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Analytical Solution of Time-Harmonic Torsional Vibration of a Cylindrical Cavity in a Half-Space

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***Abstract*—**In this article an isotropic linear elastic half-space with a cylindrical cavity of finite length is considered to be under the effect of a ring shape time-harmonic torsion force applied at an arbitrary depth on the surface of the cavity. The equation of equilibrium has been written in a cylindrical coordinate system. By means of Fourier cosine integral transform, the non-zero displacement component is obtained in the transformed domain. With the aid of the inversion theorem of the Fourier cosine integral transform, the displacement is obtained in the real domain. With the aid of boundary conditions, the involved boundary value problem for the fundamental solution is reduced to a generalized Cauchy singular integral equation. Integral representation of the stress and displacement are obtained, and it is shown that their degenerated form to the static problem coincides with existing solutions in the literature.

***Keywords*—**Cosine transform, Half space, Isotropic, Singular integral equation, Torsion

I. Introduction

INVESTIGATING half-spaces containing cylindrical cavities has been one of the great interests to many researchers. Analytical inspection of the dynamic interaction of piles with torsion moments and the pile cavities in a half-space is very important in many engineering structures such as wharves and other heavy structures. Pertaining to problems of this type, some approximate results were first obtained by for the case of hydrostatic pressure acting on an interval of an infinite cylindrical cavity extending through an infinite solid [1]. Treated the dynamic problem of a suddenly applied pressure over a finite interval of the cavity [2]. Because of the complexities encountered in the problem, the numerical results were presented only at large distance away from the location of pressure. Some interesting problems of determining the distribution of stress due to an exterior crack in an isotropic infinite elastic medium with a coaxial cylindrical cavity was studied by ([3, 4]). The response due to the application of static radial pressure and torsional ring load has been given by [5]. In addition, he proposed a quadrature method for evaluating of the singular solution for concentrated torsional and radial ring load acting on the wall of an infinite hole.

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Parnes also considered the steady-state problem of the effect of a torsional line load, with a harmonic time dependency, applied on the surface of a bore [6]. In his paper, he compared the degenerated of his dynamic results with the static case.

The solutions of the generalized problem associated with a finite cylindrical cavity in a half-space would be of even greater engineering interest and challenge. It has been found that the additional stiffness of the medium below the bottom of the hole can apparently lead to a noticeable change of the response in the upper region. [7] investigated the problem of torsional shear traction acting on an open finite cylindrical cavity in an isotropic half-space in a rigorous manner, and found the corresponding fundamental solution. They also mathematically examined the resulting load-induced as well as shape-induced singularities in the response.

In the present paper, which is an extension of the work done in [7] for dynamic case, the elastodynamic response of an isotropic half-space containing a finite open cylindrical cavity under a torsional ring load at an arbitrary depth is considered. Owing to the particular topology of the domain, it is convenient to consider the response of the elastic solid in two separate regions, which have some continuity conditions. By considering the equation of motion in each region and with the aid of Fourier cosine integral transform, the non-zero displacement component is obtained in the transformed domain. By means of the inversion theorem for Fourier cosine integral transform and the displacement compatibility conditions, the governing equation is reduced to a generalized Cauchy singular integral equation. The equation is then investigated analytically and solved numerically. Integral representation of the dynamic stress and displacement are obtained and shown to be degenerated to known existing solutions in the literature.

II. Boundary Value Problem and the Solution

An isotropic homogeneous linear elastic half-space is considered in cylindrical coordinate system (r,θ,z), with a depth-wise z-axis. A circular cylindrical cavity with radius $a>0$ and length $l>0$, as shown in Fig. 1, is assumed to be in the medium. A known time-harmonic shear stress, $\tau_{r\theta}=\mu\tau^{*}(z,\omega)e^{i\omega t}$, is considered to be applied on the wall of the cavity. Because of torsional symmetry, the displacement vector has only one non-vanishing component, which is

$u_\theta = u(r,z,t)$. We follow the research done by Pak and Abedzadeh [7]. So that it is convenient to define two different regions as indicated in Fig. 1 and find the response of each region with satisfying the boundary conditions and continuity conditions, as well.

