# Improved robust stability and stabilization conditions of discrete-time delayed system

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*Abstract*—The problem of robust stability and robust stabilization for a class of discrete-time uncertain systems with time delay is investigated. Based on Tchebychev inequality, by constructing a new augmented Lyapunov function, some improved sufficient conditions ensuring exponential stability and stabilization are established. These conditions are expressed in the forms of linear matrix inequalities (LMIs), whose feasibility can be easily checked by using Matlab LMI Toolbox. Compared with some previous results derived in the literature, the new obtained criteria have less conservatism. Two numerical examples are provided to demonstrate the improvement and effectiveness of the proposed method.

*Keywords*—Robust stabilization, robust stability, discrete-time system, time delay.

## I. INTRODUCTION

**A**S one of important sources of instability and oscillation, time delay is unavoidable in technology and nature. It extensively exists in various mechanical, biological, physical, chemical engineering, economic systems, and can make important effects on the properties of dynamic systems. Thus, the studies on stability for delayed control system are of great significance. Up to now, many different delayed systems such as delayed neural network systems, delayed switched systems, and delayed impulsive systems have been considered. And many excellent papers and monographs have been available. On the other hand, during the design of control system and its hardware implementation, the convergence of a control system may often be destroyed by its unavoidable uncertainty due to the existence of modeling error, the deviation of vital data, and so on. Generally, these unavoidable uncertainties can be classified into two types: that is, stochastic disturbances and parameter uncertainties. While modeling real control system, both of the stochastic disturbances and parameter uncertainties are probably the main resources of the performance degradations of the implemented control system. Therefore, the studies on robust convergence or mean square convergence of delayed control system have been a hot research direction. As for parameter uncertain systems, many sufficient conditions, either delay-dependent or delay-independent, have been proposed to guarantee the robust asymptotic or exponential stability for different class of delayed systems (see [1]-[7]).

It should be pointed out that most of these previous issues have been assumed to be continuous-time models. In practice, discrete-time systems play a more important role than their continuous-time counter-parts in today's digital world, such as numerical computation, computer simulation. And they can ideally keep the dynamic characteristics, functional similarity,

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and even the physical or biological reality of the continuous-time system under mild restriction. Thus, the stability analysis problems for discrete-time delayed systems have received more and more interest, and some stability criteria have been proposed (see [8]- [12]). In [8], Lee and Lee researched the stability problem for a class of discrete-time delayed systems. By using a conventional fixed method transformation that replaces the delay with the summations, some delay-dependent stability conditions were established. By using a similar technique to that in [8], the results obtained in [8] have been improved in [9]. The results obtained in [9] were further improved in [10] by constructing a new augmented Lyapunov function. However, further reduction of the conservatism in the these above results is possible.

Based on this motivation, the main aim of this paper is to establish some new improved stability and stabilization criteria. Along the technique route used in [10], a new augmented Lyapunov function is constructed, and some new improved delay-dependent sufficient conditions are obtained. Numerical examples show that these new established criteria in this paper are less conservative that those obtained in [8]-[11].

*Notation*: The notations are used in our paper except where otherwise specified. $\| \cdot \|$ denotes a vector or a matrix norm; $\mathbb{R}, \mathbb{R}^n$ are real and n-dimension real number sets, respectively; $\mathbb{N}^+$ is positive integer set. $I$ is identity matrix; $*$ represents the elements below the main diagonal of a symmetric block matrix; Real matrix $P > 0 (< 0)$ denotes $P$ is a positive definite (negative definite) matrix; $\mathbb{N}[a, b] = \{a, a+1, \cdots, b\}$; $\lambda_{min}(\lambda_{max})$ denotes the minimum (maximum) eigenvalue of a real matrix.

## II. PRELIMINARIES

Consider the following discrete-time uncertain system with time delay described by

$$\Sigma : \begin{cases} x(k+1) = A(k)x(k) + B(k)x(k-\tau) + C(k)u(k), \\ \qquad k \in \mathbb{N}^+ \\ x(\theta) = \varphi(\theta), \ \theta \in [-\tau, 0]. \end{cases}$$

$$(1)$$

where $x(k) = [x_1(k), x_2(k), \cdots, x_n(k)]^T \in \mathbb{R}^n$ denotes the state vector; $u(k) = [u_1(k), u_2(k), \cdots, u_n(k)]^T \in \mathbb{R}^n$ is the control input vector; Positive integer $\tau$ represents the transmission delay; $\varphi(\cdot)$ is vector-valued initial function and $\|\varphi\|_\tau$ is defined by $\|\varphi\|_\tau = \sup_{i \in \mathbb{N}[-\tau, 0]} \|x(i)\|$; $A(k) = A + \triangle A(k)$, $B(k) = B + \triangle B(k)$, $C(k) = C + \triangle C(k)$; $A, B, C \in \mathbb{R}^{n \times n}$ represent the weighting matrices; $\Delta A(k), \Delta B(k), \Delta C(k)$ denote the time-varying structured uncertainties which are of the

following form:

$$[\Delta A(k) \quad \Delta B(k) \quad \Delta C(k)] = GF(k)[E_a \quad E_b \quad E_c],$$

where $G, E_a, E_b, E_c$ are known real constant matrices of appropriate dimensions; $F(k)$ is unknown time-varying matrix function satisfying $F^T(k)F(k) \leq I, \forall k \in \mathbb{N}^+$.

The concerned problem is to find a pair of state feedback gain matrices $K_1, K_2$ in the control law

$$u(k) = K_1 x(k) + K_2 x(k - \tau), \quad (2)$$

so as to ensure stabilization of the closed-loop delayed system (1).

To obtain our main results, we need introduce the following definitions and lemmas.

