratic bases of both the Lagrange and Wavelet are in the same order as expected. We have revealed in this work that the linear wavelet has more advantages than the linear Lagrange when high numerical resolutions are required. The accuracy by the linear wavelet is easily improved by just increasing wavelet levels (multilevel concept) and the computations for finding coefficients are performing just the coefficients extra included in the corresponding level. This concept is different from using the tradition Lagrange bases that it is required to calculate the whole system.

Furthermore, we have presented the polynomial time basis to march numerical solutions in time. It is represented as a tensor product vector when the order of time polynomial basis is specified. By this approximation , the order of accuracy in time discretization is easily increased unlike the standard time marching scheme such as the forward Euler or the Crank-Nicolson method.

The details of this presented work are organized as follows. In Section 2, we introduce the Galerkin finite element method. Time discretization with polynomial basis is presented in Section 3.The applications of the Lagrange and wavelet basis functions to solve the heat equation are shown in Sections 4 and 5. Some numerical examples and comparisons of numerical results are demonstrated in Section 6. We have made some conclusions in Section 7.

II. Galerkin Finite Element Method

The time-dependent heat equation in terms of variable $T(x,t)$ is written in its one-dimensional form as $T(x,t)$

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad , \quad (0 \le x \le l) \quad , \tag{1}$$

where T is temperature and α is the thermal diffusivity (constant). The domain is Ω $(0 \le x \le l)$ with boundary Γ.

We give the boundary conditions as

$$T(0,t) = T(l,t) = 0 \ , \tag{2}$$

and initial conditions as

$$T(x,0) = T_0(x) \ . \tag{3}$$

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By the weighted residual method, equation (1) can be written as

$$\int_{\Omega} W\left(\frac{\partial T}{\partial t}-\alpha\frac{\partial^2 T}{\partial x^2}\right)\,\mathrm{d}\Omega = 0 \ , \qquad (4)$$

where W is a weighting function. Using integration by part, yields

$$\int_{\Omega} W\left(\frac{\partial T}{\partial t}\right)d\Omega \;+\; \int_{\Omega}\alpha\left(\frac{\partial T}{\partial x}\frac{\partial W}{\partial x}\right)d\Omega \;-\alpha W\left.\frac{\partial T}{\partial x}\right|_{\Gamma} = 0 \ . \qquad (5)$$

Let us begin by approximating the unknown function in terms of the Lagrange basis as

$$T^n = T\left(x,t_n\right) \;=\; \sum_{k=0}^{p}\sum_{i=1}^{m} N_i\left(x\right)\theta_k\left(t_n\right)c_{ik}^n \ . \qquad (6)$$

Another choice is done by assuming the unknown variable in terms of the wavelet basis as

$$T^n = T\left(x,t_n\right) \;=\; \sum_{k=0}^{p}\sum_{i=1}^{M}\sum_{j=0}^{\dim(i)} w_{ij}\left(x\right)\theta_k\left(t_n\right)c_{ijk}^n \ , \qquad (7)$$

where $T^n = T\left(x,t_n\right)$ denotes the variable's value at time $t = t_n$, $N_i(x)$ is spatial basis function, $w_{ij}\left(x\right)$ is the wavelet basis function, $\theta_k(t_n)$ is the time basis function, m is the number of elements, M is the number of level in multi-level wavelet approach, and p is the number of level for time discretization.

After setting $W = N_i\left(x\right)$ where $N_i(x)$ is the Lagrange basis function, equation (5) can be written in the matrix form as

$$\left[C_L\right]\left\{c_{ik}^n\right\}+\left[K_L\right]\left\{c_{ik}^n\right\}+\left[M_L^{++}\right]\left\{c_{ik}^n\right\}-\left[M_L^{-+}\right]\left\{c_{ik}^{n-1}\right\}=0 \ ,$$
$$\left\{\left[C_L\right]+\left[K_L\right]+\left[M_L^{++}\right]\right\}c_{ik}^n = \left[M_L^{-+}\right]c_{ik}^{n-1} \ . \qquad (8)$$

Similarly, if we set $W = w_{ij}\left(x\right)$, equation (5) can be written in the matrix form as

$$\left[C\right]\left\{c_{ijk}^n\right\}+\left[K\right]\left\{c_{ijk}^n\right\}+\left[M^{++}\right]\left\{c_{ijk}^n\right\}-\left[M^{-+}\right]\left\{c_{ijk}^{n-1}\right\}=0 \ ,$$
$$\left\{\left[C\right]+\left[K\right]+\left[M^{++}\right]\right\}c_{ijk}^n = \left[M^{-+}\right]c_{ijk}^{n-1} \ . \qquad (9)$$

The coefficients in each matrix element can be obtained. For brevity, the results are summarized in the table shown below.

In the case of the Lagrange basis function, the initial condition provides the starting unknown coefficients as

$$c_{ik}^{n-1} = \left[T_0(x)\right]\otimes\left[\theta\left(0\right)\right] \ . \qquad (10)$$

For the wavelet basis function, the initial unknown coefficients are obtained by

$$c_{ijk}^{n-1} = \left[T_0(x)\right]\otimes\left[\theta\left(0\right)\right] \ , \qquad (11)$$

where $\otimes$ is the outer tensor operation of two matrices defined below.