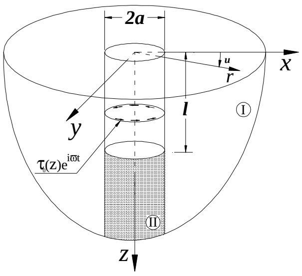

Fig. 1 A cylindrical cavity in an isotropic half-space

These two regions are defined as

$$\text{Region } 1 = \left\{ (r,\theta,z) \middle| \ r > a,\ \ 0<\theta<2\pi,\ \ z>0 \right\} \tag{1}$$

$$\text{Region } 2 = \left\{ (r,\theta,z) \middle| \ r < a,\ \ 0<\theta<2\pi,\ \ z>l \right\} \tag{2}$$

In the absence of body force, the non-zero time-harmonic equation of motion in terms of displacement is written in the form of

$$\frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r}\frac{\partial u_1}{\partial r} - \frac{u_1}{r^2} + \frac{\partial^2 u_1}{\partial z^2} = -\frac{\omega^2}{C_s^{\,2}} u_1, \quad r>a,\ z>0 \tag{3}$$

in Region 1, and

$$\frac{\partial^2 u_2}{\partial r^2} + \frac{1}{r}\frac{\partial u_2}{\partial r} - \frac{u_2}{r^2} + \frac{\partial^2 u_2}{\partial z^2} = -\frac{\omega^2}{C_s^{\,2}} u_2, \quad r<a,\ z>l \tag{4}$$

in Region 2. In the equations (3) and (4), $C_s = \sqrt{\mu/\rho}$ is the shear wave velocity, ρ the material density, μ the shear modulus of the material, and ω is the frequency of torsional excitation. The stress-displacement relations are [8]

$$\tau_{r\theta} = \mu\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) = \mu\, r\frac{\partial}{\partial r}\left(\frac{u}{r}\right), \tag{5}$$

$$\tau_{z\theta} = \mu\left(\frac{\partial u}{\partial z}\right). \tag{6}$$

The stress and displacement boundary conditions for the problem may be written as

$$\tau_{r\theta 1}(a,z,\omega) = \mu\tau^*(z,\omega), \qquad 0<z<l \tag{7}$$

$$\tau_{z\theta 1}(r,0,\omega) = 0, \qquad r>a \tag{8}$$

$$\tau_{z\theta 2}(r,l,\omega) = 0, \qquad r<a \tag{9}$$

$$\frac{\partial u_1}{\partial z}(r,0,\omega) = 0, \qquad r>a \tag{10}$$

$$\frac{\partial u_2}{\partial z}(r,l,\omega) = 0, \qquad r<a \tag{11}$$

where $\tau^*(z,\omega)$ is a prescribed non-dimensional function. Moreover, the radiation condition is given as

$$u_q(r,z,\omega) \to 0, \qquad \sqrt{(r^2+z^2)} \to \infty. \qquad q=1,\ 2 \tag{12}$$

To ensure that the solid is continuous across the common boundary of Region 1 and Region 2, it is sufficient to stipulate the compatibility conditions as

$$\tau_{r\theta 1}(a,z,\omega) = \tau_{r\theta 2}(a,z,\omega), \qquad z\ge l \tag{13}$$

$$\frac{\partial u_1}{\partial z}(a,z,\omega) = \frac{\partial u_2}{\partial z}(a,z,\omega). \qquad z\ge l \tag{14}$$

It is convenient to use an integral transform to solve the partial differential equations (3) and (4). With the aid of depth-wise Fourier cosine transforms defined as [9]

$$\tilde{f}_1(r,\xi,\omega) = \frac{2}{\pi}\int_0^\infty f_1(r,z,\omega)\cos(\xi z)\,dz, \tag{15}$$

in Region 1, and

$$\tilde{f}_2(r,\xi,\omega) = \frac{2}{\pi}\int_l^\infty f_2(r,z-l,\omega)\cos\xi(z-l)\,dz, \tag{16}$$

in Region 2, the partial differential equations (3) and (4) lead to

$$\frac{d^2\tilde{u}_q}{dr^2} + \frac{1}{r}\frac{d\tilde{u}_q}{dr} - \left(\lambda^2 + \frac{1}{r^2}\right)\tilde{u}_q = 0. \qquad q=1,\ 2 \tag{17}$$