*Definition 2.1:* The control system (1) with $u(k) = 0$ is said to be exponentially stable, if there exist two positive scalars $\alpha > 0$ and $0 < \beta < 1$ such that

$$\|x(k)\| \leq \alpha \cdot \beta^k \sup_{s \in \mathbb{N}[-\tau,0]} \|x(s)\|, \forall k \geq 0.$$

*Definition 2.2:* The control system (1) is said to be exponentially stabilized by the local control law (2), if there exist two positive scalars $\alpha > 0$ and $0 < \beta < 1$ such that

$$\|x(k)\| \leq \alpha \cdot \beta^k \sup_{s \in \mathbb{N}[-\tau,0]} \|x(s)\|, \forall k \geq 0.$$

*Lemma 2.1:* [13](Tchebychev Inequality) For any given vectors $v_i \in \mathbb{R}^n, i = 1, 2, \cdots, n$, the following inequality holds:

$$[\sum_{i=1}^n v_i]^T [\sum_{i=1}^n v_i] \leq n \sum_{i=1}^n v_i^T v_i.$$

*Lemma 2.2:* [14] Given constant symmetric matrices $\Sigma_1, \Sigma_2, \Sigma_3$, where $\Sigma_1^T = \Sigma_1$ and $0 < \Sigma_2 = \Sigma_2^T$, then $\Sigma_1 + \Sigma_3^T \Sigma_2^{-1} \Sigma_3 < 0$ if and only if

$$\begin{bmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & -\Sigma_2 \end{bmatrix} < 0 \; or \; \begin{bmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^T & \Sigma_1 \end{bmatrix} < 0.$$

*Lemma 2.3:* [1] Let $N$ and $E$ be real constant matrices with appropriate dimensions, matrix $F(k)$ satisfying $F^T(k)F(k) \leq I$, then, for any $\epsilon > 0$, $EF(k)N + N^T F^T(k)E^T \leq \epsilon^{-1} EE^T + \epsilon N^T N$.

For designing the linear feedback controller $u(k) = K_1 x(k) + K_2 x(k - \tau)$ such that the closed-loop system (1) is exponentially stabilized, we first consider the nominal system $\Sigma_0$ of $\Sigma$ defined by

$$\Sigma_0 : \begin{cases} x(k+1) = Ax(k) + Bx(k - \tau) + Cu(k), \\ \qquad k \in \mathbb{N}^+ \\ x(\theta) = \varphi(\theta), \; \theta \in [-\tau, 0]. \end{cases} \quad (3)$$

Substituting (2) into system (3) yields a closed-loop system as follows:

$$\Sigma_0 : \begin{cases} x(k+1) = (A + CK_1)x(k) + (B + CK_2)x(k - \tau), \\ \qquad k \in \mathbb{N}^+ \\ x(\theta) = \varphi(\theta), \; \theta \in [-\tau, 0]. \end{cases}$$
$$(4)$$

Then, we can obtain the following exponential stability and stabilization results.

## III. STABILITY ANALYSIS

This section considers the stability of the nominal system (3) with $u(k) = 0$ and the robust stability of system (3) with $u(k) = 0$.

### A. Stability

For the convenance of the following proof, set

$$S_1^T = \begin{bmatrix} I & 0 & I & 0 \\ 0 & I & -I & 0 \\ I & 0 & I & I \\ 0 & I & 0 & -I \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, S_2^T = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

$$\widetilde{S}_1^T = \begin{bmatrix} A_1^T & 0 & A_1^T & 0 \\ 0 & A_1^T & -I & 0 \\ I & 0 & I & I \\ 0 & I & 0 & -I \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

*Theorem 3.1:* For given positive integer $\tau > 0$, the delayed system (3) with $u(t) = 0$ is exponentially stable, if there exist positive-definite matrices $Q, R, P$, positive-definite diagonal matrices $Z_1, Z_2$, and arbitrary matrices $M, N, F, P_1, P_2, G_1, G_2$ of appropriate dimensions, such that the following LMI holds:

$$\Xi_1 = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} \\ * & \Xi_{22} & \Xi_{23} & \Xi_{24} & \Xi_{25} & \Xi_{26} \\ * & * & \Xi_{33} & \Xi_{34} & \Xi_{35} & \Xi_{36} \\ * & * & * & \Xi_{44} & \Xi_{45} & \Xi_{46} \\ * & * & * & * & \Xi_{55} & 0 \\ * & * & * & * & * & \Xi_{66} \end{bmatrix} < 0, \quad (5)$$

where $Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} \\ * & Q_{22} & Q_{23} & Q_{24} \\ * & * & Q_{33} & Q_{34} \\ * & * & * & Q_{44} \end{bmatrix},$

$\Xi_{11} = Q_{33} + Q_{13} + Q_{13}^T + P + (1 + \tau)Z_1 + M(A - I)$
$+ (A - I)^T M^T - F^T - F + P_1 + P_1^T + G_1 + G_1^T,$

