

# Banach lattices with weak Dunford-Pettis property

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*Abstract*—We introduce and study the class of weak almost Dunford-Pettis operators. As an application, we characterize Banach lattices with the weak Dunford-Pettis property. Also, we establish some sufficient conditions for which each weak almost Dunford-Pettis operator is weak Dunford-Pettis. Finally, we derive some interesting results.

*Keywords*—weak almost Dunford-Pettis operator, almost Dunford-Pettis operator, weak Dunford-Pettis operator, weak almost Dunford-Pettis operator, almost Dunford-Pettis operator, weak Dunford-Pettis operator.

## I. INTRODUCTION AND NOTATION

As many Banach spaces do not have the Dunford-Pettis property, a weak notion is introduced, called the weak Dunford-Pettis property. A Banach space (respectively, Banach lattice)  $E$  has the Dunford-Pettis (respectively, weak Dunford-Pettis) property if every weakly compact operator defined on  $E$  (and taking their values in a Banach space  $F$ ) is Dunford-Pettis (respectively, almost Dunford-Pettis, that is, the sequence  $(\|T(x_n)\|)$  converges to 0 for every weakly null sequence  $(x_n)$  consisting of pairwise disjoint elements in  $E$  [5]). It is obvious that if  $E$  has the Dunford-Pettis property, then it has the weak Dunford-Pettis property.

On the other hand, whenever Aliprantis-Burkinshaw [1] and Kalton-Saab [4] studied the domination property of Dunford-Pettis operators, they used the class of weak Dunford-Pettis operators which satisfies the domination property [4]. Let us recall from [2] that an operator  $T$  from a Banach space  $X$  into another  $Y$  is called *weak Dunford-Pettis* if the sequence  $(f_n(T(x_n)))$  converges to 0 whenever  $(x_n)$  converges weakly to 0 in  $X$  and  $(f_n)$  converges weakly to 0 in  $Y$ . Alternatively,  $T$  is weak Dunford-Pettis if  $T$  maps relatively weakly compact sets of  $X$  into Dunford-Pettis sets of  $Y$  (see Theorem 5.99 of [2]). A norm bounded subset  $A$  of a Banach lattice  $E$  is said to be *Dunford-Pettis set* if every weakly null sequence  $(f_n)$  of  $E'$  converges uniformly to zero on the set  $A$ , that is,  $\sup_{x \in A} |f_n(x)| \rightarrow 0$  (see Theorem 5.98 of [2]).

In [3], we introduced a new class of sets we call almost Dunford-Pettis set. A norm bounded subset  $A$  of a Banach lattice  $E$  is said to be *almost Dunford-Pettis set* if every disjoint weakly null sequence  $(f_n)$  of  $E'$  converges uniformly to zero on the set  $A$ , that is,  $\sup_{x \in A} |f_n(x)| \rightarrow 0$ .

As weak Dunford-Pettis operators, we introduce a new class of operators that we call *weak almost Dunford-Pettis operator*. An operator  $T$  from a Banach space  $X$  into a Banach lattice  $F$  is said to be *weak almost Dunford-Pettis* if  $T$  maps relatively weakly compact sets of  $X$  into almost Dunford-Pettis sets of  $F$ . The latter class of operators differs from

that of weak Dunford-Pettis operators. In fact, the first one is defined between Banach spaces while the second one is defined from a Banach space into a Banach lattice.

On the other hand, since each Dunford-Pettis set in a Banach lattice is almost Dunford-Pettis, then the class of weak almost Dunford-Pettis operators contains strictly that of weak Dunford-Pettis operators, that is, every weak Dunford-Pettis operator is weak almost Dunford-Pettis. But a weak almost Dunford-Pettis operator is not necessary weak Dunford-Pettis. In fact, for Wnuk (see [5], Example 1, p. 231)), the Lorentz space  $\Lambda(\omega, 1)$  has the weak Dunford-Pettis property but does not have the Dunford-Pettis property, and then its identity operator is weak almost Dunford-Pettis (because each relatively weakly compact set in a Banach lattice has the weak Dunford-Pettis property is an almost Dunford-Pettis set, see Theorem 2.8 of [3]), but it is not weak Dunford-Pettis.

The objective of this paper is to study the class of weak almost Dunford-Pettis operators. Also, we derive the following interesting consequences: some characterizations of this class of operators, some characterizations of the weak Dunford-Pettis property, the coincidence of this class of operators with that of weak Dunford-Pettis operators, the domination property of this class of operators and the duality property.