Considering the boundary conditions (10) and (11), they can be written as

$$\frac{d^2\tilde{u}_q}{d\zeta^2} + \frac{1}{\zeta}\frac{d\tilde{u}_q}{d\zeta} - \left(1 + \frac{1}{\zeta^2}\right)\tilde{u}_q = 0, \qquad q=1,2 \tag{18}$$

where $\lambda^2(\omega) = \xi^2 - \omega^2/C_s^{\,2}$, $\zeta = r\lambda$, and $\tilde{u}_q(r,\xi,\omega)$ for $q=1$ and 2 are the Fourier cosine transforms of $u_q(r,z,\omega)$ as defined in (15) and (16). The solutions of equation (18) is

$$\tilde{u}_q(r,\xi,\omega) = A_q(\xi,\omega)K_1(\lambda r) + B_q(\xi,\omega)I_1(\lambda r), q=1,2 \tag{19}$$

where $I_1(\zeta)$ and $K_1(\zeta)$ are the first order modified Bessel functions of the first and second kind, respectively.

To satisfy the radiation condition (12), $B_1(\xi,\omega)$ should be identically zero as $I_1(\lambda r)$ is unbounded when r approaches infinity. In addition, $A_2(\xi,\omega)$ must be zero for the displacement $\tilde{u}_2(r,z,\omega)$ in the Region 2 to be bounded at $r=0$.

Applying the inversion theorem for the Fourier cosine transforms , the displacement $u_q(r,z,\omega)$ for $q=1$ and 2 can be written as

$$u_1(r,z,\omega)=\int_0^{\infty} A_1(\xi,\omega)K_1(\lambda r)\cos(\xi z)\,d\xi, \quad r\ge a, z\ge 0 \quad (20)$$

$$u_2(r,z,\omega)=\int_0^{\infty} B_2(\xi,\omega)I_1(\lambda r)\cos\xi(z-l)\,d\xi. \; r\le a, z\ge l \quad (21)$$

Based on the continuity conditions (13) and (14) on the cylindrical boundary $r=a$, one may write the shear stress $\tau_{r\theta 1}$ in the Region 1 as

$$\tau_{r\theta 1}(r=a^+,z,\omega)=\lim_{r\to a^+}\mu\, r\frac{\partial}{\partial r}\left(\frac{u_1}{r}\right)=\mu\,\chi(z,\omega) \quad (22)$$

where $\chi(z,\omega)=\tau^*(z,\omega)$ for $0<z<l$ and $\chi(z,\omega)=\tau(z,\omega)$ for $z\ge l$. Moreover, the function $\tau(z,\omega)$ is unknown, which is determined from the solution and $\tau^*(z,\omega)$ is a known boundary function. Similarly, $\tau_{r\theta 2}$ at $r=a$ in Region 2 is written as

$$\tau_{r\theta 2}(r=a^-,z,\omega)=\lim_{r\to a^-}\mu\, r\frac{\partial}{\partial r}\left(\frac{u_2}{r}\right)=\mu\,\tau(z,\omega), \quad z\ge l \quad (23)$$

Substituting the displacements from (20) and (21), into (22) and (23), respectively and using the inverse theorem for Fourier cosine transform, one may find

$$A_1(\xi,\omega)=\frac{-2}{\pi\lambda K_2(\lambda a)}\int_0^{\infty}\chi(z,\omega)\cos(\xi z)\,dz, \quad (24)$$

$$B_2(\xi,\omega)=\frac{2}{\pi\lambda I_2(\lambda a)}\int_l^{\infty}\chi(z,\omega)\cos\xi(z-l)\,dz, \quad (25)$$