$\Xi_{12} = Q_{23}^T - Q_{13} - Q_{33} + F + MB - P_1$
$+ P_2^T - G_1 + G_2^T,$

$\Xi_{13} = Q_{11} + Q_{33} + Q_{13} + Q_{13}^T + Q_{14} + Q_{34} - M$
$+ (A - I)^T N^T + P_1^T + (1 + \tau)Z_1,$

$\Xi_{14} = Q_{12} + Q_{23}^T) - Q_{14} - Q_{34},$

$\Xi_{15} = Q_{33}, \Xi_{16} = Q_{34} + F - P_1 + P_2^T - G_1 + G_2^T,$

$\Xi_{22} = -Q_{23}^T - Q_{23} + Q_{33} - P - P_2 - P_2^T - G_2 - G_2^T,$

$\Xi_{23} = Q_{12}^T - Q_{13}^T + Q_{23} - Q_{33} + Q_{24}$
$- Q_{34} + B^T N^T - P_1^T,$

$\Xi_{24} = Q_{22} - Q_{23}^T - Q_{24} + Q_{34}, \Xi_{25} = Q_{33},$

$\Xi_{26} = Q_{34} - P_2 - P_2^T - G_2 - G_2^T,$

$\Xi_{33} = Q_{11} + Q_{33} + Q_{44} + Q_{13} + Q_{13}^T + Q_{14} + Q_{14}^T$

$+ Q_{34} + Q_{34}^T + \tau Z_2 + (1 + \tau)Z_1 + R - N - N^T,$

$\Xi_{34} = Q_{12} + Q_{23}^T + Q_{24}^T - Q_{14} - Q_{34} - Q_{44},$

$\Xi_{35} = Q_{13} + Q_{33} + Q_{34}^T,$

$\Xi_{36} = Q_{14} + Q_{34} + Q_{44} - P_1,$

$\Xi_{44} = Q_{22} - Q_{24} - Q_{24}^T + Q_{44} - R,$

$\Xi_{45} = Q_{23} - Q_{34}^T, \ \Xi_{46} = Q_{24} - Q_{44}^T$

$\Xi_{55} = -(1 + \tau)^{-1}Z_1,$

$\Xi_{66} = -\tau^{-1}Z_2 - P_2 - P_2^T - G_2 - G_2^T.$

*Proof.* Constructing an augmented Lyapunov-Krasovskii function candidate as follows:

$$V(k) = V_1(k) + V_2(k) + V_3(k),$$

where

$$V_1(k) = \widetilde{X}^T(k)Q\widetilde{X}(k),$$

$$\widetilde{X}^T(k) = [x^T(k), x^T(k-\tau), \sum_{i=k-\tau}^{k} x^T(i), \sum_{i=k-\tau}^{k-1} \eta^T(i)],$$

$$V_2(k) = \sum_{i=k-\tau}^{k-1} x^T(i)Px(i) + \sum_{i=k-\tau}^{k-1} \eta^T(i)R\eta(i),$$

$$V_3(k) = \sum_{j=k-\tau}^{k} \sum_{i=j}^{k} x^T(i)Z_1x(i) + \sum_{j=k-\tau}^{k-1} \sum_{i=j}^{k-1} \eta^T(i)Z_2\eta(i),$$

where $\eta(k) = x(k+1) - x(k)$. Set $X^T(k) = [x^T(k), x^T(k-\tau), \eta^T(k), \eta^T(k-\tau), \sum_{i=k-\tau}^{k} x^T(i), \sum_{i=k-\tau}^{k-1} \eta^T(i)]$, Define $\Delta V(k) = V(k+1) - V(k)$, then along the solution of system (3) we can obtain that

$$\Delta V_1(k) = \widetilde{X}^T(k+1)Q\widetilde{X}(k+1) - \widetilde{X}^T(k)Q\widetilde{X}(k)$$
$$= X^T(k)(S_1^T Q S_1 - S_2^T Q S_2)X(k), \quad (6)$$

$$\Delta V_2(k) = x^T(k)Px(k) - x^T(k-\tau)Px(k-\tau)$$
$$+ \eta^T(k)R\eta(k) - \eta^T(k-\tau)R\eta(k-\tau). \quad (7)$$

From lemma 2.1 we have

$$\Delta V_3(k)) = \sum_{j=k+1-\tau}^{k+1} \sum_{i=j}^{k+1} x^T(i)Z_1x(i) - \sum_{j=k-\tau}^{k} \sum_{i=j}^{k} x^T(i)Z_1x(i)$$
$$+ \sum_{j=k+1-\tau}^{k} \sum_{i=j}^{k} \eta^T(i)Z_2\eta(i) - \sum_{j=k-\tau}^{k-1} \sum_{i=j}^{k-1} \eta^T(i)Z_2\eta(i)$$
$$= \sum_{j=k-\tau}^{k} \sum_{i=j+1}^{k+1} x^T(i)Z_1x(i) - \sum_{j=k-\tau}^{k} \sum_{i=j}^{k} x^T(i)Z_1x(i)$$
$$+ \sum_{j=k-\tau}^{k-1} \sum_{i=j+1}^{k} \eta^T(i)Z_2\eta(i) - \sum_{j=k-\tau}^{k-1} \sum_{i=j}^{k-1} \eta^T(i)Z_2\eta(i)$$
$$= \sum_{j=k-\tau}^{k} [x^T(k+1)Z_1x(k+1) - x^T(j)Z_1x(j)]$$
$$+ \sum_{j=k-\tau}^{k-1} [\eta^T(k)Z_2\eta(k) - \eta^T(j)Z_2\eta(j)]$$
$$= (1+\tau)x^T(k+1)Z_1x(k+1) - \sum_{j=k-\tau}^{k} x^T(j)Z_1x(j)$$
$$+ \tau\eta^T(k)Z_2\eta(k) - \sum_{j=k-\tau}^{k-1} \eta^T(j)Z_2\eta(j)$$
$$= (1+\tau)x^T(k+1)Z_1x(k+1)$$
$$- \sum_{j=k-\tau}^{k} (\sqrt{Z_1}x(j))^T \sqrt{Z_1}x(j)$$
$$+ \tau\eta^T(k)Z_2\eta(k) - \sum_{j=k-\tau}^{k-1} (\sqrt{Z_2}\eta(j))^T \sqrt{Z_2}\eta(j)$$
$$\leq (1+\tau)x^T(k+1)Z_1x(k+1)$$
$$- \frac{1}{1+\tau}[\sum_{j=k-\tau}^{k} x(j)]^T Z_1[\sum_{j=k-\tau}^{k} x(j)]$$
$$+ \tau\eta^T(k)Z_2\eta(k) - \frac{1}{\tau}[\sum_{j=k-\tau}^{k-1} \eta(j)]^T Z_2[\sum_{j=k-\tau}^{k-1} \eta(j)]. \quad (8)$$