To state our results, we need to fix some notation and recall some definitions. A Banach lattice is a Banach space  $(E, \|\cdot\|)$  such that  $E$  is a vector lattice and its norm satisfies the following property: for each  $x, y \in E$  such that  $|x| \leq |y|$ , we have  $\|x\| \leq \|y\|$ . Note that if  $E$  is a Banach lattice, its topological dual  $E'$ , endowed with the dual norm and the dual order, is also a Banach lattice. A norm  $\|\cdot\|$  of a Banach lattice  $E$  is order continuous if for each generalized sequence  $(x_\alpha)$  such that  $x_\alpha \downarrow 0$  in  $E$ ,  $(x_\alpha)$  converges to 0 for the norm  $\|\cdot\|$  where the notation  $x_\alpha \downarrow 0$  means that  $(x_\alpha)$  is decreasing, its infimum exists and  $\inf(x_\alpha) = 0$ .

A linear mapping  $T$  from a vector lattice  $E$  into a vector lattice  $F$  is called a lattice homomorphism, if  $x \wedge y = 0$  in  $E$  implies  $T(x) \wedge T(y) = 0$  in  $F$ . An operator  $T : E \rightarrow F$  between two Banach lattices is a bounded linear mapping. It is positive if  $T(x) \geq 0$  in  $F$  whenever  $x \geq 0$  in  $E$ . If  $T : E \rightarrow F$  is a positive operator between two Banach lattices, then its adjoint  $T' : F' \rightarrow E'$ , defined by  $T'(f)(x) = f(T(x))$  for each  $f \in F'$  and for each  $x \in E$ , is also positive. We refer the reader to [2] for unexplained terminologies on Banach lattice theory and positive operators.

## II. MAIN RESULTS

Recall from [5] that an operator from a Banach lattice  $E$  into a Banach space  $X$  is said to be almost Dunford-Pettis if the sequence  $(\|T(x_n)\|)$  converges to 0 for every weakly null sequence  $(x_n)$  consisting of pairwise disjoint elements in  $E$ .

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The following result gives a characterizations of weak almost Dunford-Pettis operators from a Banach space into a Banach lattice in term of weakly compact operators and the adjoint of almost Dunford-Pettis operators.

*Theorem 2.1:* For an operator  $T$  from a Banach space  $X$  into a Banach lattice  $F$ , the following statements are equivalent:

- 1)  $T$  is weak almost Dunford-Pettis operator.
- 2) If  $S$  is a weakly compact operator from an arbitrary Banach space  $Z$  into  $X$ , then the adjoint of the operator product  $T \circ S$  is almost Dunford-Pettis.
- 3) If  $S$  is a weakly compact operator from  $\ell^1$  into  $X$ , then the adjoint of the operator product  $T \circ S$  is almost Dunford-Pettis.
- 4) For all weakly null sequence  $(x_n)_n \subset X$ , and for all disjoint weakly null sequence  $(f_n)_n \subset F'$  it follows that  $f_n(T(x_n)) \rightarrow 0$ .

*Proof:* (1)  $\Rightarrow$  (2) Let  $(f_n)$  be a disjoint weakly null sequence in  $F'$ , we have to prove that  $((T \circ S)'(f_n))$  converges to 0 for the norm of  $Z'$ . If not, then there exist a sequence  $(z_n)$  in the closed unit ball  $B_Z$  of  $Z$ , a subsequence of  $((T \circ S)'(f_n))$  (which we shall denote by  $((T \circ S)'(f_n))$  again), and some  $\varepsilon > 0$  satisfying  $|f_n(T(S(z_n)))| > \varepsilon$  for all  $n$ . Since  $S$  is weakly compact, the set  $A = \{S(z_1), S(z_2), \dots\}$  is relatively weakly compact subset of  $E$ , and then the set  $T(A)$  is an almost Dunford-Pettis (because  $T$  carries weakly relatively compact sets of  $X$  to almost Dunford-Pettis sets of  $F$ ). Hence we obtain

$$|f_n(T(S(z_n)))| \leq \sup_{x \in T(A)} |f_n(x)| \rightarrow 0.$$

Then  $|f_n(T(S(z_n)))| \rightarrow 0$ , which is impossible with  $|f_n(T \circ S(x_n))| > \varepsilon$  for all  $n$ . Thus, the sequence  $((T \circ S)'(f_n))$  converges to 0 for the norm of  $Z'$ , and so the adjoint  $(T \circ S)'$  is almost Dunford-Pettis.

(2)  $\Rightarrow$  (3) Obvious.