By virtue of (20), (21), (24) and (25) it can be shown that

$$\begin{aligned}\frac{\partial u_1}{\partial z}(r,z,\omega)=\frac{1}{\pi}\int_0^{\infty}[&\varphi_k(r,z-\zeta,\omega)\\ &+\varphi_k(r,z+\zeta,\omega)]\chi(\zeta,\omega)\,d\zeta, \quad r>a, z\ge 0\end{aligned} \quad (26)$$

$$\begin{aligned}\frac{\partial u_2}{\partial z}(r,z,\omega)=-\frac{1}{\pi}\int_l^{\infty}[&\varphi_I(r,z-\zeta,\omega)\\ &+\varphi_I(r,z+\zeta-2l,\omega)]\chi(\zeta,\omega)\,d\zeta, \quad r<a, z\ge l\end{aligned} \quad (27)$$

where $\varphi_k(r,d,\omega)$ and $\varphi_I(r,d,\omega)$ define as follow

$$\varphi_k(r,d,\omega)=\int_0^{\infty}\frac{\xi K_1(\lambda r)}{\lambda K_2(\lambda a)}\sin(\xi d)\,d\xi, \quad (28)$$

$$\varphi_I(r,d,\omega)=\int_0^{\infty}\frac{\xi I_1(\lambda r)}{\lambda I_2(\lambda a)}\sin(\xi d)\,d\xi. \quad (29)$$

Then, the continuity condition (14) can be stated in the form of

$$\begin{aligned}&\int_l^{\infty}(\varphi_I(a^-,z-\zeta,\omega)+\varphi_I(a^-,z+\zeta-2l,\omega))\\ &\quad\times\tau(\zeta,\omega)\,d\zeta+\int_l^{\infty}(\varphi_k(a^+,z-\zeta,\omega)\\ &+\varphi_k(a^+,z+\zeta,\omega))\,\tau(\zeta,\omega)\,d\zeta=-\int_0^{l}(\varphi_k(a^+,z-\zeta,\omega)\\ &\quad+\varphi_k(a^+,z+\zeta,\omega))\,\tau^*(\zeta,\omega)\,d\zeta. \qquad z\ge l\end{aligned} \quad (30)$$

Equation (30) is a single integral equation for determining $\tau(\zeta,\omega)$.

III. Integral Equation

The equation (30) is a generalized Cauchy singular integral equation [10, 11]. The solution of several problems of mathematical physics and specialty plane elasticity and fluid mechanics can be reduced to the solution of Cauchy type singular integral equation. For further analysis of the integral equations, it is useful to consider the equations (28) and (29). The equation (30) can be written as

$$\begin{aligned}&\int_{\hat l}^{\infty}\left[\frac{2}{\hat\zeta-\hat z}-\frac{1}{\hat\zeta+\hat z-2\hat l}-\frac{1}{\hat\zeta+\hat z}\right]\hat\tau(\hat\zeta,\omega)\,d\hat\zeta\\ &+\int_{\hat l}^{\infty}[k_1(\hat\zeta-\hat z,\omega)-k_1(\hat\zeta+\hat z-2\hat l,\omega)\\ &+k_2(\hat\zeta-\hat z,\omega)-k_2(\hat\zeta+\hat z,\omega)]\,\hat\tau(\hat\zeta,\omega)\,d\hat\zeta=f(\hat z,\omega), \quad \hat z\ge\hat l\end{aligned} \quad (31)$$

where

$$k_1(d,\omega)=\int_0^{\infty}\left(\frac{\hat\xi I_1(\hat\lambda)}{\hat\lambda I_2(\hat\lambda)}-1\right)\sin(\hat\xi d)\,d\hat\xi, \quad (32)$$

$$k_2(d,\omega)=\int_0^{\infty}\left(\frac{\hat\xi K_1(\hat\lambda)}{\hat\lambda K_2(\hat\lambda)}-1\right)\sin(\hat\xi d)\,d\hat\xi, \quad (33)$$

$$\begin{aligned}f(\hat z,\omega)=\int_0^{\hat l}[k_2(\hat z-\hat\zeta,\omega)+\frac{1}{\hat z-\hat\zeta}+k_2(\hat\zeta+\hat z,\omega)+\frac{1}{\hat\zeta+\hat z}]\\ \times\hat\tau^*(\hat\zeta,\omega)\,d\hat\zeta\end{aligned} \quad (34)$$

with

$$\hat z=\frac{z}{a}, \quad \hat l=\frac{l}{a}, \quad \hat\zeta=\frac{\zeta}{a}, \quad \hat\xi=\xi a, \quad \hat\lambda=\lambda a, \quad (35)$$

where $\tau(z,\omega)=\hat\tau(\hat z,\omega)$, $\hat\lambda=\sqrt{\hat\xi^2-\omega_0^2}$ and $\omega_0=\dfrac{\omega a}{C_s}$ is non-dimension frequency.