On the other hand, for arbitrary matrices $M, N, F$ of appropriate dimensions, we have

$$0 = 2x^T(k)F[x(k-\tau) + \sum_{i=k-\tau}^{k-1} \eta(i) - x(k)],$$

$$0 = 2\eta^T(k)N[(A-I)x(k) + Bx(k-\tau) - \eta(k)],$$

$$0 = 2x^T(k)M[(A-I)x(k) + Bx(k-\tau) - \eta(k)]. \quad (9)$$

Since $x(k) - \sum_{i=k-\tau}^{k-1} \eta(i) - x(k-\tau) = 0$, for arbitrary matrices $P_1, P_2, G_1, G_2$ of appropriate dimensions, we can obtain that

$$0 = 2\widetilde{X}_1^T \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \widetilde{X}_2, \ 0 = \overline{X}_1^T \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \widetilde{X}_2, \quad (10)$$

where $\widetilde{X}_1^T(k) = [\eta^T(k) + x^T(k), \sum_{i=k-\tau}^{k-1} \eta^T(i) + x^T(k-\tau)]$, $\widetilde{X}_2^T = x^T(k) - \sum_{i=k-\tau}^{k-1} \eta^T(i) - x^T(k-\tau)$, $\overline{X}_1^T = [x^T(k), \sum_{i=k-\tau}^{k-1} \eta^T(i) + x^T(k-\tau)]$. Combining (6)-(10), we get

$$\Delta V(k) \leq X^T(k)\Xi_1 X(k). \quad (11)$$

If the LMI (5) holds, it follows that there exists a sufficient small positive scalar $\varepsilon > 0$ such that

$$\Delta V(k) \leq -\varepsilon\|x(k)\|^2. \quad (12)$$

On the other hand, one can easily get that

$$V(k) \leq \alpha_1\|x(k)\|^2 + \alpha_2 \sum_{i=k-\tau}^{k-1} \|x(i)\|^2, \qquad (13)$$

where $\alpha_1 = (1+3\tau)\lambda_{max}(Q) + \lambda_{max}(R) + \tau\lambda_{max}(Z_2)$, $\alpha_2 = (1+3\tau)\lambda_{max}(Q) + \lambda_{max}(P) + 2\lambda_{max}(R) + (1+\tau)\lambda_{max}(Z_1) + 2\tau\lambda_{max}(Z_2)$,

For any $\theta > 1$, it follows from (13) that

$$\begin{aligned}
\theta^{j+1}V(j+1) - \theta^j V(j) &= \theta^{j+1}\Delta V(j) + \theta^j(\theta-1)V(j) \\
&\leq \theta^j[-\varepsilon\theta\|x(j)\|^2 + (\theta-1)\alpha_1\|x(j)\|^2 \\
&\quad + (\theta-1)\alpha_2 \sum_{i=j-\tau}^{j-1} \|x(i)\|^2]. \qquad (14)
\end{aligned}$$

Summing up both sides of (14) from 0 to $k-1$ we can obtain

$$\begin{aligned}
\theta^k V(k) - V(0) &\leq [\alpha_1(\theta-1) - \varepsilon\theta] \sum_{j=0}^{k-1} \theta^j\|x(j)\|^2 \\
&\quad + \alpha_2(\theta-1) \sum_{j=0}^{k-1} \sum_{i=j-\tau}^{j-1} \theta^{j-1}\|x(i)\|^2 \\
&\leq \mu_1(\theta) \sup_{j\in\mathbb{N}[-\tau,0]} \|x(j)\|^2 \\
&\quad + \mu_2(\theta) \sum_{j=0}^{k} \theta^k\|x(j)\|^2, \qquad (15)
\end{aligned}$$

where $\mu_1(\theta) = \alpha_2(\theta-1)\tau^2\theta^\tau$, $\mu_2(\theta) = \alpha_2(\theta-1)\tau\theta^\tau + \alpha_1(\theta-1) - \varepsilon\theta$. Since $\mu_2(1) = -\varepsilon < 0$, there must exist a positive $\theta_0 > 1$ such that $\mu_2(\theta_0) < 0$. Thus, we have

$$\begin{aligned}
V(k) &\leq \mu_1(\theta_0)(\frac{1}{\theta_0})^k \sup_{j\in\mathbb{N}[-\tau,0]} \|x(j)\|^2 + (\frac{1}{\theta_0})^k V(0) \\
&= (\frac{1}{\theta_0})^k(\mu_1(\theta_0)\|\varphi\|_\tau^2 + V(0)), \forall k \geq 1. \qquad (16)
\end{aligned}$$

Set $\varpi = \alpha_1 + \tau\alpha_2$, we can obtain

$$V(0) \leq \varpi \sup_{j\in\mathbb{N}[-\tau,0]} \|x(j)\|^2 \text{ and } V(k) \geq \lambda_{min}(Q)\|x(k)\|^2. \quad (17)$$

It follows that $\|x(k)\| \leq \alpha \cdot \beta^k \sup_{j\in\mathbb{N}[-\tau_M,0]} \|x(j)\|$, where $\beta = (\theta_0)^{-1/2}$, $\alpha = \sqrt{\frac{\mu_1(\theta_0)+\varpi}{\lambda_{min}(Q)}}$. By Definition 2.1, system (3) is exponentially stable, which complete the proof of Theorem 3.1.