The coefficients in each matrix element can be obtained.
Lagrange basis functions :

$$\left[C_L\right] = \int_{\Omega} W\frac{\partial T}{\partial t}d\Omega = \int_0^l\int_{t_{n-1}}^{t_n}\left(N_i\otimes\theta\right)\left(N_i\otimes\frac{\partial\theta}{\partial t}\right)^T dtdx$$
$$= \int_0^l\left(N_iN_i^T\right)dx\otimes\int_{t_{n-1}}^{t_n}\left(\theta\frac{\partial\theta^T}{\partial t}\right)dt \qquad (12)$$

$$\left[K_L\right] = \int_{\Omega}\alpha\frac{\partial T}{\partial x}\frac{\partial W}{\partial x}d\Omega = \alpha\int_0^l\int_{t_{n-1}}^{t_n}\left(\frac{\partial N_i}{\partial x}\otimes\theta\right)\left(\frac{\partial N_i}{\partial x}\otimes\theta\right)^T dtdx$$
$$= \alpha\int_0^l\left(\frac{\partial N_i}{\partial x}\frac{\partial N_i}{\partial x}^T\right)dx\otimes\int_{t_{n-1}}^{t_n}\left(\theta\theta^T\right)dt \qquad (13)$$

$$\left[M_L^{++}\right] = \int_0^l N_i\left(\theta^+\right)\left(N_i\left(\theta^+\right)\right)^T dx = \int_0^l N_iN_i^T dx\otimes\left[\left(\theta^+\right)\left(\theta^+\right)^T\right] \qquad (14)$$

$$\left[M_L^{-+}\right] = \int_0^l N_i\left(\theta^-\right)\left(N_i\left(\theta^+\right)\right)^T dx = \int_0^l N_iN_i^T dx\otimes\left[\left(\theta^-\right)\left(\theta^+\right)^T\right] \qquad (15)$$

Wavelet basis functions :

$$\left[C\right] = \int_{\Omega} W\frac{\partial T}{\partial t}d\Omega = \int_0^l\int_{t_{n-1}}^{t_n}\left(w_{ij}\otimes\theta\right)\left(w_{ij}\otimes\frac{\partial\theta}{\partial t}\right)^T dtdx$$
$$= \int_0^l\left(w_{ij}w_{ij}^T\right)dx\otimes\int_{t_{n-1}}^{t_n}\left(\theta\frac{\partial\theta^T}{\partial t}\right)dt \qquad (16)$$

$$\left[K\right] = \int_{\Omega}\alpha\frac{\partial T}{\partial x}\frac{\partial W}{\partial x}d\Omega = \alpha\int_0^l\int_{t_{n-1}}^{t_n}\left(\frac{\partial w_{ij}}{\partial x}\otimes\theta\right)\left(\frac{\partial w_{ij}}{\partial x}\otimes\theta\right)^T dtdx$$
$$= \alpha\int_0^l\left(\frac{\partial w_{ij}}{\partial x}\frac{\partial w_{ij}}{\partial x}^T\right)dx\otimes\int_{t_{n-1}}^{t_n}\left(\theta\theta^T\right)dt \qquad (17)$$

$$\left[M^{++}\right] = \int_0^l w_{ij}\left(\theta^+\right)\left(w_{ij}\left(\theta^+\right)\right)^T dx = \int_0^l w_{ij}w_{ij}^T dx \otimes \left[\left(\theta^+\right)\left(\theta^+\right)^T\right] \tag{18}$$

$$\left[M^{-+}\right] = \int_0^l w_{ij}\left(\theta^-\right)\left(w_{ij}\left(\theta^+\right)\right)^T dx = \int_0^l w_{ij}w_{ij}^T dx \otimes \left[\left(\theta^-\right)\left(\theta^+\right)^T\right] \tag{19}$$

For example, when matrices A and B are given in dimension of 2×2, the outer tensor operation is defined as follows.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \ , \ B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \ ,$$

$$A \otimes B = \begin{bmatrix} Ab_{11} & Ab_{12} \\ Ab_{21} & Ab_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{11} & a_{112}b_{12} & a_{12}b_{12} \\ a_{21}b_{11} & a_{22}b_{11} & a_{21}b_{12} & a_{22}b_{12} \\ a_{11}b_{21} & a_{12}b_{21} & a_{11}b_{22} & a_{12}b_{22} \\ a_{21}b_{21} & a_{22}b_{21} & a_{21}b_{22} & a_{22}b_{22} \end{bmatrix} . \tag{20}$$

Finally, we have the systems of equation in equation (8), or in equation (9) that can be solved to find the coefficients $\left\{c_{ik}^n\right\}$, or $\left\{c_{ijk}^n\right\}$, and hence we know the approximate values T^n. Note that the system of linear equation is solved iteratively by the Gauss-Seidel method in this work.

III. Time Discretization

For the discretization in time, we give the basis function in time as

$$\theta_k\left(t\right) = \left(\left(t - t_{n-1}\right)/\Delta t\right)^k \ , \tag{21}$$

where

$$\theta = \begin{bmatrix} \theta_0 & \theta_1 & \theta_2 & \cdots & \theta_p \end{bmatrix} = \begin{bmatrix} 1 & \dfrac{t-t_{n-1}}{\Delta t} & \left(\dfrac{t-t_{n-1}}{\Delta t}\right)^2 & \cdots & \left(\dfrac{t-t_{n-1}}{\Delta t}\right)^p \end{bmatrix}^T . \tag{22}$$

We give the notations $\theta^+ = \theta\left(t_{n-1}^+\right)$ and $\theta^- = \theta\left(t_{n-1}^-\right)$ referring to the right and left limits at time t_{n-1} respectively where $\Delta t = t_{n+1} - t_n$.

The coefficients of matrices $\left[\int_{t_{n-1}}^{t_n}\left(\theta\dfrac{\partial\theta^T}{\partial t}\right)dt\right]$, $\left[\int_{t_{n-1}}^{t_n}\left(\theta\theta^T\right)dt\right]$, $\left[\left(\theta^+\right)\left(\theta^+\right)^T\right]$ and $\left[\left(\theta^-\right)\left(\theta^+\right)^T\right]$ can be calculated by

$$\int_{t_{n-1}}^{t_n}\left(\theta\frac{\partial\theta^T}{\partial t}\right)dt = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & \frac{1}{2} & \frac{2}{3} & \cdots & \frac{p}{p+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \frac{1}{p+1} & \frac{2}{p+2} & \cdots & \frac{p}{2p} \end{bmatrix} , \tag{23}$$

$$\int_{t_{n-1}}^{t_n}\left(\theta\theta^T\right)dt = \Delta t\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{p+1} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{p+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{1}{p+1} & \frac{1}{p+2} & \frac{1}{p+3} & \cdots & \frac{1}{2p+1} \end{bmatrix} , \tag{24}$$

$$\left[\left(\theta^+\right)\left(\theta^+\right)^T\right] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} , \tag{25}$$

$$\left[\left(\theta^-\right)\left(\theta^+\right)^T\right] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} . \tag{26}$$

So, we can derive the full form of all matrices resulting to the full system that can be solved iteratively to obtain approximate solutions when the initial and boundary conditions are specified.

