

Variation of Uncertainty in Steady And Non-Steady Processes Of Queuing Theory

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Abstract—Probabilistic measures of uncertainty have been obtained as functions of time and birth and death rates in a queuing process. The variation of different entropy measures has been studied in steady and non-steady processes of queuing theory.

Keywords—Uncertainty, steady state, non-steady state, traffic intensity, monotonicity

I. INTRODUCTION

WE consider a simple birth-death process under the following assumptions:

(i) The probability of a birth in a small time interval Δt per unit individual of a population is $\lambda \Delta t + o(\Delta t)$, where $o(\Delta t)$ is an infinitesimal of a higher order than Δt .

(ii) The probability of death in a small time-interval Δt per unit individual of a population is $\mu \Delta t + o(\Delta t)$.

(iii) The probability of a more than one birth or death in a small time-interval Δt per unit individual of a population is $o(\Delta t)$.

Let $p_n(t)$ denotes the probability of there being n persons in the population at time t and let n_0 denotes the number of persons at time $t = 0$, then by using the theorems of total and compound probabilities, Medhi [3] gave an expression for $p_n(t)$.

In fact, if we define the probability generating function as

$$\phi(s, t) = \sum_{i=1}^n p_n(t) s^n \quad (1)$$

We get the following result,

$$\phi(s, t) = \left[\frac{(\lambda - \mu)s + \mu(x-1)}{(\lambda - \lambda x)s + (\lambda x - 1)} \right]^{n_0}, \lambda \neq \mu \quad (2)$$

$$= \left[\frac{\lambda t - (\lambda t - 1)s}{1 - \lambda t - \lambda t s} \right]^{n_0}, \lambda = \mu \quad (3)$$

where $x = \exp(\lambda - \mu)t$

By expanding $\phi(s, t)$ in powers of s , we can find $p_n(t)$.

In a queuing system, if λ and μ denote arrival and

service rates in steady case, that is, when $p_n(t)$ is independent of t , then we have

$$p_n = (1 - \rho)\rho^n, n = 0, 1, 2, \dots, \rho = \frac{\lambda}{\mu}$$

At any time t , the number of persons in the system can be $n = 0, 1, 2, \dots$, so that there is uncertainty about the number of persons in the system. We want to have a measure of this uncertainty and we want to discuss how this uncertainty varies w.r.t. λ, μ and t .

The most important and widely used measure of uncertainty has been introduced by Shannon [4]. It is given by

$$S(\lambda, \mu, t) = - \sum_{i=1}^n p_n(t) \log p_n(t) \quad (4)$$

Many other contributions to the literature of information measures have been made by Asadi, Ebrahimi, Hamedani, and Soofi [1], Cai, Kulkarni and Verdu [2], Chakrabarti [3], Chen [4], Garbaczewski [5], Nanda and Paul [9], Paninski and Yajima [10], Piera and Parade [11], Zyczkowski [14] etc.

In section 2, the variation of entropy in steady-state has been discussed where as in section 3, the variation of entropy in non-steady state has been discussed.

II. VARIATION OF ENTROPY IN THE STEADY STATE

In this section, we study the variation of different measures of entropy in the steady state queuing processes.

A. Variation in Sharma and Taneja's [5] entropy

We know that Sharma and Taneja's [5] measure of entropy of degree α and β is given by

$$\begin{aligned} S_1^{\alpha, \beta}(\lambda, \mu) &= \frac{1}{\beta - \alpha} \left[\sum_{n=0}^{\infty} p_n^\alpha - \sum_{n=0}^{\infty} p_n^\beta \right] \\ &= \frac{1}{\beta - \alpha} \left[\sum_{i=0}^{\infty} \rho^{n\alpha} (1 - \rho)^\alpha - \sum_{i=0}^{\infty} \rho^{n\beta} (1 - \rho)^\beta \right] \\ &= \frac{(1 - \rho)^\alpha (1 - \rho^\beta) - (1 - \rho)^\beta (1 - \rho^\alpha)}{(\beta - \alpha)(1 - \rho^\alpha)(1 - \rho^\beta)} \end{aligned} \quad (5)$$

Taking $\alpha \rightarrow \beta$, we get

$$S_1^\beta(\lambda, \mu) = \frac{(1 - \rho)^\beta \log(1 - \rho) + \rho^\beta \log \rho}{-(1 - \rho^\beta)}$$

Again taking $\beta \rightarrow 1$, we get

$$S_1(\lambda, \mu) = \frac{-(1 - \rho) \log(1 - \rho) - \rho \log \rho}{1 - \rho} \quad (6)$$

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which is a result developed by Kapur[2].

