

# A Modified Laplace Decomposition Algorithm Solution for Blasius' Boundary Layer Equation of the Flat Plate in a Uniform Stream

M. A. Koroma, Z. Chuangyi, A. F., Kamara, and A. M. H. Conteh

**Abstract**—In this work, we apply the Modified Laplace decomposition algorithm in finding a numerical solution of Blasius' boundary layer equation for the flat plate in a uniform stream. The series solution is found by first applying the Laplace transform to the differential equation and then decomposing the nonlinear term by the use of Adomian polynomials. The resulting series, which is exactly the same as that obtained by Weyl 1942a, was expressed as a rational function by the use of diagonal padé approximant.

**Keywords**— Modified Laplace decomposition algorithm, Boundary layer equation, Padé approximant, Numerical solution.

## I. INTRODUCTION

SINCE its introduction by G. Adomian, the Adomian Decomposition Algorithm has been used in finding numerical solutions to a wide variety of problems in Mathematics, Physics and Engineering. It is fundamentally based on providing a solution in the form of series and decomposing nonlinear terms by the use of Adomian polynomials [1]-[2]. In recent years Adomian's algorithm has been modified to make it more effective in providing solutions to differential and integral equations.

Yahya Qaid Hassan and Liu Ming Zhu [3]-[4] used a Modified Adomian Decomposition method in solving singular boundary value problems of higher ordinary differential equations. S. N. Venkatarangan and T. R. Sivakumar [5] adopted a Modified Decomposition method for Boundary Value problems. A. M. Wazwaz [6] combined the Adomian Decomposition method and Pade Approximants in solving Flierl-petviashvili Equation and its variants. In 2001, Khuri [7] proposed the Laplace Decomposition method which was later developed by Yusufoglu [8] in 2006. A reliable modification of this decomposition method was achieved by Khan [9] in 2009. Our current work is motivated by Yasir Khan and Naem Farz

[10] and Rashidi, M. M. [11] who applied the modified Laplace decomposition method in solving boundary layer equation.

In this work we applied the Laplace Decomposition method to obtain a series solution of Blasius' boundary layer equation for the flat plate. It is interesting to note here that the series solution obtained by our method is exactly the same as that obtained by Weyl 1942a, see [12].

## II. MODIFIED LAPLACE DECOMPOSITION ALGORITHM

We illustrate the Laplace decomposition algorithm for solving Blasius' Boundary layer equation as follows:

Consider the homogenous nonlinear differential equation

$$f'''(\eta) + g(\eta) = 0 \quad (1)$$

$$f(0) = a, f'(0) = b, f''(0) = c \quad (2)$$

By the Laplace decomposition algorithm, proposed by Khuri [7] 2001 and Yusufoglu 2006 [8], we have  $L[f'''(\eta)] + L[g(\eta)] = 0$ , where  $L$  denotes the Laplace transform.

Thus

$$s^3 L[f(\eta)] - s^2 f(0) - s f'(0) - f''(0) + L[g(\eta)] = 0$$

Applying the boundary conditions in (2) gives

$$s^3 L[f(\eta)] - as^2 - bs - c + L[g(\eta)] = 0 \quad (3)$$

Let  $H(s) = as^2 + bs + c$  so that (3) becomes

$$L[f(\eta)] = \frac{1}{s^3} H(s) - \frac{1}{s^3} L[g(\eta)] \quad (4)$$

By the Adomian decomposition algorithm the solution to  $f(\eta)$  in (4) is given by the series

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$$f(\eta) = \sum_{n=0}^{\infty} f_n(\eta) \tag{5}$$

and the nonlinear term  $g(\eta)$  is given by the Adomian polynomials

$$g(\eta) = \sum_{n=0}^{\infty} A_n \tag{6}$$

where  $n = 0, 1, 2, 3, \dots$

Substituting (5) and (6) in (4) gives

$$L\left[\sum_{n=0}^{\infty} f_n\right] = \frac{1}{s^3} H(s) - \frac{1}{s^3} L\left[\sum_{n=0}^{\infty} A_n\right] \tag{7}$$