For the discretization in time of level 2,

we set $\theta = \begin{bmatrix} 1 & \dfrac{t - t_{n-1}}{\Delta t} \end{bmatrix}^T$. So, we can find,

$$\int_{t_{n-1}}^{t_n}\left(\theta\frac{\partial\theta^T}{\partial t}\right)dt = \begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{2} \end{bmatrix} , \tag{27}$$

$$\int_{t_{n-1}}^{t_n}\left(\theta\theta^T\right)dt = \Delta t\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} , \tag{28}$$

$$\left[\left(\theta^+\right)\left(\theta^+\right)^T\right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} , \tag{29}$$

$$\left[\left(\theta^-\right)\left(\theta^+\right)^T\right] = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} . \tag{30}$$

IV. Lagrange Basis Functions

In this section, we will show in details the derivation of matrix coefficients by two classes of the Lagrange basis function which are linear and quadratic bases.

A. Linear Lagrange Basis Function

We begin defining nodal points in the domain $0 \le x \le l$ with K elements of uniform element size. Thus, there are $K+1$ nodes corresponding to the coordinates $x_1, x_2, ..., x_{K+1}$ as shown in Fig.1.

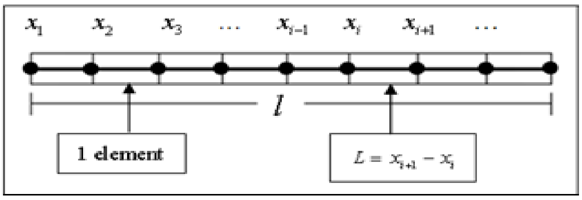

Fig. 1 Partition of the domain in linear element grid

In this case, we assume the approximate solution in equation (6) as

$$T^n(x,t_n) = \sum_{k=0}^{p}\left(N_1(x)\theta_k(t_n)c_{1k}^n + N_2(x)\theta_k(t_n)c_{2k}^n\right), \quad (31)$$

where

$$N_1(x) = 1-\frac{x}{L}, \quad (32)$$

$$N_2(x) = \frac{x}{L}. \quad (33)$$

These are the well-known linear Lagrange basis functions. Their variations in an element are shown in Fig. 2.

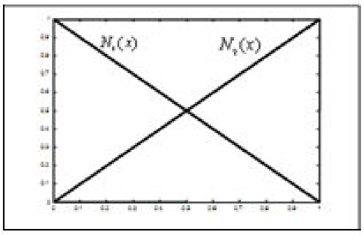

Fig. 2 Linear basis functions

Hence, some parts in the matrices $[C_L]$, $[K_L]$, $[M_L^{++}]$ and $[M_L^{-+}]$ can be evaluated as follows

$$\int_0^L \left(N_i N_i^T\right)dx = \int_0^L \begin{Bmatrix} 1-\frac{x}{L} \\ \frac{x}{L} \end{Bmatrix}\begin{bmatrix} 1-\frac{x}{L} & \frac{x}{L} \end{bmatrix}dx = \frac{L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad (34)$$

$$\int_0^L \left(\frac{\partial N_i}{\partial x}\frac{\partial N_i}{\partial x}^T\right)dx = \int_0^L \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix}\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}dx = \frac{1}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \quad (35)$$

The full definition of these matrices are obtained by using the outer tensor operator $\otimes$ in equation (20).

The region $0 \le x \le l$ has been divided by K elements of equal length $L = \frac{l}{K}$. After assembling all elements together, we obtain the full system of linear equation as

$$[C_L] = \left[\frac{L}{6}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}_{(K+1)\times(K+1)} \otimes \begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}\right]_{2(K+1)\times 2(K+1)}, \quad (36)$$

$$[K_L] = \alpha\left[\frac{1}{L}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}_{(K+1)\times(K+1)} \otimes \Delta t\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}\right]_{2(K+1)\times 2(K+1)}, \quad (37)$$

$$[M_L^{++}] = \left[\frac{L}{6}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}_{(K+1)\times(K+1)} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right]_{2(K+1)\times 2(K+1)}, \quad (38)$$

$$[M_L^{-+}] = \left[\frac{L}{6}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}_{(K+1)\times(K+1)} \otimes \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right]_{2(K+1)\times 2(K+1)}. \quad (39)$$

B. Quadratic Lagrange Basis Function

The nodal notation used in this case is shown in Fig.4. There are three nodes in one element. The ith element is defined on $x_{2i-1} \le x \le x_{2i+1}$, $i = 1,2,...,K$ and its element size is given by $L = x_{2i+1} - x_{2i-1}$.