(3)  $\Rightarrow$  (4) Let  $(f_n)$  be a disjoint weakly null sequence in  $F'$ , and let  $(x_n)$  be a weakly null sequence in  $X$ . Consider the operator  $S : \ell^1 \rightarrow X$  defined by

$$S((\lambda_i)_{i=1}^\infty) = \sum_{i=1}^\infty \lambda_i x_i \text{ for each } (\lambda_i)_{i=1}^\infty \in \ell^1.$$

Then  $S$  is weakly compact (Theorem 5.26 of [2]), and so by our hypothesis  $(T \circ S)' = S' \circ T'$  is an almost Dunford-Pettis operator. Thus  $\|(T \circ S)'(f_n)\| \rightarrow 0$  and the desired conclusion follows from the inequality

$$\begin{aligned} |f_n(T(x_n))| &= |f_n(T(S(e_n)))| \\ &\leq \sup_{(\lambda_i) \in B_{\ell^1}} |f_n(T(S((\lambda_i)_{i=1}^\infty)))| \\ &= \|(T \circ S)'(f_n)\| \end{aligned}$$

for each  $n$ , where  $(e_i)_{i=1}^\infty$  is the canonical basis of  $\ell^1$ .

(4)  $\Rightarrow$  (1) Let  $W$  be a relatively weakly compact subset of  $X$ , and let  $(f_n)$  be a disjoint weakly null sequence in  $F'$ . If  $(f_n)$  does not converge uniformly to zero on  $T(W)$ , then there exist a sequence  $(x_n)$  of  $W$ , a subsequence of  $(f_n)$  (which we shall denote by  $(f_n)$  again), and some  $\varepsilon > 0$  satisfying  $|f_n(T(x_n))| > \varepsilon$  for all  $n$ .

Since  $W$  is weakly compact, we can assume that  $x_n \rightarrow x$  weakly in  $X$ . Then  $T(x_n) \rightarrow T(x)$  weakly in  $F$  and so,

by our hypothesis, we have  $0 < \varepsilon < |f_n(T(x_n))| \leq |f_n(T(x_n - x))| + |f_n(T(x))| \rightarrow 0$ , which is impossible. Thus,  $(f_n)$  converges uniformly to zero on  $T(W)$ , and this shows that  $T(W)$  is an almost Dunford-Pettis set. This ends the proof of the Theorem. ■

Let us recall that, an operator  $T$  from a Banach lattice  $E$  into a Banach lattice  $F$  is said to be order bounded if for each  $z \in E^+$ , the set  $T([-z, z])$  is order bounded set in  $F$ . An operator  $T$  from a Banach lattice  $E$  into a Banach lattice  $F$  is said to be regular if it can be written as a difference of two positive operators. Note that, every regular operator is order bounded but an order bounded operator is not necessary regular (see [2], Example 1.16, p. 13).

*Remark 2.2:* Each order interval  $[-z, z]$  of a Banach lattice  $E$  is an almost Dunford-Pettis set for each  $z \in E^+$ . In fact, if  $(f_n)$  be a disjoint weakly null sequence in  $E'$ , then by Remark 1 of Wnuk [5],  $(|f_n|)$  is a disjoint weakly null sequence in  $E'$ . Hence  $\sup_{x \in [-z, z]} |f_n(x)| = |f_n|(z) \rightarrow 0$  for each  $z \in E^+$ . As a consequence, if  $T : E \rightarrow F$  is an order bounded operator from a Banach lattice  $E$  into another  $F$ , then  $T([-z, z])$  is an almost Dunford-Pettis set in  $F$ , and then  $|f_n \circ T|(z) = \sup_{y \in T([-z, z])} |f_n(y)| \rightarrow 0$  for each  $z \in E^+$ .

We will need the following characterizations, which are just Theorem 2.4 of [3].

*Theorem 2.3:* [3] Let  $T : E \rightarrow F$  be an order bounded operator from a Banach lattice  $E$  into another Banach lattice  $F$ , and let  $A$  be a norm bounded solid subset of  $E$ . The following statements are equivalent:

- 1)  $T(A)$  is an almost Dunford-Pettis set.
- 2)  $\{T(x_n), n \in N\}$  is an almost Dunford-Pettis set for each disjoint sequence  $(x_n)$  in  $A^+ = A \cap E^+$ .
- 3)  $f_n(T(x_n)) \rightarrow 0$  for each disjoint sequence  $(x_n)$  in  $A^+$  and for every disjoint weakly null sequence  $(f_n)$  of  $E'$ .

*Proof:* (1)  $\Rightarrow$  (2) Obvious.

(2)  $\Rightarrow$  (3) Obvious.