Differentiating (5) w.r.t. ρ , we get

$$\begin{aligned} \frac{d}{d\rho} S_1^{\alpha,\beta}(\lambda, \mu) &= \frac{-\alpha(1-\rho)^{\alpha-1}(1-\rho^\alpha) + \alpha(1-\rho)^\alpha \rho^{\alpha-1}}{(\beta-\alpha)(1-\rho^\alpha)^2} \\ &\quad - \frac{-\beta(1-\rho)^{\beta-1}(1-\rho^\beta) + \beta(1-\rho)^\beta \rho^{\beta-1}}{(\beta-\alpha)(1-\rho^\beta)^2} \\ &= \frac{-\alpha(1-\rho)^{\alpha-1}(1-\rho^\alpha)}{(\beta-\alpha)(1-\rho^\alpha)^2} + \frac{\beta(1-\rho)^{\beta-1}(1-\rho^\beta)}{(\beta-\alpha)(1-\rho^\beta)^2} \\ &= \frac{-\alpha(1-\rho)^{\alpha-1}(1-\rho^\alpha)(1-\rho^\beta)^2}{(\beta-\alpha)(1-\rho^\alpha)^2(1-\rho^\beta)^2} \\ &\quad + \frac{\beta(1-\rho)^{\beta-1}(1-\rho^\beta)(1-\rho^\alpha)^2}{(\beta-\alpha)(1-\rho^\alpha)^2(1-\rho^\beta)^2} \end{aligned}$$

Taking $\alpha \rightarrow \beta$, we get

$$\begin{aligned} \frac{d}{d\rho} S_1^\beta(\lambda, \mu) &= \frac{-\alpha(1-\rho)^{\alpha-1}(1-\rho^\alpha)(1-\rho^\beta)^2}{(\beta-\alpha)(1-\rho^\alpha)^2(1-\rho^\beta)^2} \\ &\quad + \frac{\beta(1-\rho)^{\beta-1}(1-\rho^\beta)(1-\rho^\alpha)^2}{(\beta-\alpha)(1-\rho^\alpha)^2(1-\rho^\beta)^2} \\ &= \frac{-2\beta(1-\rho)^{\beta-1}(1-\rho^\beta)(1-\rho^\beta)\rho^\beta \log \rho}{-(1-\rho^\beta)^4} \end{aligned}$$

Again, taking $\beta \rightarrow 1$, we get

$$\frac{d}{d\rho} S_1(\lambda, \mu) = -\frac{\log \rho}{(1-\rho)^2} > 0$$

which means that in steady-state queuing process, the uncertainty increases monotonically from 0 to ∞ as ρ increases from 0 to unity. Thus in this case, the uncertainty measure increases if the traffic intensity or utilization factor increases.

B. Variation in Kapur's [1] entropy

We know that Kapur's [1] measure of entropy of order α and β is given by

$$\begin{aligned} S_2^{\alpha,\beta}(\lambda, \mu) &= \frac{1}{\alpha + \beta - 2} \left[\sum_{n=0}^{\infty} P_n^\alpha + \sum_{n=0}^{\infty} P_n^\beta - 2 \right], \\ &\quad \alpha \neq \beta \\ &= \frac{1}{\alpha + \beta - 2} \left[\sum_{n=0}^{\infty} \rho^{n\alpha} (1-\rho)^\alpha + \sum_{n=0}^{\infty} \rho^{n\beta} (1-\rho)^\beta - 2 \right] \\ &= \frac{1}{\alpha + \beta - 2} \left[\frac{(1-\rho)^\alpha}{(1-\rho^\alpha)} - 1 + \frac{(1-\rho)^\beta}{(1-\rho^\beta)} - 1 \right] \\ &= \frac{1}{\alpha + \beta - 2} \left[\frac{(1-\rho)^\alpha - (1-\rho^\alpha)}{(1-\rho^\alpha)} + \frac{(1-\rho)^\beta - (1-\rho^\beta)}{(1-\rho^\beta)} \right] \end{aligned} \tag{7}$$