*Remark 1.* In previous literature, the disposal for items $\sum_{j=k-\tau}^k x^T(j)Z_1x(j)$, $\sum_{j=k-\tau}^{k-1} \eta^T(j)Z_2\eta(j)$ is complex (see [15], [16]). However, by applying Tchebychev inequality, the method used in this paper is relatively more simple.

*Remark 2.* In Theorem 3.1, we proposed $V_1$ which takes $x^T(k), x^T(k-\tau), \sum_{i=k-\tau}^k x^T(i), \sum_{i=k-\tau}^{k-1} \eta^T(i)$ as augmented state. The proposed augmented Lyapunov function $V_1$ may reduce the conservatism of the delay-dependent result. It should pointed out that $X^T(k)$ in (11) takes $x^T(k), x^T(k-\tau), \eta^T(k), \eta^T(k-\tau), \sum_{i=k-\tau}^k x^T(i), \sum_{i=k-\tau}^{k-1} \eta^T(i)$ as state vector, which is different from [10].

*Remark 3.* Free-weighting matrices $M, N, F, P_1, P_2, G_1, G_2$ are introduced so as to reduce the conservatism of the delay-dependent result further. On the other hand, it is worth pointing out that nineteen free-weighting matrices were introduced in [10], which increase the computational demand. In our paper, there are only seven free-weighting matrices in Theorem 3.1, but it is less conservative than the results in [10](More details see example 1, and example2).

Decomposing weighting matrix $A$ as $A = A_1 + A_2$, we can obtain the following stability result.

*Theorem 3.2:* For given positive integer $\tau > 0$, the delayed system (3) with $u(t) = 0$ is exponentially stable, if there exist positive-definite matrices $Q, R, P$, positive-definite diagonal matrices $Z_1, Z_2$, and arbitrary matrices $M, N, F, P_1, P_2, G_1, G_2$ of appropriate dimensions, such that the following LMI holds:

$$\Xi_2 = \begin{bmatrix}
\widetilde{\Xi}_{11} & \widetilde{\Xi}_{12} & \widetilde{\Xi}_{13} & \widetilde{\Xi}_{14} & \widetilde{\Xi}_{15} & \widetilde{\Xi}_{16} \\
* & \widetilde{\Xi}_{22} & \widetilde{\Xi}_{23} & \widetilde{\Xi}_{24} & \widetilde{\Xi}_{25} & \widetilde{\Xi}_{26} \\
* & * & \Xi_{33} & \Xi_{34} & \Xi_{35} & \Xi_{36} \\
* & * & * & \Xi_{44} & \Xi_{45} & \Xi_{46} \\
* & * & * & * & \Xi_{55} & \Xi_{56} \\
* & * & * & * & * & \Xi_{66}
\end{bmatrix} < 0, \quad (18)$$

where $Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} \\ * & Q_{22} & Q_{23} & Q_{24} \\ * & * & Q_{33} & Q_{34} \\ * & * & * & Q_{44} \end{bmatrix}$,

$\widetilde{\Xi}_{11} = A_1^T(Q_{11}+Q_{33}+Q_{13}+Q_{13}^T)A_1 - Q_{11} + P + (1+\tau)Z_1$
$+MA_2 + A_2^T M^T - A_1^T F^T - FA_1 + (P_1+G_1)A_1 + A_1^T(P_1^T+G_1^T)$,

$\widetilde{\Xi}_{12} = A_1^T(Q_{12}+Q_{23}^T)A_1 - Q_{12} - A_1^T(Q_{13}+Q_{33})$
$+F + MB - P_1 - G_1 + A_1^T(P_2^T+G_2^T)$,

$\widetilde{\Xi}_{13} = A_1^T(Q_{11}+Q_{33}+Q_{13}+Q_{13}^T+Q_{14}+Q_{34})$
$-M + A_2^T N^T + A_1^T P_1^T + (1+\tau)Z_1$,

$\widetilde{\Xi}_{14} = A_1^T(Q_{12}+Q_{23}^T-Q_{14}-Q_{34})$,

$\widetilde{\Xi}_{15} = A_1^T(Q_{13}+Q_{33}) - Q_{13}$
$+(P_1+G_1)(I-A_1) - F(I-A_1)$,

$\widetilde{\Xi}_{16} = A_1^T(Q_{14}+Q_{34}) - Q_{14}$
$+F - P_1 - G_1 + A_1^T(P_2^T+G_2^T)$,

$\widetilde{\Xi}_{22} = A_1^T Q_{22}A_1 - Q_{23}^T A_1 - A_1^T Q_{23}$
$+Q_{33} - Q_{22} - P - P_2 - P_2^T - G_2 - G_2^T$,

$\widetilde{\Xi}_{23} = A_1^T(Q_{12}^T+Q_{23}+Q_{24}) - Q_{13}^T$
$-Q_{33} - Q_{34} + B^T N^T - P_1^T$,

$\widetilde{\Xi}_{24} = A_1^T(Q_{22}-Q_{24}) - Q_{23}^T + Q_{34}$,

$\widetilde{\Xi}_{25} = A_1^T Q_{23} - Q_{33} - Q_{23} + (P_2+G_2)(I-A_1)$,

$\widetilde{\Xi}_{26} = A_1^T Q_{24} - Q_{34} - Q_{24} - P_2 - P_2^T - G_2 - G_2^T$,

$\widetilde{\Xi}_{35} = Q_{13} + Q_{33} + Q_{34}^T + P_1(I-A_1)$,

$\widetilde{\Xi}_{56} = (I-A_1)^T(P_2+G_2)^T$.