(3)  $\Rightarrow$  (1) To prove that  $T(A)$  is an almost Dunford-Pettis set, it suffice to show that  $\sup_{x \in A} |f_n(T(x))| \rightarrow 0$  for every disjoint weakly null sequence  $(f_n)$  of  $F'$ . Otherwise, there exists a sequence  $(f_n) \subset E'$  satisfying  $\sup_{x \in A} |f_n(T(x))| > \varepsilon$  for some  $\varepsilon > 0$  and all  $n$ . For every  $n$  there exists  $z_n$  in  $A^+$  such that  $|T'(f_n)|(z_n) > \varepsilon$ . Since  $|T'(f_n)|(z) \rightarrow 0$  for every  $z \in E^+$  (see Remark 2.2), then by an easy inductive argument shows that there exist a subsequence  $(y_n)$  of  $(z_n)$  and a subsequence  $(g_n)$  of  $(f_n)$  such that

$$|T'(g_{n+1})(y_{n+1})| > \varepsilon \text{ and } |T'(g_{n+1})|(4^n \sum_{i=1}^n y_i) < \frac{1}{n}$$

for all  $n \geq 1$ . Put  $x = \sum_{i=1}^\infty 2^{-i} y_i$  and  $x_n = (y_{n+1} - 4^n \sum_{i=1}^n y_i - 2^{-n} x)^+$ . By Lemma 4.35 of [2] the sequence  $(x_n)$  is disjoint. Since  $0 \leq x_n \leq y_{n+1}$  for every  $n$ , and  $(y_{n+1})$  in  $A^+$  then  $(x_n) \subset A^+$ .

From the inequalities

$$\begin{aligned} |T'(g_{n+1})(x_n)| &\geq |T'(g_{n+1})|(y_{n+1} - 4^n \sum_{i=1}^n y_i - 2^{-n} x) \\ &\geq \varepsilon - \frac{1}{n} - 2^{-n} |T'(g_{n+1})|(x) \end{aligned}$$

we see that  $|T'(g_{n+1})(x_n)| > \frac{\varepsilon}{2}$  must hold for all  $n$  sufficiently large (because  $2^{-n}|T'(g_{n+1})(x)| \rightarrow 0$ ).

In view of  $|T'(g_{n+1})(x_n)| = \sup\{|g_{n+1}(T(z))| : |z| \leq x_n\}$ , for each  $n$  sufficiently large there exists some  $|z_n| \leq x_n$  with  $|g_{n+1}(T(z_n))| > \frac{\varepsilon}{2}$ . Since  $(z_n^+)$  and  $(z_n^-)$  are both norm bounded disjoint sequence in  $A^+$ , it follows from our hypothesis that

$$\frac{\varepsilon}{2} < |g_{n+1}(T(z_n))| \leq |g_{n+1}(T(z_n^+))| + |g_{n+1}(T(z_n^-))| \rightarrow 0$$

which is impossible. This proves that  $T(A)$  is an almost Dunford-Pettis set. ■

For order bounded operators between two Banach lattices, we give a characterization of weak almost Dunford-Pettis operators.

**Theorem 2.4:** Let  $T$  be an order bounded operator from a Banach lattice  $E$  into another  $F$ . Then the following assertions are equivalent:

- 1)  $T$  is weak almost Dunford-Pettis operator.
- 2)  $f_n(T(x_n)) \rightarrow 0$  for all weakly null sequence  $(x_n)$  in  $E$  consisting of pairwise disjoint terms, and for all weakly null sequence  $(f_n)$  in  $F'$  consisting of pairwise disjoint terms.

*Proof:* (1)  $\Rightarrow$  (2) Obvious.

(2)  $\Rightarrow$  (1) Let  $(x_n)$  be a weakly null sequence in  $E$ , and let  $(f_n)$  be a disjoint weakly null sequence in  $F'$ . We have to prove that  $f_n(T(x_n)) \rightarrow 0$ .

Let  $A$  be the solid hull of the weak relatively compact subset  $\{x_n, n \in N\}$  of  $E$ , by Theorem 4.34 of [2],  $(z_n) \rightarrow 0$  weakly for each disjoint sequence  $(z_n)$  in  $A^+$  and so, by our hypothesis, we have  $g_n(T(z_n)) \rightarrow 0$  for each disjoint weakly null sequence  $(g_n)$  in  $F'$  and for each disjoint sequence  $(z_n)$  in  $A^+$ , then Theorem 2.3, implies that  $T(A)$  is an almost Dunford-Pettis set, and hence  $\sup_{y \in T(A)} |f_n(y)| \rightarrow 0$ . Therefore,

$$|f_n(T(x_n))| \leq \sup_{x \in A} |f_n(T(x))| \leq \sup_{y \in T(A)} \|f_n(y)\| \rightarrow 0$$

holds and the proof is finished. ■

Now for positive operators between two Banach lattices, we give other characterizations of weak almost Dunford-Pettis operators.