Taking $\alpha \rightarrow \beta$, we get

$$S_2^\beta(\lambda, \mu) = \frac{(1-\rho)^\beta - (1-\rho^\beta)}{(\beta-1)(1-\rho^\beta)}$$

Taking $\beta \rightarrow 1$, we get

$$S_2(\lambda, \mu) = \frac{(1-\rho) \log(1-\rho) + \rho \log \rho}{1-\rho} \tag{8}$$

which is a result developed by Kapur [2]

Differentiating (8) w.r.t. ρ , we get

$$\begin{aligned} \frac{d}{d\rho} S_2^{\alpha,\beta}(\lambda, \mu) &= \frac{-\alpha(1-\rho^\alpha)(1-\rho)^{\alpha-1} + \alpha\rho^{\alpha-1}(1-\rho)^\alpha}{(\alpha + \beta - 2)(1-\rho^\alpha)^2} \\ &\quad + \frac{-\beta(1-\rho^\beta)(1-\rho)^{\beta-1} + \beta\rho^{\beta-1}(1-\rho)^\beta}{(\alpha + \beta - 2)(1-\rho^\beta)^2} \end{aligned}$$

Taking $\alpha \rightarrow \beta$, we get

$$\begin{aligned} \frac{d}{d\rho} S_2^\beta(\lambda, \mu) &= \frac{-\beta(1-\rho^\beta)(1-\rho)^{\beta-1} + \beta\rho^{\beta-1}(1-\rho)^\beta}{(\beta-1)(1-\rho^\beta)^2} \\ &= \frac{-\beta(1-\rho)^{\beta-1}(1-\rho^{\beta-1})}{(\beta-1)(1-\rho^\beta)^2} \end{aligned}$$

Taking $\beta \rightarrow 1$, we get

$$S_2(\lambda, \mu) = -\frac{\log \rho}{(1-\rho)^2} > 0$$

Thus, we see that in this case also, the uncertainty increases monotonically from 0 to ∞ , as ρ increases from 0 to unity. Hence we conclude that in both cases, the variation of entropy remains same, that is, entropy always increases monotonically.

III. VARIATION OF ENTROPY IN THE NON-STEADY STATE

In this section, we study the variation of different measures of entropy in the non-steady processes of queuing theory.

Using (2) and (3) in (1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} P_n(t) s^n &= \frac{\lambda t}{1 + \lambda t} \left(1 - \frac{\lambda t - 1}{\lambda t} s \right) \left(1 - \frac{\lambda t}{1 + \lambda t} s \right)^{-1} \\ &= \frac{\lambda t}{1 + \lambda t} \left(1 - \frac{\lambda t - 1}{\lambda t} s \right) \sum_{n=0}^{\infty} \left(\frac{\lambda t}{1 + \lambda t} \right)^n s^n \end{aligned}$$

$$\text{so that } P_n(t) = \begin{cases} \frac{(\lambda t)^{n-1}}{(1 + \lambda t)^{n+1}}, & n \geq 1 \\ \frac{\lambda t}{1 + \lambda t}, & n = 0 \end{cases} \tag{9}$$

We now study the different variations.

A. Variation in Sharma and Taneja's [5] entropy

We know that Sharma and Taneja's [5] measure of entropy of degree α and β is given by