The solution of the singular integral equation (31) is the unknown shear stress distribution at $r=a$ over $z\ge l$. The kernel $\frac{2}{(\hat\zeta-\hat z)}-\frac{1}{(\hat\zeta+\hat z-2\hat l)}$ is a generalized Cauchy kernel [11]. In this kernel, the terms $\frac{2}{(\hat\zeta-\hat z)}$ and $\frac{1}{(\hat\zeta+\hat z-2\hat l)}$ become unbounded as either $\hat{\zeta}$ approaches $\hat{z}$ or both $\hat{\zeta}$ and $\hat{z}$ approach the end point of the cavity, $\hat{l}$.

IV. Numerical Solution Of Singular Integral Equation

Because of complex kernel function exists in the integrand of the integral introduced in previous section, the integral cannot be determined analytically, and thus a numerical procedure is needed, which is presented in this section. To this end, it is convenient to write the integral equation (31) as

$$\int_0^1 G(x,v)\hat{\tau}(\hat{l}/v,\omega)\,dv = g(x,\omega), \qquad 0 \le v \le 1 \tag{36}$$

where

$$\begin{aligned} G(x,v) = &\left[\frac{2}{x-v} - \frac{2x-1}{v+x-2vx} + \frac{1}{x+v}\right] \\ &+ \left(\frac{\hat{l}}{v^2}\right)\left[k_1\left(\frac{\hat{l}}{v} - \frac{\hat{l}}{x},\omega\right) - k_1\left(\frac{\hat{l}}{v} + \frac{\hat{l}}{x} - 2\hat{l},\omega\right)\right. \\ &\left. + k_2\left(\frac{\hat{l}}{v} - \frac{\hat{l}}{x},\omega\right) - k_2\left(\frac{\hat{l}}{v} + \frac{\hat{l}}{x},\omega\right)\right], \end{aligned} \tag{37}$$

and $v = \hat{l}/\hat{\zeta}$, $x = \hat{l}/\hat{z}$ and $g(x,\omega) = f(\hat{l}/x,\omega)$. For normalizing the variable, it is convenient to consider an even extension of $\hat{\tau}(\hat{l}/v,\omega)$ with respect to the origin so that the limits of integration in (36) changes to the symmetric interval of $(-1,1)$, which results in

$$\frac{1}{2}\int_{-1}^{1} G(|x|,|v|)\,\hat{\tau}(\hat{l}/|v|,\omega)\,dv = g(|x|,\omega), \qquad -1 \le x \le 1 \tag{38}$$

In recognition of the singular behavior of $\tau(\hat{z},\omega)$ it is useful to separate the regular and singular part of the unknown function and by means of two sectionally analytic function which are statement in [7], $\hat{\tau}(\hat{l}/x,\omega)$ is expressed as [7, 11]

$$\hat{\tau}(\hat{l}/x,\omega) = T(x,\omega)/(1-x^2)^{\frac{1}{3}}, \tag{39}$$

where $T(x,\omega)$ is a regular and analytic unknown function with respect to x, which of course is bounded in the interval $-1 < x < 1$, and $(1-x^2)^{-\frac{1}{3}}$ is considered as a weighting function. Since the weighting function is the same as weighting function for Jacobi polynomials $P_n^{(\gamma,\beta)}(x)$ for $\gamma = \beta = \dfrac{1}{3}$ [12], one may use a numerical procedure based on Gauss-Jacobi integration formula to solve the integral equation (36). In the term of (39), (38) can be written as [11]:

$$\frac{1}{2}\int_{-1}^{1} G(|x|,|v|)\frac{T(|v|,\omega)}{(1-v^2)^{1/3}}\,dv = g(|x|,\omega). \qquad -1 \le x \le 1 \tag{40}$$