*Proof.* Constructing an augmented Lyapunov-Krasovskii function as the same in Theorem 3.1. Set $\eta(k) = x(k+1) - A_1x(k)$, we have

$$\sum_{i=k-\tau}^{k-1} \eta(i) = A_1x(k) - x(k-\tau) + (I-A_1)\sum_{i=k-\tau}^{k} x(i).$$

$$\begin{aligned}
\Delta V_1(k) &= \widetilde{X}^T(k+1)Q\widetilde{X}(k+1) - \widetilde{X}^T(k)Q\widetilde{X}(k) \\
&= X^T(k)(\widetilde{S}_1^T Q\widetilde{S}_1 - S_2^T QS_2)X(k), \qquad (19)
\end{aligned}$$

$$\begin{aligned}
0 &= 2x^T(k)F[\sum_{i=k-\tau}^{k-1} \eta(i) - A_1x(k) \\
&\quad + x(k-\tau) - (I-A_1)\sum_{i=k-\tau}^{k} x(i)],
\end{aligned}$$

$$0 = 2\eta^T(k)N[A_2x(k) + Bx(k-\tau) - \eta(k)],$$

$$0 = 2x^T(k)M[A_2x(k) + Bx(k-\tau) - \eta(k)], \qquad (20)$$

$$0 = \widetilde{X}_1^T \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \widetilde{X}_2', \quad 0 = \overline{X}_1^T \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \widetilde{X}_2', \qquad (21)$$

where $\widetilde{X}_2'^T = x^T(k)A_1^T - x^T(k-\tau) + \sum_{i=k-\tau}^{k} x^T(i)(I - A_1)^T - \sum_{i=k-\tau}^{k-1} \eta^T(i)$. Similar to the proof of Theorem 3.1, one can easily derives this result, which is omitted here.

*Remark 4.* The decomposition of matrix $A = A_1 + A_2$ makes the conservatism of delay-dependent result reduced further (details see example 1, 2). But, what is the optimal decomposition of matrix $A$ is an important and interesting problem need to be solved.

### B. Robust Stability

*Theorem 3.3:* For given positive integer $\tau > 0$, the delayed system (1) with $u(t) = 0$ is robustly and exponentially stable, if there exist positive-definite matrices $Q, R, P$, positive-definite diagonal matrices $Z_1, Z_2$, arbitrary matrices $M, N, F, P_1, P_2, G_1, G_2$ of appropriate dimensions, and positive scalar $\epsilon > 0$, such that the following LMI holds:

$$\Xi_3 = \begin{bmatrix} \Xi_1 & \xi_1 & \epsilon\xi_2^T \\ * & -\epsilon I & 0 \\ * & * & -\epsilon I \end{bmatrix} < 0, \qquad (22)$$

where $\xi_1^T = [G^T M^T, 0, G^T N^T, 0, 0, 0]$, $\xi_2 = [E_a, E_b, 0, 0, 0, 0]$.

*Proof.* Replacing $A, B$ in inequality (5) with $A + GF(t)E_a$, and $B + GF(t)E_b$, respectively. Inequality (22) for system (1) is equivalent to $\Xi_1 + \xi_1 F(t)\xi_2 + \xi_2^T F^T(t)\xi_1^T < 0$. From lemma 2.2 and lemma 2.3, one can easily obtain this result, which complete the proof. Similarly, we have

*Theorem 3.4:* For given positive integer $\tau > 0$, the delayed system (1) with $u(t) = 0$ is robustly and exponentially stable, if there exist positive-definite matrices $Q, R, P$, positive-definite diagonal matrices $Z_1, Z_2$, arbitrary matrices $M, N, F, P_1, P_2, G_1, G_2$ of appropriate dimensions, and positive scalar $\epsilon > 0$, such that the following LMI holds:

$$\Xi_4 = \begin{bmatrix} \Xi_2 & \xi_1 & \epsilon\xi_2^T \\ * & -\epsilon I & 0 \\ * & * & -\epsilon I \end{bmatrix} < 0, \qquad (23)$$

*Proof.* Replacing $A_2, B$ in inequality (18) with $A_2 + GF(t)E_a$, and $B + GF(t)E_b$, respectively. Inequality (23) for system (1) is equivalent to $\Xi_2 + \xi_1 F(t)\xi_2 + \xi_2^T F^T(t)\xi_1^T < 0$. From lemma 2.2 and lemma 2.3, one can easily obtain this result, which complete the proof.

### IV. STATE FEEDBACK STABILIZATION

Based on the results obtained in section 3, we can easily give out the design method for the state feedback controller $u(k)$ as follows.

### A. Stabilization

*Theorem 4.1:* For given positive integer $\tau > 0$, the control law (2) stabilizes system (3), if there exist positive-definite matrices $Q, R, P$, positive-definite diagonal matrices $Z_1, Z_2$, and arbitrary matrices $F, P_1, P_2, G_1, G_2$ of appropriate dimensions, such that the following LMI holds:

$$\Xi_5 = \begin{bmatrix} \Xi_{11}' & \Xi_{12}' & \Xi_{13}' & \Xi_{14} & \Xi_{15} & \Xi_{16} \\ * & \Xi_{22} & \Xi_{23}' & \Xi_{24} & \Xi_{25} & \Xi_{26} \\ * & * & \Xi_{33}' & \Xi_{34} & \Xi_{35} & \Xi_{36} \\ * & * & * & \Xi_{44} & \Xi_{45} & \Xi_{46} \\ * & * & * & * & \Xi_{55} & 0 \\ * & * & * & * & * & \Xi_{66} \end{bmatrix} < 0, \qquad (24)$$