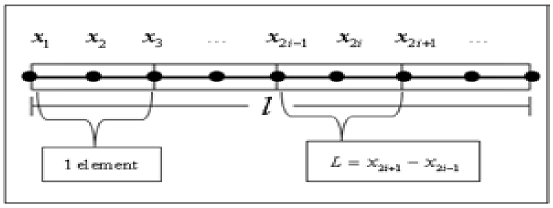

Fig. 3 Partition of the domain in Quadratic element grid

By this basis function, we assume the approximate $T^n(x,t_n)$ as

$$T^n(x,t_n) = \sum_{k=0}^{p}\begin{pmatrix} N_1(x)\theta_k(t_n)c_{1k}^n + N_2(x)\theta_k(t_n)c_{2k}^n \\ +N_3(x)\theta_k(t_n)c_{3k}^n \end{pmatrix}, \quad (40)$$

and the quadratic Lagrange basis functions are

$$N_1(x) = 1 - \frac{3x}{L} + 2\left(\frac{x}{L}\right)^2 , \tag{41}$$

$$N_2(x) = \frac{4x}{L}\left(1 - \frac{x}{L}\right) , \tag{42}$$

$$N_3(x) = \frac{x}{L}\left(\frac{2x}{L} - 1\right) . \tag{43}$$

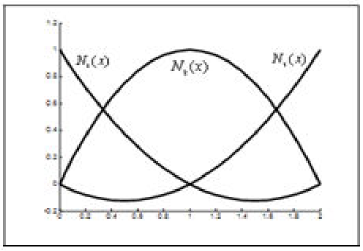

Fig. 4 Quadratic basis functions

Hence, some parts in the matrices $\left[C_L\right]$, $\left[K_L\right]$, $\left[M_L^{++}\right]$ and $\left[M_L^{-+}\right]$ can be evaluated as

$$\begin{aligned}\int_0^L \left(N_i N_i^T\right) dx &= \int_0^L \begin{Bmatrix} 1-\frac{3x}{L}+2\left(\frac{x}{L}\right)^2 \\ \frac{4x}{L}\left(1-\frac{x}{L}\right) \\ \frac{x}{L}\left(\frac{2x}{L}-1\right) \end{Bmatrix} \left[1-\frac{3x}{L}+2\left(\frac{x}{L}\right)^2 \quad \frac{4x}{L}\left(1-\frac{x}{L}\right) \quad \frac{x}{L}\left(\frac{2x}{L}-1\right)\right] dx \\ &= \frac{L}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} ,\end{aligned} \tag{44}$$

$$\begin{aligned}\int_0^L \left(\frac{\partial N_i}{\partial x}\frac{\partial N_i^T}{\partial x}\right) dx &= \int_0^L \begin{Bmatrix} \frac{1}{L}\left(\frac{4x}{L}-3\right) \\ \frac{4}{L}\left(1-\frac{2x}{L}\right) \\ \frac{1}{L}\left(\frac{4x}{L}-1\right) \end{Bmatrix} \left[\frac{1}{L}\left(\frac{4x}{L}-3\right) \quad \frac{4}{L}\left(1-\frac{2x}{L}\right) \quad \frac{1}{L}\left(\frac{4x}{L}-1\right)\right] dx \\ &= \frac{1}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} .\end{aligned} \tag{45}$$

Using the same time polynomial basis as in the linear approximation and after assembling all elements together, we obtain the full linear system as

$$\left[C_L\right] = \left[\frac{L}{30}\begin{bmatrix} 4 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 2 & 16 & 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & 8 & 2 & -1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & -1 & 2 & 8 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 & 16 & 2 \\ 0 & 0 & \cdots & 0 & -1 & 2 & 4 \end{bmatrix}_{(2K+1)\times(2K+1)} \otimes \begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{2} \end{bmatrix} \right]_{2(2K+1)\times 2(2K+1)} , \tag{46}$$

$$\left[K_L\right] = \alpha \left[\frac{1}{3L}\begin{bmatrix} 7 & -8 & 1 & 0 & \cdots & 0 & 0 \\ -8 & 16 & -8 & 1 & 0 & \cdots & 0 \\ -1 & -8 & 14 & -8 & 1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & 1 & -8 & 14 & -8 & 1 \\ 0 & \cdots & 0 & 1 & -8 & 16 & 2 \\ 0 & 0 & \cdots & 0 & 1 & -8 & 7 \end{bmatrix}_{(2K+1)\times(2K+1)} \otimes \Delta t \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} \right]_{2(2K+1)\times 2(2K+1)} , \tag{47}$$

$$\left[M_L^{++}\right] = \left[\frac{L}{30}\begin{bmatrix} 4 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 2 & 16 & 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & 8 & 2 & -1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & -1 & 2 & 8 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 & 16 & 2 \\ 0 & 0 & \cdots & 0 & -1 & 2 & 4 \end{bmatrix}_{(2K+1)\times(2K+1)} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right]_{2(2K+1)\times 2(2K+1)} , \tag{48}$$

$$\left[M_L^{-+}\right] = \left[\frac{L}{30}\begin{bmatrix} 4 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 2 & 16 & 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & 8 & 2 & -1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & -1 & 2 & 8 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 & 16 & 2 \\ 0 & 0 & \cdots & 0 & -1 & 2 & 4 \end{bmatrix}_{(2K+1)\times(2K+1)} \otimes \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right]_{2(2K+1)\times 2(2K+1)} . \tag{49}$$

This system can be solved iteratively to find the coefficients $\left\{c_{ik}^n\right\}$.

V. Wevelet Basis Functions

The construction of the wavelet basis functions used in the Galerkin method follows the derivations proposed by [1]. We construct multi-scale orthonormal bases for the Sobolev space on the unit interval $I := [0,1]$. Specifically, we let m be a fixed positive integer and $H_0^m(I)$ denoted the Sobolev spaces of element that T satisfies the homogeneous boundary conditions of $T^{(j)}(0) = T^{(j)}(1) = 0$, $j \in Z_m$, where $Z_n := \{0,1,2,...,n-1\}$. For any nonnegative integer n, we denote by X_n the subspace of $H_0^m(I)$ whose elements are the piecewise polynomials of order k with knots j/μ^n, $j-1 \in Z_{\mu^n - 1}$, when $k > 2m$ and $\mu > 1$ be a fixed positive integer. We have that $X_0 = \mathrm{span}\left\{x^{m+j}(1-x)^m : j \in Z_{k-2m}\right\}$, so we let W_n be the orthonormal complement of X_{n+1} in X_n, i.e., $X_n = X_{n-1} \oplus_m W_n$ and thus, repeatedly using this decomposition leads to $X_n = X_0 \oplus_m W_1 \oplus_m \cdots \oplus_m W_n$. Spaces W_n can be recursively constructed once W_1 has been given. To describe the construction, the family of affine mappings $\Phi_\mu := \left\{\phi_e : e \in Z_\mu\right\}$ is required where $\phi_e(x) = (x+e)/\mu$; $e \in Z_\mu$.