$$\begin{aligned} S_1^{\alpha,\beta}(\lambda, t) &= \frac{1}{\beta - \alpha} \left[\sum_{n=0}^{\infty} P_n^\alpha - \sum_{n=0}^{\infty} P_n^\beta \right] \\ &= \frac{1}{\beta - \alpha} \left[\left(\frac{\lambda t}{1 + \lambda t} \right)^\alpha + \left(\frac{(\lambda t)^0}{(1 + \lambda t)^2} \right)^\alpha + \left(\frac{(\lambda t)^1}{(1 + \lambda t)^3} \right)^\alpha + \dots \infty \right] \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{\beta-\alpha} \left[\left(\frac{\lambda t}{1+\lambda t} \right)^\beta + \left(\frac{(\lambda t)^0}{(1+\lambda t)^2} \right)^\beta + \left(\frac{(\lambda t)^1}{(1+\lambda t)^3} \right)^\beta + \dots \right] \\
 & = \frac{1+(\lambda t)^\alpha [(1+\lambda t)^\alpha - (\lambda t)^\alpha]}{(\beta-\alpha)(1+\lambda t)^\alpha [(1+\lambda t)^\alpha - (\lambda t)^\alpha]} \\
 & \quad - \frac{1+(\lambda t)^\beta [(1+\lambda t)^\beta - (\lambda t)^\beta]}{(\beta-\alpha)(1+\lambda t)^\beta [(1+\lambda t)^\beta - (\lambda t)^\beta]}
 \end{aligned}$$

$$\begin{aligned}
 & (\lambda t)^\alpha [1+(1+\lambda t)^\alpha - (\lambda t)^\alpha] [(1+\lambda t)^\beta ((1+\lambda t)^\beta - (\lambda t)^\beta)] \\
 & = \frac{-(\lambda t)^\beta [1+(1+\lambda t)^\beta - (\lambda t)^\beta] [(1+\lambda t)^\alpha ((1+\lambda t)^\alpha - (\lambda t)^\alpha)]}{(\beta-\alpha)(1+\lambda t)^\alpha (1+\lambda t)^\beta [(1+\lambda t)^\alpha - (\lambda t)^\alpha] [(1+\lambda t)^\beta - (\lambda t)^\beta]}
 \end{aligned}$$

Taking $\alpha \rightarrow \beta$, equation (11) gives

$$\begin{aligned}
 S_1^{\alpha,\beta}(\lambda, t) & \left[\begin{array}{l} (\lambda t)^\beta [(1+\lambda t)^\beta \log(1+\lambda t) \\ - (\lambda t)^\beta \log \lambda t \\ + (\lambda t)^\beta \log \lambda t [(1+\lambda t)^\beta - (\lambda t)^\beta] \end{array} \right] \left[(1+\lambda t)^\beta ((1+\lambda t)^\beta - (\lambda t)^\beta) \right] \\
 & = \frac{-[1+(\lambda t)^\beta ((1+\lambda t)^\beta - (\lambda t)^\beta)] (1+\lambda t)^\beta \left[\begin{array}{l} (1+\lambda t)^\beta \log(1+\lambda t) \\ - (\lambda t)^\beta \log \lambda t + \log \\ (1+\lambda t)((1+\lambda t)^\beta - (\lambda t)^\beta) \end{array} \right]}{-(1+\lambda t)^{2\beta} ((1+\lambda t)^\beta - (\lambda t)^\beta)^2}
 \end{aligned}$$

Taking $\beta \rightarrow 1$ (12) becomes

$$S_1(\lambda, t) = \frac{2[(1+\lambda t) \log(1+\lambda t) - \lambda t \log \lambda t]}{(1+\lambda t)} \tag{13}$$

which is a result developed by Kapur [2].

Differentiating (10) w.r.t. λt , we get

$$\begin{aligned}
 \frac{d}{d(\lambda t)} S_1^{\alpha,\beta}(\lambda, t) & = \frac{1}{\beta-\alpha} \frac{d}{d(\lambda t)} \frac{1+(\lambda t+(\lambda t)^2)^\alpha - (\lambda t)^{2\alpha}}{(1+\lambda t)^{2\alpha} - (\lambda t+(\lambda t)^2)^\alpha} \\
 & \quad - \frac{1}{\beta-\alpha} \frac{d}{d(\lambda t)} \frac{1+(\lambda t+(\lambda t)^2)^\beta - (\lambda t)^{2\beta}}{(1+\lambda t)^{2\beta} - (\lambda t+(\lambda t)^2)^\beta} \\
 & = \frac{\alpha(1+\lambda t)^{3\alpha-1} (\lambda t)^{\alpha-1} - 2\alpha(1+\lambda t)^{2\alpha-1} (\lambda t)^{2\alpha-1} + \alpha(1+\lambda t)^{\alpha-1} (\lambda t)^{3\alpha-1} - 2\alpha(1+\lambda t)^{2\alpha-1} + \alpha(1+\lambda t)^{\alpha-1} (\lambda t)^{\alpha-1} + 2\alpha(1+\lambda t)^{\alpha-1} (\lambda t)^\alpha}{(\beta-\alpha)[(1+\lambda t)^{2\alpha} - (1+\lambda t)^\alpha (\lambda t)^\alpha]^2}
 \end{aligned}$$