**Theorem 2.5:** Let  $E$  and  $F$  be two Banach lattices. For every positive operator  $T$  from  $E$  into  $F$ , the following assertions are equivalent:

- 1)  $T$  is weak almost Dunford-Pettis.
- 2) If  $S$  is a weakly compact operator from an arbitrary Banach space  $Z$  into  $E$ , then the adjoint of the operator product  $T \circ S$  is almost Dunford-Pettis.
- 3) If  $S$  is a weakly compact operator from  $\ell^1$  into  $E$ , then the adjoint of the operator product  $T \circ S$  is almost Dunford-Pettis.
- 4) For all weakly null sequence  $(x_n)_n \subset E$ , and for all disjoint weakly null sequence  $(f_n)_n \subset F'$  it follows that  $f_n(T(x_n)) \rightarrow 0$ .
- 5)  $f_n(T(x_n)) \rightarrow 0$  for every weakly null sequence  $(x_n)$  in  $E^+$  and for all disjoint weakly null sequence  $(f_n)$  in  $F'$ .

6)  $f_n(T(x_n)) \rightarrow 0$  for all weakly null sequence  $(x_n)$  in  $E$  consisting of pairwise disjoint terms, and for all weakly null sequence  $(f_n)$  in  $F'$  consisting of pairwise disjoint terms.

7) For all disjoint weakly null sequences  $(x_n)_n \subset E^+$ ,  $(f_n)_n \subset (F')^+$  it follows that  $f_n(T(x_n)) \rightarrow 0$ .

8)  $f_n(T(x_n)) \rightarrow 0$  for every disjoint weakly null sequence  $(x_n)$  in  $E^+$  and for all weakly null sequence  $(f_n)$  in  $F'$ .

9)  $f_n(T(x_n)) \rightarrow 0$  for every disjoint weakly null sequence  $(x_n)$  in  $E^+$  and for all weakly null sequence  $(f_n)$  in  $(F')^+$ .

10)  $f_n(T(x_n)) \rightarrow 0$  for every weakly null sequence  $(x_n)$  in  $E$  and for all weakly null sequence  $(f_n)$  in  $(F')^+$ .

11)  $f_n(T(x_n)) \rightarrow 0$  for every weakly null sequence  $(x_n)$  in  $E^+$  and for all weakly null sequence  $(f_n)$  in  $(F')^+$ .

12)  $f_n(T(x_n)) \rightarrow 0$  for every weakly null sequence  $(x_n)$  in  $E^+$  and for all weakly null sequence  $(f_n)$  in  $F'$ .

*Proof:* (1)  $\Leftrightarrow$  (2)  $\Leftrightarrow$  (3)  $\Leftrightarrow$  (4) Follows from Theorem 2.1.

(6)  $\Leftrightarrow$  (4) Follows from Theorem 2.4.

(4)  $\Rightarrow$  (5) Obvious.

(5)  $\Rightarrow$  (6) Let  $(x_n)$  be a weakly null sequence in  $E$  consisting of pairwise disjoint elements, and let  $(f_n)$  be a weakly null sequence in  $F'$ , consisting of pairwise disjoint elements, it follows from Remark 1 of Wnuk [5] that  $x_n^+ \rightarrow 0$  and  $x_n^- \rightarrow 0$  weakly in  $E^+$ . Hence by (5),  $f_n(T(x_n)) = f_n(T(x_n^+)) - f_n(T(x_n^-)) \rightarrow 0$ .

(6)  $\Rightarrow$  (7) Obvious.

(7)  $\Rightarrow$  (8) Assume by way of contradiction that there exists a disjoint weakly null sequence  $(x_n) \subset E^+$  and a weakly null sequence  $(f_n) \subset F'$  such that  $f_n(T(x_n)) \not\rightarrow 0$ . The inequality  $|f_n(T(x_n))| \leq |f_n|(T(x_n))$  implies  $|f_n|(T(x_n)) \not\rightarrow 0$ . Then there exists some  $\varepsilon > 0$  and a subsequence of  $|f_n|(T(x_n))$  (which we shall denote by  $|f_n|(T(x_n))$  again) satisfying  $|f_n|(T(x_n)) > \varepsilon \forall n$ .