Taking the inverse Laplace of both sides of (7) gives

$$\sum_{n=0}^{\infty} f_n = L^{-1}\left[\frac{1}{s^3} H(s)\right] - L^{-1}\left[\frac{1}{s^3} L\left[\sum_{n=0}^{\infty} A_n\right]\right] \tag{8}$$

As a first approximation to  $f(\eta)$  in (8), the Laplace decomposition algorithm assumes

$$f_0 = L^{-1}\left[\frac{1}{s^3} H(s)\right] \tag{9}$$

Higher iterates of  $f(\eta)$  are obtained from the recurrence relation

$$f_{n+1}(\eta) = -L^{-1}\left[\frac{1}{s^3} L(A_n)\right] \tag{10}$$

$n \geq 0.$

### III. SOLUTION TO BLASIUS' BOUNDARY LAYER EQUATION FOR THE FLAT PLATE IN A UNIFORM STREAM BY THE LAPLACE DECOMPOSITION ALGORITHM

The equation of Blasius for the flat plate in a uniform stream is

$$f''' + ff'' = 0 \quad f = f(\eta) \tag{11}$$

with boundary conditions

$$f(0) = f'(0) = 0; \quad f''(0) = 1 \tag{12}$$

Applying Laplace transform to (1) gives

$$s^3 L(f) - s^2 f(0) - s f'(0) - f''(0) + L(ff'') = 0$$

Using the boundary conditions in (12) gives

$$f = L^{-1}\left\{\frac{1}{s^3} - \frac{1}{s^3} L(ff'')\right\} \tag{13}$$

$$f = \frac{\eta^2}{2} - L^{-1}\left\{\frac{1}{s^3} L(ff'')\right\}$$

By the Adomian decomposition algorithm we can express (13) as

$$f(\eta) = \sum_{n=0}^{\infty} f_n(\eta) = \frac{\eta^2}{2} - L^{-1}\left\{\frac{1}{s^3} L\left[\sum_{n=0}^{\infty} A_n\right]\right\} \tag{14}$$

where  $ff'' = \sum_{n=0}^{\infty} A_n$  and the Adomian polynomials  $A_n$  are obtained from

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left\{ \left( \sum_{i=1}^{\infty} \lambda^i f_i \right) \left( \sum_{i=1}^{\infty} \lambda^i f_i'' \right) \right\}_{\lambda=0} \tag{15}$$

From (15) we have

$$\begin{aligned} A_0 &= f_0 f_0'' \\ A_1 &= f_1 f_0'' + f_0 f_1'' \\ A_2 &= f_2 f_0'' + f_1 f_1'' + f_0 f_2'' \\ A_3 &= f_3 f_0'' + f_2 f_1'' + f_1 f_2'' + f_0 f_3'' \\ A_4 &= f_4 f_0'' + f_3 f_1'' + f_2 f_2'' + f_1 f_3'' + f_0 f_4'' \dots \end{aligned} \tag{16}$$

As a first approximation to  $f(\eta)$  in (14) we take

$$f_0(\eta) = \frac{\eta^2}{2} \tag{17}$$

Subsequent iterates are obtained from the recurrence relation

$$f_{n+1}(\eta) = -L^{-1}\left\{\frac{1}{s^3} L(A_n)\right\} \quad \text{For } n \geq 0 \tag{18}$$