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Finally, the wavelet basis function can be constructed by the composition as follows.

$$w_{ij} = \tau_{\mathrm{e}} w_{1l} = \mu^{n\left(\frac{1}{2}-m\right)} w_{1l} \circ \phi_{\mathrm{e}}^{-1}(x) \; ; \; \mathrm{e} \in Z_{\mu}^{i-1}. \qquad (50)$$

This construction will be applied to obtain both linear and quadratic wavelet bases in the next sections.

A. Linear Wevelet Basis Function

From equation (50), we set $m=1$, $\mu=2$, and $r=1$. We also give l , Z_{μ} , and e by $l \in Z_r = \{0\}$, $e \in Z_{\mu} = \{0,1\}$ and $\phi_0(x) = \dfrac{x}{2}$, $\phi_1(x) = \dfrac{x+1}{2}$.

The desired basis of W_1 (level 1) is obtained by

$$w_{10}(x) = \begin{cases} x & ; x \in [0,1/2] \\ 1-x & ; x \in [1/2,1] \end{cases} . \qquad (51)$$

The wavelet basis function of W_2 (level 2) is given by $\left(n=1,\ \mathrm{e} \in Z_2^1 = \{0,1\}\right)$

$$w_{2j} = \mu^{1\left(\frac{1}{2}-1\right)} w_{1l} \circ \phi_{\mathrm{e}}^{-1}(x) = \frac{1}{\sqrt{2}} w_{1l} \circ \phi_{\mathrm{e}}^{-1}(x) \;, \qquad (52)$$

$$w_{20}(x) = \begin{cases} 2x/\sqrt{2} & ; x \in [0,1/4] \\ (1-2x)/\sqrt{2} & ; x \in [1/4,1/2] \end{cases} , \qquad (53)$$

$$w_{21}(x) = \begin{cases} (2x-1)/\sqrt{2} & ; x \in [1/2,3/4] \\ (2-2x)/\sqrt{2} & ; x \in [3/4,1] \end{cases} . \qquad (54)$$

The wavelet basis function of W_3 (level 3) is that $\left(n=2,\ \mathrm{e} \in Z_2^2 = \{(0,0),(0,1),(1,0),(1,1)\}\right)$

$$w_{3j} = \mu^{2\left(\frac{1}{2}-1\right)} w_{1l} \circ \phi_{\mathrm{e}}^{-1}(x) = \frac{1}{2} w_{1l} \circ \phi_{\mathrm{e}}^{-1}(x) \;, \qquad (55)$$

$$w_{30}(x) = \begin{cases} 4x/2 & ; x \in [0,1/8] \\ (1-4x)/2 & ; x \in [1/8,1/4] \end{cases} , \qquad (56)$$

$$w_{31}(x) = \begin{cases} (4x-1)/2 & ; x \in [1/4,3/8] \\ (2-4x)/2 & ; x \in [3/8,1/2] \end{cases} , \qquad (57)$$

$$w_{32}(x) = \begin{cases} (4x-2)/2 & ; x \in [1/2,5/8] \\ (3-4x)/2 & ; x \in [5/8,3/4] \end{cases} , \qquad (58)$$

$$w_{33}(x) = \begin{cases} (4x-3)/2 & ; x \in [3/4,7/8] \\ (4-4x)/2 & ; x \in [7/8,1] \end{cases} . \qquad (59)$$

The profiles of these three linear wavelet basis functions W_1 , W_2 and W_3 are shown in Fig.5. In practice, any levels of the linear wavelet basis can be obtained recursively by the same process of this construction.

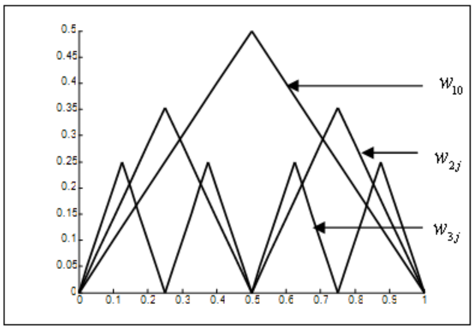

Fig. 5 Linear Wavelet basis functions

For example, using the linear wavelet functions for the first three levels which are composed of $\mathrm{W}_1, \mathrm{W}_2$ and W_3, some parts of the coefficients in matrices $[C]$, $[K]$, $\left[M^{++}\right]$ and $\left[M^{-+}\right]$ can be evaluated as

$$\int_0^L \left(w_{ij} w_{ij}{}^T\right) dx = \begin{bmatrix} \frac{1}{12} & \frac{\sqrt{2}}{64} & \frac{\sqrt{2}}{64} & \frac{1}{256} & \frac{3}{256} & \frac{3}{256} & \frac{1}{256} \\ \frac{\sqrt{2}}{64} & \frac{1}{48} & 0 & \frac{\sqrt{2}}{256} & \frac{\sqrt{2}}{256} & 0 & 0 \\ \frac{\sqrt{2}}{64} & 0 & \frac{1}{48} & 0 & 0 & \frac{\sqrt{2}}{256} & \frac{\sqrt{2}}{256} \\ \frac{1}{256} & \frac{\sqrt{2}}{256} & 0 & \frac{1}{192} & 0 & 0 & 0 \\ \frac{3}{256} & \frac{\sqrt{2}}{256} & 0 & 0 & \frac{1}{192} & 0 & 0 \\ \frac{3}{256} & 0 & \frac{\sqrt{2}}{256} & 0 & 0 & \frac{1}{192} & 0 \\ \frac{1}{256} & 0 & \frac{\sqrt{2}}{256} & 0 & 0 & 0 & \frac{1}{192} \end{bmatrix} , \qquad (60)$$

$$\int_0^L \left(\frac{\partial w_{ij}}{\partial x} \frac{\partial w_{ij}{}^T}{\partial x}\right) dx = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} . \qquad (61)$$

Then we can apply the outer tensor operator $\otimes$ in equation (20) to find the full form of matrices**.**

B. Quadratic Wevelet Basis Function

In this case, we set $m=1$, $\mu=2$ and $r=2$, thus l , Z_{μ}, e are given by $l \in Z_r = \{0,1\}$, $e \in Z_{\mu} = \{0,1\}$ and $\phi_0(x) = \dfrac{x}{2}$, $\phi_1(x) = \dfrac{x+1}{2}$.