On the other hand, since  $(x_n) \rightarrow 0$  weakly in  $E$ , then  $T(x_n) \rightarrow 0$  weakly in  $F$ . Now an easy inductive argument shows that there exist a subsequence  $(z_n)$  of  $(x_n)$  and a subsequence  $(g_n)$  of  $(f_n)$  such that  $\forall n \geq 1$

$$|g_n|(T(z_n)) > \varepsilon \text{ and } (4^n \sum_{i=1}^n |g_i|)(T(z_{n+1})) < \frac{1}{n}$$

Put  $h = \sum_{n=1}^{\infty} 2^{-n} |g_n|$  and  $h_n = (|g_{n+1}| - 4^n \sum_{i=1}^n |g_i| - 2^{-n} h)^+$ . By Lemma 4.35 of [2] the sequence  $(h_n)$  is disjoint. Since  $0 \leq h_n \leq |g_{n+1}|$  for all  $n \geq 1$  and  $(g_n) \rightarrow 0$  weakly in  $F'$  then it follows from Theorem 4.34 of [2] that  $(h_n) \rightarrow 0$  weakly in  $F'$ .

From the inequalities

$$\begin{aligned} h_n(T(z_{n+1})) &\geq (|g_{n+1}| - 4^n \sum_{i=1}^n |g_i| - 2^{-n} h)(T(z_{n+1})) \\ &\geq \varepsilon - \frac{1}{n} - 2^{-n} h(T(z_{n+1})) \end{aligned}$$

we see that  $h_n(T(z_{n+1})) > \frac{\varepsilon}{2}$  must hold for all  $n$  sufficiently large (because  $2^{-n} h(T(z_{n+1})) \rightarrow 0$ ), which contradicts with our hypothesis (7).

(8)  $\Rightarrow$  (9) Obvious.

(9)  $\Rightarrow$  (10) Assume by way of contradiction that there exists a weakly null sequence  $(x_n) \subset E$  and a weakly null sequence  $(f_n) \subset (F')^+$  such that  $f_n(T(x_n)) \not\rightarrow 0$ . The inequality  $|f_n(T(x_n))| \leq f_n(T(|x_n|))$  implies  $f_n(T(|x_n|)) \not\rightarrow 0$ . Then there exists some  $\varepsilon > 0$  and a subsequence of  $f_n(T(|x_n|))$  (which we shall denote by  $f_n(T(|x_n|))$  again) satisfying  $f_n(T(|x_n|)) > \varepsilon$  for all  $n$ .

On the other hand, since  $(f_n) \rightarrow 0$  weakly in  $F'$ , then  $T'(f_n) \rightarrow 0$  weakly in  $E'$ . Now an easy inductive argument shows that there exist a subsequence  $(z_n)$  of  $(|x_n|)$  and a subsequence  $(g_n)$  of  $(f_n)$  such that  $\forall n \geq 1$

$$T'(g_n)(z_n) > \varepsilon \text{ and } T'(g_{n+1})\left(4^n \sum_{i=1}^n z_i\right) < \frac{1}{n}$$

Put  $z = \sum_{n=1}^{\infty} 2^{-n} z_n$  and  $y_n = (z_{n+1} - 4^n \sum_{i=1}^n z_i - 2^{-n} z)^+$ . By Lemma 4.35 of [2] the sequence  $(y_n)$  is disjoint. Since  $0 \leq y_n \leq z_{n+1}$  for all  $n \geq 1$  and  $(z_n) \rightarrow 0$  weakly in  $E$ , then it follows from Theorem 4.34 of [2] that  $(y_n) \rightarrow 0$  weakly in  $E$ .

From the inequalities

$$\begin{aligned} T'(g_{n+1})(y_n) &\geq T'(g_{n+1})\left(z_{n+1} - 4^n \sum_{i=1}^n z_i - \frac{z}{2^n}\right) \\ &\geq \varepsilon - \frac{1}{n} - 2^{-n} T'(g_{n+1})(z) \end{aligned}$$

we see that  $g_{n+1}(T(y_n)) = T'(g_{n+1})(y_n) > \frac{\varepsilon}{2}$  must hold for all  $n$  sufficiently large (because  $2^{-n} T'(g_{n+1})(z) \rightarrow 0$ ), which contradicts with our hypothesis (9).

(10)  $\Rightarrow$  (11) Obvious.