Thus

$$f_1 = -L^{-1}\left\{\frac{1}{s^3} L(A_0)\right\} = -L^{-1}\left\{\frac{1}{s^3} L\left(\frac{\eta^2}{2}\right)\right\} = -\frac{\eta^5}{5!} \tag{19}$$

$$f_2 = -L^{-1}\left\{\frac{1}{s^3} L\left[-\frac{\eta^5}{5!} - \frac{10\eta^5}{5!}\right]\right\} = \frac{11\eta^8}{8!} \tag{20}$$

$$f_3 = -L^{-1} \left\{ \frac{1}{s^3} L \left[ \frac{11\eta^8}{8!} + \frac{\eta^8}{6!} + \frac{28 \times 11\eta^8}{8!} \right] \right\} = \frac{-375\eta^{11}}{11!}$$

$$f_4 = -L^{-1} \left\{ \frac{1}{s^3} L \left[ -\frac{375\eta^{11}}{11!} - \frac{11\eta^{11}}{6 \times 8!} - \frac{77\eta^{11}}{15 \times 8!} - \frac{20625\eta^{11}}{11!} \right] \right\}$$

$$= L^{-1} \left\{ \frac{1}{s^3} L \left[ \frac{1}{11!} (375 + 1815 + 5082 + 20625) \eta^{11} \right] \right\}$$

$$= \frac{27897\eta^{14}}{14!} \tag{21}$$

$$f_5 = -L^{-1} \left\{ \frac{1}{s^3} L \left( \frac{3817137\eta^{14}}{14!} \right) \right\}$$

$$= -\frac{3817137\eta^{17}}{17!} \tag{22}$$

$$f_6 = -L^{-1} \left\{ \frac{1}{s^3} L \left( \frac{-865874115\eta^{17}}{17!} \right) \right\} \tag{23}$$

$$f_6 = \frac{865874115\eta^{20}}{20!}$$

$$f_7 = -L^{-1} \left\{ \frac{1}{s^3} L \left( \frac{303083960103\eta^{20}}{20!} \right) \right\}$$

$$f_7 = -\frac{303083960103\eta^{23}}{23!} \tag{24}$$

TABLE I  
THE [10,10] PADE APPROXIMANT SOLUTION OF (11)

$\eta$	$f(0)$	$f'(0)$	$f''(0)$	$\eta$	$f(0)$	$f'(0)$	$f''(0)$
0	0	0	1	2.6	2.501198	1.172317	-1.816971
0.2	0.019998	0.199933	0.998668	2.8	2.389023	0.893722	-2.611255
0.4	0.079915	0.398937	0.989396	3.0	1.919599	0.595299	-3.252987
0.6	0.179357	0.594660	0.964702	3.2	1.397027	0.306177	-3.736494
0.8	0.317314	0.783380	0.918533	3.4	1.157629	0.033964	-4.099834
1.0	0.491930	0.960413	0.847581	3.6	1.153555	-0.222775	-4.379842
1.2	0.700367	1.120782	0.752298	3.8	1.240840	-0.467063	-4.602194
1.4	0.938827	1.259952	0.636289	4.0	1.343252	-0.701283	-4.783355
1.6	1.202674	1.374089	0.501647	4.2	1.433771	-0.926975	-4.933887
1.8	1.486436	1.458718	0.336329	4.4	1.505622	-1.145108	-5.060800
2.0	1.782948	1.504160	0.091963	4.6	1.559043	-1.356328	-5.168962
2.2	2.079493	1.488726	-0.326230	4.8	1.596361	-1.561122	5.261905
2.4	2.345381	1.381721	-0.990116	5.0	1.620271	-1.759904	-5.342293

Substituting (24) in (14) gives

$$f(\eta) = \frac{\eta^2}{2!} - \frac{\eta^5}{5!} + \frac{11\eta^8}{8!} - \frac{375\eta^{11}}{11!} + \frac{27897\eta^{14}}{14!}$$

$$- \frac{3817137\eta^{17}}{17!} + \frac{865874115\eta^{20}}{20!}$$

$$- \frac{303083960103\eta^{23}}{23!} + \dots \tag{25}$$

Equation (25) can be expressed in the form

$$f(\eta) = \sum_{n=0}^{\infty} (-1)^n C_n \eta^{3n+2} \tag{26}$$

where  $C_n$  satisfies the relation

$$(3n+2)(3n+1)3nC_n = \sum_{i=0}^{n-1} (3i+2)(3i+1)C_i C_{n-1-i} \tag{27}$$

and  $C_0 = \frac{1}{2!}, C_1 = \frac{1}{5!}$ .