$$\begin{aligned}
 & \beta(1+\lambda t)^{3\beta-1} (\lambda t)^{\beta-1} - 2\beta(1+\lambda t)^{2\beta-1} (\lambda t)^{2\beta-1} \\
 & \quad + \beta(1+\lambda t)^{\beta-1} (\lambda t)^{3\beta-1} - 2\beta(1+\lambda t)^{2\beta-1} \\
 & \quad + \beta(1+\lambda t)^{\beta-1} (\lambda t)^{\beta-1} + 2\beta(1+\lambda t)^{\beta-1} (\lambda t)^\beta \\
 & = \frac{(\beta-\alpha)[(1+\lambda t)^{2\beta} - (1+\lambda t)^\beta (\lambda t)^\beta]^2}{}
 \end{aligned}$$

Taking $\alpha \rightarrow \beta \rightarrow 1$, we get

$$\frac{d}{d(\lambda t)} S_1(\lambda, t) = \frac{-2 \log \lambda t}{((1-\lambda t)^2)^2} > 0 \quad \text{iff } \lambda t < 1$$

which means that the uncertainty increases if $\lambda t < 1$ and decreases if $\lambda t \geq 1$. Also the maximum uncertainty occurs when $\lambda t = 1$ and in this case,

$$\text{Max } S_1(\lambda, t) = 2 \log 2$$

Further when $t = 0$, the uncertainty is zero. And when $t \rightarrow \infty$, we have from (13)

$$\begin{aligned}
 \lim_{t \rightarrow \infty} S_1(\lambda, t) & = \lim_{t \rightarrow \infty} \frac{2[(1+\lambda t) \log(1+\lambda t) - \lambda t \log \lambda t]}{(1+\lambda t)} \\
 & = 0
 \end{aligned}$$

Thus in this case, the uncertainty starts with zero value at $t = 0$ and ends with zero value as $t \rightarrow \infty$, and in between, it attains the maximum value at $\lambda t = 1$ that is $t = \frac{1}{\lambda}$.

B. Variation in Kapur's [1] entropy

We know that Kapur's [1] measure of entropy of order α and β is given by

$$\begin{aligned}
 S_2^{\alpha,\beta}(\lambda, t) & = \frac{1}{\alpha+\beta-2} \left[\sum_{n=0}^{\infty} P_n^\alpha + \sum_{n=0}^{\infty} P_n^\beta - 2 \right] \\
 & = \frac{1}{\alpha+\beta-2} \left[\left(\left(\frac{\lambda t}{1+\lambda t} \right)^\alpha + \left(\frac{(\lambda t)^0}{(1+\lambda t)^2} \right)^\alpha + \left(\frac{(\lambda t)^1}{(1+\lambda t)^3} \right)^\alpha + \dots \right)^\alpha \right. \\
 & \quad \left. + \left(\left(\frac{\lambda t}{1+\lambda t} \right)^\beta + \left(\frac{(\lambda t)^0}{(1+\lambda t)^2} \right)^\beta + \left(\frac{(\lambda t)^1}{(1+\lambda t)^3} \right)^\beta + \dots \right)^\beta - 2 \right] \\
 & = \frac{1}{\alpha+\beta-2} \left[\frac{(\lambda t)^\alpha}{(1+\lambda t)^\alpha} + \frac{1}{(1+\lambda t)^\alpha [(1+\lambda t)^\alpha - (\lambda t)^\alpha]} - 1 \right] \\
 & \quad + \frac{1}{\alpha+\beta-2} \left[\frac{(\lambda t)^\beta}{(1+\lambda t)^\beta} + \frac{1}{(1+\lambda t)^\beta [(1+\lambda t)^\beta - (\lambda t)^\beta]} - 1 \right] \\
 & = \frac{1 - [(1+\lambda t)^\alpha - (\lambda t)^\alpha]^2}{(\alpha+\beta-2)[(1+\lambda t)^{2\alpha} - (\lambda t)^\alpha (1+\lambda t)^\alpha]} \\
 & \quad + \frac{1 - [(1+\lambda t)^\beta - (\lambda t)^\beta]^2}{(\alpha+\beta-2)[(1+\lambda t)^{2\beta} - (\lambda t)^\beta (1+\lambda t)^\beta]}
 \end{aligned}$$