where

$$\Xi_{11}' = Q_{33} + Q_{13} + Q_{13}^T + P + (1+\tau)Z_1$$
$$+(A-I)+(A-I)^T+CK_1+K_1^T C^T-F^T-F+P_1+P_1^T+G_1+G_1^T,$$

$$\Xi_{12}' = Q_{23}^T - Q_{13} - Q_{33} + F + B + CK_2$$
$$-P_1 + P_2^T - G_1 + G_2^T,$$

$$\Xi_{13}' = Q_{11} + Q_{33} + Q_{13} + Q_{31}^T + Q_{14} + Q_{34}^T - I$$
$$+(A - I)^T + K_1^T C^T + P_1^T + (1+\tau)Z_1,$$

$$\Xi_{23}' = Q_{12}^T - Q_{13}^T + Q_{23} - Q_{33} + Q_{24}$$
$$-Q_{34} + B^T + K_2^T C^T - P_1^T,$$

$$\Xi_{33}' = Q_{11} + Q_{33} + Q_{44} + Q_{13} + Q_{13}^T + Q_{14}$$
$$+Q_{14}^T + Q_{34} + Q_{34}^T + \tau Z_2 + (1+\tau)Z_1 + R - 2I.$$

*Proof.* Similar to the proof of Theorem 3.1, one can easily obtain this result, where is omitted here. Similarly, we have

*Theorem 4.2:* For given positive integer $\tau > 0$, the control law (2) stabilizes system (3), if there exist positive-definite matrices $Q, R, P$, positive-definite diagonal matrices $Z_1, Z_2$, and arbitrary matrices $F, P_1, P_2, G_1, G_2$ of appropriate dimensions, such that the following LMI holds:

$$\Xi_6 = \begin{bmatrix} \widetilde{\Xi}_{11}' & \widetilde{\Xi}_{12}' & \widetilde{\Xi}_{13}' & \widetilde{\Xi}_{14} & \widetilde{\Xi}_{15} & \widetilde{\Xi}_{16} \\ * & \widetilde{\Xi}_{22} & \widetilde{\Xi}_{23}' & \widetilde{\Xi}_{24} & \widetilde{\Xi}_{25} & \widetilde{\Xi}_{26} \\ * & * & \widetilde{\Xi}_{33}' & \widetilde{\Xi}_{34} & \widetilde{\Xi}_{35} & \widetilde{\Xi}_{36} \\ * & * & * & \widetilde{\Xi}_{44} & \widetilde{\Xi}_{45} & \widetilde{\Xi}_{46} \\ * & * & * & * & \widetilde{\Xi}_{55} & \widetilde{\Xi}_{56} \\ * & * & * & * & * & \widetilde{\Xi}_{66} \end{bmatrix} < 0, \qquad (25)$$

where

$$\widetilde{\Xi}_{11}' = A_1^T(Q_{11} + Q_{33} + Q_{13} + Q_{13}^T)A_1 - Q_{11}$$
$$+P + (1+\tau)Z_1 + A_2 + A_2^T + CK_1 + K_1^T C^T - A_1^T F^T$$
$$-FA_1 + (P_1 + G_1)A_1 + A_1^T(P_1^T + G_1^T),$$

$$\widetilde{\Xi}_{12}' = A_1^T(Q_{12} + Q_{23}^T)A_1 - Q_{12} - A_1^T(Q_{13}$$
$$+Q_{33}) + F + B + CK_2 - P_1 - G_1 + A_1^T(P_2^T + G_2^T),$$

$$\widetilde{\Xi}_{13}' = A_1^T(Q_{11} + Q_{33} + Q_{13} + Q_{31}^T + Q_{14} + Q_{34}^T)$$
$$-I + A_2^T + K_1^T C^T + A_1^T P_1^T + (1+\tau)Z_1,$$

$$\widetilde{\Xi}_{23}' = A_1^T(Q_{12}^T+Q_{23}+Q_{24})-Q_{13}^T-Q_{33}-Q_{34}+B^T+K_2^T C^T-P_1^T,$$

*Remark 5.* From inequality (24) or (25), the feedback gain matrices $K_1, K_2$ can be solved through Matlab LMI Toolbox directly, which avoid the computation of inverse matrix like in [10]. This make the design for controller $u(k) = K_1x(k) + K_2x(k - \tau)$ become more easy. On the other hand, from Lemma 2.3, 2.4, similar to the proof of Theorem 3.3, and Theorem 3.4, we can obtain the following results.

### B. Robust Stabilization

*Theorem 4.3:* For given positive integer $\tau > 0$, the control law (2) stabilizes the uncertain system (1), if there exist positive-definite matrices $Q, R, P$, positive-definite diagonal matrices $Z_1, Z_2$, arbitrary matrices $F, P_1, P_2, G_1, G_2$ of appropriate dimensions, and positive scalar $\epsilon > 0$, such that the following LMI holds:

$$\Xi_7 = \begin{bmatrix} \Xi_5 & \widetilde{\xi}_1 & \epsilon\widetilde{\xi}_2^T \\ * & -\epsilon I & 0 \\ * & * & -\epsilon I \end{bmatrix} < 0, \qquad (26)$$

where $\widetilde{\xi}_1^T = [G^T, 0, G^T, 0, 0, 0]$, $\widetilde{\xi}_2 = [E_a + E_c K_1, E_b + E_c K_2, 0, 0, 0, 0]$.