It is interesting to note that (26) is exactly the same as the numerical series solution proposed by Weyl 1942a, see [12].

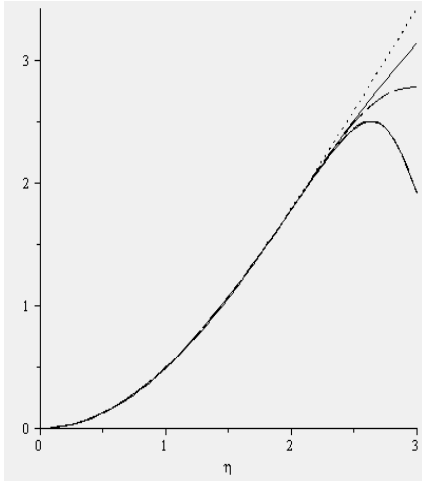


Fig. 1 Pade approximant solutions of  $f(\eta)$  for  $0 \leq \eta \leq 3$  dot=[8,8], dash=[9,9], solid=[10,10], ticks=[11,11], long dash=[12,12]

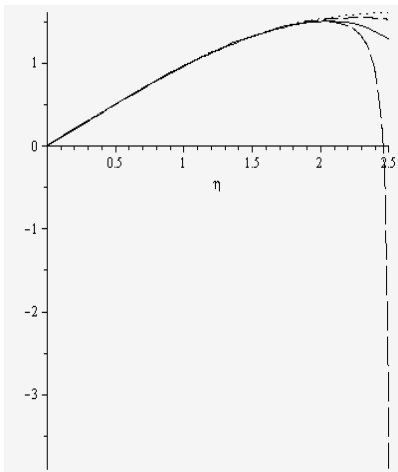


Fig. 2 Pade approximant solutions of  $f'(\eta)$  for  $0 \leq \eta \leq 2.5$  dot=[8,8], dash=[9,9], solid=[10,10], ticks=[11,11], long dash=[12,12]

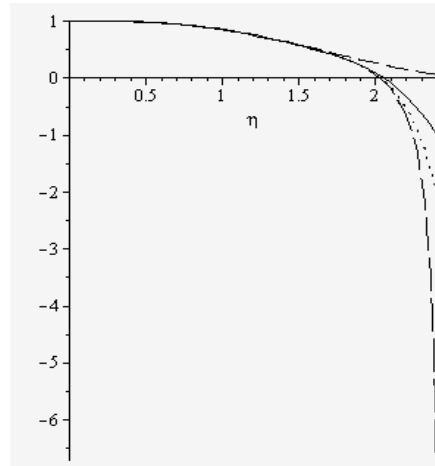


Fig. 3 Pade approximant solutions of  $f''(\eta)$  for  $0 \leq \eta \leq 2.4$  Dash=[8,8], solid=[10,10], ticks=[11,11], long dash=[12,12]

IV. CONCLUSION

In this work we obtained an alternative procedure to a series solution of the Boundary layer equation of a flat plate in a uniform stream by means of the Laplace decomposition algorithm. This algorithm, as demonstrated in equations (15) to (27) above, proved to be an easy and efficient tool in finding a numerical solution to the boundary layer equation (11). The solution obtained by our method above is exactly the same as that obtained by Weyl 1942a. It is worth noting that the series solution (25) is only valid for small values of  $\eta$ , as noted by Weyl. The graphs in Figures 1 to 3 demonstrate the behavior of  $f, f'$  and  $f''$  for small values of  $\eta$ . These solutions are of great importance in understanding the physical problem represented by (11). For instance,  $f(\eta)$  is useful in finding the stream function,  $f'(\eta)$  is useful in determining the velocity distribution of flow and the displacement thickness and  $f''(\eta)$  is useful in finding the shearing stress and the coefficient of drag along the plate.

This Algorithm is very useful in providing numerical approximations to boundary layer equations.

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