The desired bases of W_0 and W_1 are given by

$$w_{00}\left(x\right)=\sqrt{3}x(1-x)\quad ;x\in\left[0,1\right] \; , \tag{62}$$

$$w_{10}\left(x\right)=\begin{cases} x\left(1-3x\right) & ;x\in\left[0,1/2\right] \\ \left(1-x\right)\left(3x-2\right) & ;x\in\left[1/2,1\right] \end{cases} \; , \tag{63}$$

$$w_{11}\left(x\right)=\begin{cases} \sqrt{3}x\left(1-2x\right) & ;x\in\left[0,1/2\right] \\ \sqrt{3}\left(1-x\right)\left(1-2x\right) & ;x\in\left[1/2,1\right] \end{cases} \; . \tag{64}$$

The quadratic wavelet basis of level two, W_2 is given by $\left(n=1,\ \mathrm{e}\in Z_2^1=\left\{0,1\right\}\right)$,

$$w_{2j} = \mu^{1\left(\frac{1}{2}-1\right)} w_{1l}\circ\phi_{\mathrm{e}}^{-1}\left(x\right)=\frac{1}{\sqrt{2}}w_{1l}\circ\phi_{\mathrm{e}}^{-1}\left(x\right) \; , \tag{65}$$

$$w_{20}\left(x\right)=\begin{cases} \frac{2x}{\sqrt{2}}\left(1-6x\right) & ;x\in\left[0,1/4\right] \\ \frac{2}{\sqrt{2}}\left(1-2x\right)\left(3x-1\right); & x\in\left[1/4,1/2\right] \end{cases} \; , \tag{66}$$

$$w_{21}\left(x\right)=\begin{cases} \sqrt{6}x\left(1-4x\right) & ;x\in\left[0,1/4\right] \\ \frac{\sqrt{6}}{2}\left(1-2x\right)\left(3x-1\right); & x\in\left[1/4,1/2\right] \end{cases} \; , \tag{67}$$

$$w_{22}\left(x\right)=\begin{cases} \frac{1}{\sqrt{2}}\left(4-6x\right)\left(2x-1\right); & x\in\left[1/2,3/4\right] \\ \frac{1}{\sqrt{2}}\left(2-2x\right)\left(6x-5\right); & x\in\left[3/4,1\right] \end{cases} \; , \tag{68}$$

$$w_{23}\left(x\right)=\begin{cases} \frac{\sqrt{6}}{2}\left(2x-1\right)\left(3-4x\right); & x\in\left[1/2,3/4\right] \\ \frac{\sqrt{6}}{2}\left(2-2x\right)\left(3-4x\right); & x\in\left[3/4,1\right] \end{cases} \; . \tag{69}$$

The profiles of quadratic wavelet functions W_0, W_1 and W_2 are shown in Fig.6.

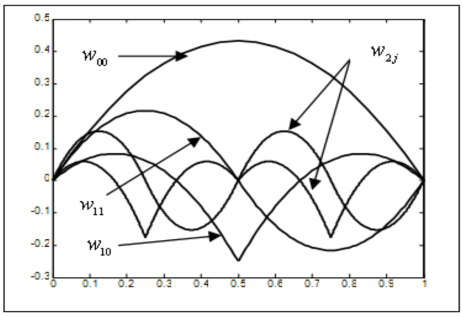

Fig. 6 Quadratic Wavelet basis functions

For example, after using the quadratic wavelet functions for the first two levels, some parts of the coefficients in matrices $\left[C\right]$, $\left[K\right]$, $\left[M^{++}\right]$ and $\left[M^{-+}\right]$ can be evaluated as

$$\int_0^L\left(w_{ij}w_{ij}^{\;T}\right)d\omega=\begin{bmatrix} \frac{1}{10} & \frac{-\sqrt{3}}{240} & 0 & \frac{-\sqrt{6}}{3840} & \frac{-\sqrt{2}}{256} & \frac{-\sqrt{6}}{3840} & \frac{\sqrt{2}}{256} \\ \frac{-\sqrt{3}}{240} & \frac{1}{120} & 0 & \frac{-\sqrt{2}}{1280} & \frac{\sqrt{6}}{768} & \frac{-\sqrt{2}}{1280} & \frac{-\sqrt{6}}{768} \\ 0 & 0 & \frac{1}{40} & \frac{-\sqrt{6}}{1920} & 0 & \frac{\sqrt{6}}{1920} & 0 \\ \frac{-\sqrt{6}}{3840} & \frac{-\sqrt{2}}{1280} & \frac{-\sqrt{6}}{1920} & \frac{1}{480} & 0 & 0 & 0 \\ \frac{-\sqrt{2}}{256} & \frac{\sqrt{6}}{768} & 0 & 0 & \frac{1}{160} & 0 & 0 \\ \frac{-\sqrt{6}}{3840} & \frac{-\sqrt{2}}{1280} & \frac{\sqrt{6}}{1920} & 0 & 0 & \frac{1}{480} & 0 \\ \frac{\sqrt{2}}{256} & \frac{-\sqrt{6}}{768} & 0 & 0 & 0 & 0 & \frac{1}{160} \end{bmatrix} \; , \tag{70}$$

$$\int_0^L\left(\frac{\partial w_{ij}}{\partial x}\frac{\partial w_{ij}}{\partial x}^{T}\right)d\omega=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \; . \tag{71}$$

Then we can apply the outer tensor operator $\otimes$ in equation (20) to find the full form of matrices.