*Theorem 4.4:* For given positive integer $\tau > 0$, the control law (2) stabilizes the uncertain system (1), if there exist positive-definite matrices $Q, R, P$, positive-definite diagonal matrices $Z_1, Z_2$, arbitrary matrices $F, P_1, P_2, G_1, G_2$ of appropriate dimensions, and positive scalar $\epsilon > 0$, such that the following LMI holds:

$$\Xi_8 = \begin{bmatrix} \Xi_6 & \widetilde{\xi}_1 & \epsilon\widetilde{\xi}_2^T \\ * & -\epsilon I & 0 \\ * & * & -\epsilon I \end{bmatrix} < 0. \qquad (27)$$

TABLE I
ALLOWABLE UPPER BOUNDS $\tau$ FOR THE STABILITY (EXAMPLE 1)

| $\alpha$ | 0.9 | 1.0 | 1.1 | 1.2 | 1.3 |
|---|---|---|---|---|---|
| By [8] | 1 | 1 | 1 | 0 | 0 |
| By [9] | 4 | 4 | 3 | 2 | 1 |
| By [10] | 4 | 4 | 5 | 3 | 2 |
| By Theorem 3.1 | $\tau_M$ | $\tau_M$ | $\tau_M$ | 7(7.675) | 4(4.082) |
| By Theorem 3.2 | $\tau_M$ | $\tau_M$ | $\tau_M$ | 11(11.549) | 4(4.381) |

| $\alpha$ | 1.374 | 1.401 | 2.943 |
|---|---|---|---|
| By [8] | 0 | failed | failed |
| By [9] | 0 | failed | failed |
| By [10] | 2 | failed | failed |
| By Theorem 3.1 | 3(3.061) | failed | failed |
| By Theorem 3.2 | 3 (3.143) | 2(2.721) | 0(0.681) |

TABLE II
ALLOWABLE UPPER BOUNDS $\tau$ FOR THE ROBUST STABILITY (EXAMPLE 2)

| $\beta$ | 0.05 | 0.1 | 0.15 | 0.3 |
|---|---|---|---|---|
| By [11] | 2 | 2 | 1 | failed |
| By [10] | 4 | 4 | 3 | failed |
| By Theorem 3.3 | 39(39.314) | 15(15.944) | 9(9.155) | 3(3.876) |
| By Theorem 3.4 | 52(52.23) | 47(47.55) | 45(45.39) | 44(44.14) |

| $\beta$ | 0.35 | 0.40 | 0.45 |
|---|---|---|---|
| By [11] | failed | failed | failed |
| By [10] | failed | failed | failed |
| By Theorem 3.3 | 3(3.870) | 3(3.868) | 3(3.865) |
| By Theorem 3.4 | 44(44.135) | 44(44.104) | 44(44.017) |

## V. NUMERICAL EXAMPLES

In this section, two numerical examples will be presented to show the validity of the main results derived above.

*Example 1*. Consider delayed discrete-time system [10] in (3) with parameters given by

$$A = \begin{bmatrix} \alpha & 0.3 \\ -0.1 & 0.7 \end{bmatrix}, B = \begin{bmatrix} -0.4 & -0.2 \\ 0.2 & -0.1 \end{bmatrix}, u(k) = 0$$

Set $A_1 = A_2 = 0.5A$, it can be verified that LMI (5), (18) are feasible. For $\alpha = 0.9, 1.0, 1.1, 1.2, 1.3, 1.374, 1.401, 2.943$, Table 1 gives out the allowable upper bound of $\tau$, respectively, which shows that Theorem 3.1, Theorem 3.2 obtained in this paper are less conservative than the results obtained in [8]-[10].

*Example 2*. Consider delayed discrete-time system [10] in (1) with parameters given by

$$A = \begin{bmatrix} 0.8 & 0.3 \\ -0.1 & 0.7 \end{bmatrix}, B = \begin{bmatrix} -0.4 & -0.2 \\ 0.2 & -0.1 \end{bmatrix},$$

$$Ea = Eb = \beta I, G = I, u(k) = 0$$

Set $A_1 = A_2 = 0.5A$, it can be verified that LMI (22), (23) are feasible. For $\beta = 0.05, 0.1, 0.15, 0.3, 0.35, 0.40, 0.45$. Table 2 gives out the allowable upper bound of $\tau$, respectively, which shows that Theorem 3.3, Theorem 3.4 obtained in this paper are less conservative than the results obtained in [10]-[11].

*Example 3*. Consider delayed discrete-time system in (1) with parameters given by

$$A = \begin{bmatrix} 0.8 & 0.3 \\ -0.1 & 0.7 \end{bmatrix}, B = \begin{bmatrix} -0.4 & -0.2 \\ 0.2 & -0.1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

$$Ea = Eb = 0.3I, G = I, \tau = 3$$

It can be verified that the LMI (26), (27) are feasible. In views of LMI (26), the state feedback gain matrices $K_1, K_2$ is given by

$$K_1 = \begin{bmatrix} -349.5754 & -104.2675 \\ -0.4688 & -158.9516 \end{bmatrix}, K_2 = \begin{bmatrix} 134.4109 & -7.0794 \\ -21.7179 & -6.1164 \end{bmatrix}.$$

Set $A_1 = A_2 = 0.5A$, in views of LMI (27), the state feedback gain matrices $K_1, K_2$ are given by

$$K_1 = \begin{bmatrix} -319.2314 & -89.8405 \\ 13.8002 & -161.5962 \end{bmatrix}, K_2 = \begin{bmatrix} 152.1089 & 41.5243 \\ -41.5186 & 8.6483 \end{bmatrix}.$$

## VI. CONCLUSION

Combined with linear matrix inequality (LMI) technique, a new augmented Lyapunov-Krasovskii function is constructed, and some improved delay-dependent sufficient conditions ensuring exponential stability and stabilization are obtained. Based on these new conditions, some robust stability and robust stabilization results are also given. Numerical examples show that these new results obtained in this paper are less conservative that some previous results established in the literature cited therein.

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