(11)  $\Rightarrow$  (6) Let  $(x_n)$  be a weakly null sequence in  $E$  consisting of pairwise disjoint elements, and let  $(f_n)$  be a weakly null sequence in  $F'$ , consisting of pairwise disjoint elements, it follows from Remark 1 of Wnuk [5] that  $|x_n| \rightarrow 0$  in  $\sigma(E, E')$ , and  $|f_n| \rightarrow 0$  in  $\sigma(F', F'')$ . Hence by (11),  $|f_n|(T(|x_n|)) \rightarrow 0$ . Now, from  $|f_n(T(x_n))| \leq |f_n|(T(|x_n|))$  for each  $n$ , we derive that  $f_n(T(x_n)) \rightarrow 0$ .

(12)  $\Rightarrow$  (8) Obvious.

(5)  $\Rightarrow$  (12) The proof is similar of the proof (7)  $\Rightarrow$  (8). ■

An application of Theorem 2.5, gives other characterizations of Banach lattices with the weak Dunford-Pettis property.

Corollary 2.6: For a Banach lattice  $E$  the following statements are equivalent:

- 1)  $E$  has the weak Dunford-Pettis property.
- 2) The identity operator  $Id_E : E \rightarrow E$  is weak almost Dunford-Pettis, that is, every relatively weakly compact set of  $E$  is almost Dunford-Pettis set.
- 3) Every weakly compact operator  $T$  from an arbitrary Banach space  $X$  to  $E$  has an adjoint  $T' : E' \rightarrow X'$  which is almost Dunford-Pettis.
- 4) Every weakly compact operator  $T : \ell^1 \rightarrow E$  has an adjoint  $T'$  which is almost Dunford-Pettis.
- 5) For all weakly null sequence  $(x_n)_n \subset E$ , and for all disjoint weakly null sequence  $(f_n)_n \subset E'$  it follows that  $f_n(x_n) \rightarrow 0$ .
- 6)  $f_n(x_n) \rightarrow 0$  for every weakly null sequence  $(x_n)_n$  in  $E^+$  and for all disjoint weakly null sequence  $(f_n)_n$  in  $E'$ .
- 7) For all disjoint weakly null sequences  $(f_n)_n \subset E'$ ,  $(x_n)_n \subset E$  it follows that  $f_n(x_n) \rightarrow 0$ .

8) For all disjoint weakly null sequences  $(f_n)_n \subset (E')^+$ ,  $(x_n)_n \subset E^+$  it follows that  $f_n(x_n) \rightarrow 0$ .

9)  $f_n(x_n) \rightarrow 0$  for every disjoint weakly null sequence  $(x_n)$  in  $E^+$  and for all weakly null sequence  $(f_n)$  in  $E'$ .

10)  $f_n(x_n) \rightarrow 0$  for every disjoint weakly null sequence  $(x_n)$  in  $E^+$  and for all weakly null sequence  $(f_n)$  in  $(E')^+$ .

11)  $f_n(x_n) \rightarrow 0$  for every weakly null sequence  $(x_n)$  in  $E$  and for all weakly null sequence  $(f_n)$  in  $(E')^+$ .

12)  $f_n(x_n) \rightarrow 0$  for every weakly null sequence  $(x_n)_n$  in  $E^+$  and for all weakly null sequence  $(f_n)$  in  $(E')^+$ .

13)  $f_n(x_n) \rightarrow 0$  for every weakly null sequence  $(x_n)$  in  $E^+$  and for all weakly null sequence  $(f_n)$  in  $E'$ .

Proof: (1)  $\Leftrightarrow$  (8) Follows from Proposition 1 of Wnuk [5].

(2)  $\Leftrightarrow$  (3)  $\Leftrightarrow \dots \Leftrightarrow$  (13) Follows from Theorem 2.5. ■

The following consequence of Theorem 2.5 gives a sufficient conditions under which the class of positive weak almost Dunford-Pettis operators coincide with that of positive weak Dunford-Pettis operators.

Corollary 2.7: Let  $E$  and  $F$  be two Banach lattices. Then each positive weak almost Dunford-Pettis operator from  $E$  into  $F$  is weak Dunford-Pettis if one of the following assertions is valid:

- 1) The lattice operation of  $E$  are weak sequentially continuous;
- 2) The lattice operation of  $F'$  are weak sequentially continuous.

Proof: (1) Assume that  $T : E \rightarrow F$  is a positive weak almost Dunford-Pettis operator. Let  $(x_n)$  be a weakly null sequence in  $E$ , and let  $(f_n)$  be a weakly null sequence in  $F'$ . We have to prove that  $f_n(T(x_n)) \rightarrow 0$ .