For a given initial condition , $T\left(x,0\right)=T_0\left(x\right)$, and we have assumed that

$$T^n\left(x,t_n\right)=\sum_{k=0}^{p}\sum_{i=1}^{M}\sum_{j=0}^{\dim(i)} w_{ij}\left(x\right)\theta_k\left(t_n\right)c_{ijk}^n \; ,$$

$$\text{so,}\quad T\left(x,0\right)=\sum_{k=0}^{p}\sum_{i=1}^{M}\sum_{j=0}^{\dim(i)} w_{ij}\left(x\right)\theta_k\left(0\right)c_{ijk}^0 \; . \tag{72}$$

Thus, in the case of wavelet basis function, we can find the coefficients $\left\{c_{ijk}^0\right\}$ from the system

$$\left[w_{ij}\left(x_s\right)\right]\left[c_{ijk}^0\right]=\left[T_0\left(x_s\right)\right] \; , \tag{73}$$

where $x_s=s\Delta x$, $\Delta x=1/n$, $s=1,2,...,n+1$, and $n+1$ is the number of knots.

VI. Numerical Results

The time-dependent heat equation in terms of temperature $T\left(x,t\right)$ is

$$\frac{\partial T}{\partial t}=\frac{k}{\rho c}\frac{\partial^2 T}{\partial x^2} \; , \tag{74}$$

where $\partial T/\partial t$ is the rate of change of temperature. We set the thermal diffusivity as $k/\rho c=\alpha=1$.
The boundary and initial conditions are given by

$$T\left(0,t\right)=T\left(1,t\right)=0 \; , \tag{75}$$

$$T\left(x,0\right)=\sin\left(\pi x\right) \; . \tag{76}$$

The exact solution for this problem is $T(x,t) = e^{-\pi^2 t}\sin(\pi x)$.

For discretization in time, we set $\theta = \begin{bmatrix} 1 & \dfrac{t - t_{n-1}}{\Delta t} \end{bmatrix}^T$ which corresponds to $k = 0$ and 1. The time step is $\Delta t = 0.005$ for all calculations.To check the accuracy of the presented numerical schemes, we use the RMS error defined by

$$RMS = \sqrt{\frac{\sum_{i=1}^{N}\left(T_i - T_{Exact}\right)^2}{N}} . \qquad (77)$$

The profiles of numerical solutions at various time steps (0.005, 0.01, 0.015, 0.02, 0.025, 0.03, 0.035, 0.04, 0.045, 0.05) are shown in Fig 8. The temperature profile decreases dramatically as time increases. In this figure, the numerical solutions are obtained by the finite element method based on linear Lagrange basis with 8 elements. The numerical results are in good agreement with the exact solutions even though we have used a small number of elements.

To investigate the convergent rate of numerical schemes for various types of basis functions, if we have observed in two cases of the element size, Δx and $\Delta x/2$, the rate of convergence (r) of the numerical method would be defined by

$$r = \frac{\log\left(RMS_{\Delta x} / RMS_{\Delta x/2}\right)}{\log(2)} . \qquad (78)$$

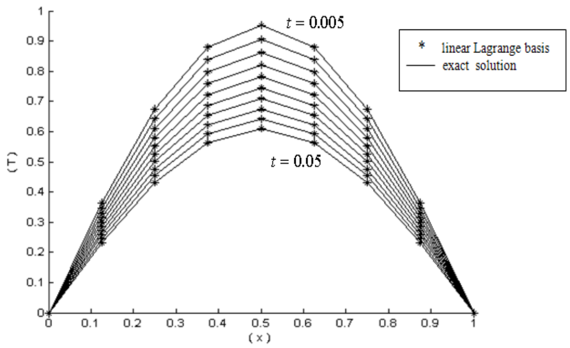

Fig. 7 The finite element results with Linear Lagrange basis and exact solution

The numerical solutions at various time steps are shown in Tables I- IV. Comparing Table I with Table II, the RMS errors by the linear Lagrange basis are almost the same as the RMS errors by the linear wavelet basis. This implies that the accuracy is the same for these two types of basis function when we use the same element size. The rate of convergence is approximately 2.1 as expected for the linear basis. The RMS errors and the rate of convergences for the quadratic Lagrange and wavelet bases are shown in Tables III and IV respectively. The rates of convergence for the quadratic Lagrange and the quadratic wavelet are approximately 4.1 and 4.4 respectively. The plots of rate of convergence are shown in Fig. 8.

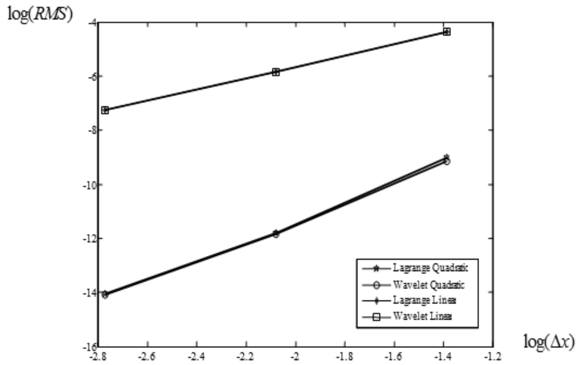

Fig. 8 Rate of convergence for Lagrange and Wavelet bases

TABLE I
RMS ERROR GALERKIN FINITE ELEMENT METHOD (LINEAR LAGRANGE BASIS FUNCTION)