Taking $\alpha \rightarrow \beta$, we get

$$S_2^\beta(\lambda, t) = \frac{1 - [(1 + \lambda t)^\beta - (\lambda t)^\beta]^2}{(\beta - 1)[(1 + \lambda t)^{2\beta} - (\lambda t)^{2\beta} + (1 + \lambda t)^\beta (\lambda t)^\beta]} \\ = \frac{1 - (1 + \lambda t)^{2\beta} - (\lambda t)^{2\beta} + 2(1 + \lambda t)^\beta (\lambda t)^\beta}{(\beta - 1)[(1 + \lambda t)^{2\beta} - (\lambda t)^{2\beta} + (1 + \lambda t)^\beta (\lambda t)^\beta]} \quad (14)$$

Taking $\beta \rightarrow 1$, we get

$$S_2(\lambda, t) = \frac{2[(1 + \lambda t) \log(1 + \lambda t) - \lambda t \log \lambda t]}{(1 + \lambda t)} \quad (15)$$

which is again a result developed by Kapur[2].

Differentiating (14) w.r.t. λt , we get

$$\frac{d}{d(\lambda t)} S_2^\beta(\lambda, t) = \frac{-2\beta(1 + \lambda t)^{2\beta-1}(\lambda t)^{2\beta-1} + \beta(1 + \lambda t)^{3\beta-1}(\lambda t)^{\beta-1} + \beta(1 + \lambda t)^{\beta-1}(\lambda t)^{3\beta-1} + \beta(1 + \lambda t)^\beta(\lambda t)^{\beta-1} + \beta(1 + \lambda t)^{\beta-1}(\lambda t)^\beta - 2\beta(1 + \lambda t)^{2\beta-1}}{(\beta - 1)[(1 + \lambda t)^{2\beta} - (\lambda t)^{2\beta} + (1 + \lambda t)^\beta (\lambda t)^\beta]^2} \quad (13)$$

Taking $\beta \rightarrow 1$, we get

$$\frac{d}{d(\lambda t)} S_2(\lambda, t) = \frac{2 \log \lambda t}{(1 + \lambda t)^2} > 0 \text{ if } \lambda t > 1$$

Thus, uncertainty increases if $\lambda t > 1$ and decreases if $\lambda t \leq 1$.

Also from (3.7), we have

$$\text{Max } S_2(\lambda, t) = 2 \log 2, \text{ when } \lambda t = 1.$$

Further when $t = 0$, the uncertainty is zero and when $t \rightarrow \infty$, we have from(15)

$$\lim_{t \rightarrow \infty} S_1(\lambda, t) = \lim_{t \rightarrow \infty} \frac{2[(1 + \lambda t) \log(1 + \lambda t) - \lambda t \log \lambda t]}{(1 + \lambda t)} = 2 \log 1 = 0$$

Thus in this case also, the uncertainty starts with zero value at $t = 0$ and ends with zero value at $t \rightarrow \infty$, and in between,

it attains the maximum value at $\lambda t = 1$ that is $t = \frac{1}{\lambda}$.

IV. CONCLUSION

In the steady state queuing process, it is shown that as the arrival rate increases relatively to service rate, the uncertainty increases whereas in the case of non-steady birth-death process, the uncertainty first increases and attains its maximum value and then with the passage of time, it decreases and attains its minimum value. These results find total compatibility with the real life situations and hence are interesting.

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