Since the lattice operation of  $E$  are weak sequentially continuous, then the positive sequences  $(x_n^+)$  and  $(x_n^-)$  converge weakly to zero. Thus, Theorem 2.5 (12) imply that

$$f_n(T(x_n^+)) \rightarrow 0 \text{ and } f_n(T(x_n^-)) \rightarrow 0.$$

Finally, from  $f_n(T(x_n)) = f_n(T(x_n^+)) - f_n(T(x_n^-))$  for each  $n$ , we conclude that  $f_n(T(x_n)) \rightarrow 0$ . This shows that  $T$  is weak Dunford-Pettis.

(2) Assume that  $T : E \rightarrow F$  is a positive weak almost Dunford-Pettis operator. Let  $(x_n)$  be a weakly null sequence in  $E$ , and let  $(f_n)$  be a weakly null sequence in  $F'$ . We have to prove that  $f_n(T(x_n)) \rightarrow 0$ .

Since the lattice operation of  $F'$  are weak sequentially continuous, then the positive sequences  $(f_n^+)$  and  $(f_n^-)$  converge weakly to zero. Thus, Theorem 2.5 (10) imply that  $f_n^+(T(x_n)) \rightarrow 0$  and  $f_n^-(T(x_n)) \rightarrow 0$ . Finally, from  $f_n(T(x_n)) = f_n^+(T(x_n)) - f_n^-(T(x_n))$  for each  $n$ , we conclude that  $f_n(T(x_n)) \rightarrow 0$ . This shows that  $T$  is weak Dunford-Pettis. ■

The preceding Corollary, gives a sufficient conditions under which the weak Dunford-Pettis property and the Dunford-Pettis property coincide.

Corollary 2.8: Let  $E$  be a Banach lattice. Then  $E$  has the Dunford-Pettis property if and only if it has the weak Dunford-Pettis property, if one of the following assertions is valid:



- 1) The lattice operation of  $E$  are weak sequentially continuous;
- 2) The lattice operation of  $E'$  are weak sequentially continuous.

Our consequence of Theorem 2.5 we obtain the domination property for weak almost Dunford-Pettis operators.

*Corollary 2.9:* Let  $E$  and  $F$  be two Banach lattices. If  $S$  and  $T$  are two positive operators from  $E$  into  $F$  such that  $0 \leq S \leq T$  and  $T$  is weak almost Dunford-Pettis operator, then  $S$  is also weak almost Dunford-Pettis operator.

*Proof:* Let  $(x_n)_n$  be a weakly null sequence in  $E^+$  and  $(f_n)$  be a weakly null sequence in  $(F')^+$ . According to (11) of Theorem 2.5, it suffices to show that  $f_n(S(x_n)) \rightarrow 0$ . Since  $T$  is weak almost Dunford-Pettis, then Theorem 2.5 implies that  $f_n(T(x_n)) \rightarrow 0$ . Now, by using the inequalities  $0 \leq f_n(S(x_n)) \leq f_n(T(x_n))$  for each  $n$ , we see that  $f_n(S(x_n)) \rightarrow 0$ . ■

Now, we look at the duality property of the class of positive weak almost Dunford-Pettis operators.

*Theorem 2.10:* Let  $E$  and  $F$  be two Banach lattices and let  $T$  be a positive operator from  $E$  into  $F$ . If the adjoint  $T'$  is weak almost Dunford-Pettis from  $F'$  into  $E'$ , then  $T$  itself is weak almost Dunford-Pettis.

*Proof:* Let  $(x_n)$  be a weakly null sequence in  $E^+$ , and let  $(f_n)$  be a weakly null sequence in  $(F')^+$ . We have to prove that  $f_n(T(x_n)) \rightarrow 0$ .

Let  $\tau : E \rightarrow E''$  be the canonical injection of  $E$  into its topological bidual  $E''$ . Since  $\tau$  is a lattice homomorphism, the sequence  $(\tau(x_n))$  is weakly null in  $(E'')^+$ . And as the adjoint  $T'$  is weak almost Dunford-Pettis from  $F'$  into  $E'$ , we deduce by Theorem 2.1 that  $\tau(x_n)(T'(f_n)) \rightarrow 0$ . But  $\tau(x_n)(T'(f_n)) = T'(f_n)(x_n) = f_n(T(x_n))$  for each  $n$ . Hence  $f_n(T(x_n)) \rightarrow 0$  and this ends the proof. ■

We end this paper by a consequence of Theorem 2.10, we obtain Proposition 2 of Wnuk [5].

*Corollary 2.11:* Let  $E$  be a Banach lattice. If  $E'$  has the weak Dunford-Pettis property, then  $E$  itself has the weak Dunford-Pettis.

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