Elements \Time	0.02	0.03	0.04	0.05
8 element	1.5802240973e-003	2.1461995067e-003	2.5910096924e-003	2.9325087600e-003
16 element	3.8073552774e-004	5.1734772873e-004	6.2486925191e-004	7.0756615170e-004
32 element	9.3709947764e-005	1.2734891922e-004	1.5383393948e-004	1.7421284498e-004
64 element	2.3482876353e-005	3.1923205356e-005	3.8576448904e-005	4.3704067644e-005

($\Delta t = 0.005$, TOL = 10^(-8))

Approximate convergence rate = 2.1

TABLE II
RMS ERROR GALERKIN FINITE ELEMENT METHOD (LINEAR WAVELET BASIS FUNCTION)

Level\ Time	0.02	0.03	0.04	0.05
W_3	1.58022e-003	2.14620e-003	2.59101e-003	2.93251e-003
W_4	3.80734e-004	5.17346e-004	6.24867e-004	7.07563e-004
W_5	9.37094e-005	1.27348e-004	1.53833e-004	1.74212e-004
W_6	2.33765e-005	3.17686e-005	3.83766e-005	4.34614e-005

($\Delta t = 0.005$, TOL = 10^(-8))

Approximate convergence rate = 2.1

We have also investigated the rate of convergence of time basis. Two cases of time basis levels are considered. We set the final time as 0.8, and $\alpha = 0.5$. Numerical results are shown in Tables 5-7. The RMS errors resulted from using time basis level $1(\theta_1)$ and by the linear Lagrange and wavelet bases are almost the same as those errors by the quadratic Lagrange and wavelet bases. The rates of convergence are approximately 1.1. Similarly, we have found that the rates of convergence are approximately in order between 2 and 3 when using the time basis of level two. This shows an advantage of using this type of time basis that the order of accuracy in time can be improved just increasing the dimension of time matrix. It is unlike other standard schemes such as the Euler method or the Runge-Kutta scheme that the order of accuracy is fixed after derivation. Using this presented time basis is more flexible than the standard approach.

TABLE III
RMS ERROR GALERKIN FINITE ELEMENT METHOD (QUADRATIC LAGRANGE BASIS FUNCTION)

Elements \Time	0.02	0.03	0.04	0.05
8 element	1.18419e-003	1.61835e-003	1.96275e-003	2.22884e-003
16 element	7.39192e-005	9.10600e-005	1.07179e-004	1.20620e-004
32 element	4.73808e-006	5.70161e-006	6.65048e-006	7.45794e-006
64 element	4.29003e-007	5.66735e-007	6.84214e-007	7.78069e-007

($\Delta t = 0.005$, TOL = 10^(-12))

Approximate convergence rate = 4.1

TABLE IV
RMS ERROR GALERKIN FINITE ELEMENT METHOD (QUADRATIC WAVELET BASIS FUNCTION)

Level\ Time	0.02	0.03	0.04	0.05
W_1	9.70882e-004	1.32503e-003	1.60590e-003	1.82285e-003
W_2	6.23686e-005	7.73105e-005	9.23327e-005	1.05184e-004
W_3	4.57742e-006	5.50828e-006	6.42497e-006	7.20506e-006
W_4	4.22027e-007	5.57519e-007	6.73088e-007	7.65417e-007

($\Delta t = 0.005$, TOL = 10^(-12))

Approximate convergence rate = 4.4

VII. CONCLUSION

In this work, we have presented the Galerkin finite element method to solve numerically the one-dimensional heat equation. The purpose is to show and compare the order of accuracy in space and time for wavelet basis. Two types of basis functions which are the Lagrange (for comparing) and wavelet bases are employed to derive the full matrix system. We consider both linear and quadratic bases. Also, we have introduced a time basis for the time discretization process. When the initial and boundary conditions are specified, the full system matrices can be solved iteratively by the Gauss-Seidel method. Our numerical results show that the rate of convergence for the Linear Lagrange and the Linear Wavelet is the same in order of 2 while the rate of convergence for the quadratic Lagrange and the quadratic wavelet is approximately in order of 4. These two rates are in expected as theoretical results follow the Lagrange basis. The numerical resolutions can be increased by increasing the number of wavelet basis levels. This shows an advantage of the wavelet basis over using the Lagrange basis. By this point of investigation, we can apply the presented wavelet bases with multilevel approach to further solving other types of differential equation, especially the singularly perturbed problem. Some of the results will be reported elsewhere further.

TABLE V
RMS ERROR GALERKIN FINITE ELEMENT METHOD
(LINEAR WAVELET BASIS AND LINEAR LAGRANGE BASIS FUNCTION)

discretization in time in level (θ)	Level\ Time	0.8 $\Delta t = 0.4$	0.8 $\Delta t = 0.2$	0.8 $\Delta t = 0.1$	r
(θ_1)	W_4	6.81299 e-002	3.24632 e-002	1.51638 e-002	1.1
Level 1	16 element	6.81299 e-002	3.24632 e-002	1.51638 e-002	1.1
(θ_2)	W_4	4.55056 e-003	7.85250 e-004	2.60605 e-004	2.1
Level 2	16 element	4.55056 e-003	7.85250 e-004	2.60605 e-004	2.1

($\alpha = 0.5$, TOL = 10^(-12))

TABLE VI
RMS ERROR GALERKIN FINITE ELEMENT METHOD
(LINEAR WAVELET BASIS AND LINEAR LAGRANGE BASIS FUNCTION)

discretization in time in level (θ)	Level\ Time	0.8 $\Delta t = 0.4$	0.8 $\Delta t = 0.2$	0.8 $\Delta t = 0.1$	r
(θ_1)	W_5	6.72812 e-002	3.21552 e-002	1.51012 e-002	1.1
Level 1	32 element	6.72812 e-002	3.21552 e-002	1.51012 e-002	1.1
(θ_2)	W_5	4.34828 e-003	6.41622 e-004	1.25339 e-004	2.6
Level 2	32 element	4.34828 e-003	6.41622 e-004	1.25339 e-004	2.6

($\alpha = 0.5$, TOL = 10^(-12))

r = Approximate convergence rate

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