With the aid of the Gauss-Jacobi numerical integration rule and using collocation method [11], (40) can be reduced to

$$\frac{1}{2}\sum_{k=1}^{2N} W_k G(|x_j|,|v_k|)T(|v_k|,\omega) = g(|x_j|,\omega), \qquad j = 1,\ldots,2N-1 \tag{41}$$

where

$$W_k = \frac{-(4N+\gamma+\beta+2)\,2^{\gamma+\beta}}{(2N+1)!(2N+\gamma+\beta+1)\;P_{2N+1}^{(\gamma,\beta)}(v_k)\,dP_{2N}^{(\gamma,\beta)}(v_k)/dv} \times \frac{\Gamma(2N+\gamma+1)\Gamma(2N+\beta+1)}{\Gamma(2N+\gamma+\beta+1)} \tag{42}$$

and x_j, v_k are the roots of Jacobi's functions

$$P_{2N-1}^{(2/3,2/3)}(x_j) = 0, \qquad j = 1,\ldots,2N-1 \tag{43}$$

$$P_{2N}^{(-1/3,-1/3)}(v_k) = 0, \qquad k = 1,\ldots,2N \tag{44}$$

respectively. As the roots x_j and v_k are symmetric with respect to the origin, (41) can be reduced to a simpler system of equations as

$$\sum_{k=1}^{N} W_k G(x_j,v_k)T(v_k,\omega) = g(x_j,\omega). \qquad j = 1,\ldots,N \tag{45}$$

Equation (45) provides n equations to determine n unknowns $T(v_k,\omega)$ at n collocation points v_k, $k = 1$ to n. The last equation in (45) involves the root $x_N = 0$, which corresponds to infinity for z.

With the relationship between $T(x,\omega)$ and $\hat{\tau}(\hat{z},\omega)$, and the representations (20), (21), (24) and (25), the response of the half-space to an arbitrary distributed load, $\tau^*(z,\omega)$, can be determined completely. For many applications, the displacement and shear stress variation along the cavity wall are of particular interest. To determine these two functions, we recall equation (20), which can be written at $r = a$ as

$$u(a,z,\omega) = \frac{a}{\pi}\left\{Q(\hat{z},\omega) + \int_{\hat{l}}^{\infty}\left[q_2(\hat{z}+\hat{\zeta},\omega) + q_2(\hat{z}-\hat{\zeta},\omega)\right]\hat{\tau}(\hat{\zeta},\omega)\,d\hat{\zeta}\right\} \tag{46}$$

where

$$q_2(d,\omega) = -\int_0^{\infty}\frac{K_1(\hat{\lambda})}{\hat{\lambda}K_2(\hat{\lambda})}\cos(\hat{\xi}d)\,d\hat{\xi} \tag{47}$$

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$$Q(\hat{z},\omega)=\int_0^{\hat{l}}[q_2(\hat{z}+\hat{\zeta},\omega)+q_2(\hat{z}-\hat{\zeta},\omega)]\hat{\tau}^*(\hat{\zeta},\omega)\,d\hat{\zeta} \tag{48}$$

$$\hat{\tau}^*(\hat{z},\omega)=\tau^*(z,\omega) \tag{49}$$

In terms of the solution $T(x,\omega)$, one may write (46) as

$$\frac{u(a,z,\omega)}{a}=\frac{1}{\pi}\{Q(\hat{z},\omega)+\sum_{k=1}^{N}W_k[q_2(\hat{z}+\frac{\hat{l}}{v_k},\omega)+q_2(\hat{z}-\frac{\hat{l}}{v_k},\omega)](\frac{\hat{l}}{v_k^{\,2}})T(v_k,\omega)\},\qquad \hat{z}\le\hat{l} \tag{50}$$

which can be computed numerically. With the aid of (26), the shear stress $\tau_{z\theta}(a,z,\omega)$ can be expressed as

$$\frac{\tau_{z\theta}(a,z,\omega)}{\mu}=\frac{1}{\pi}\{f(\hat{z},\omega)+\int_{\hat{l}}^{\infty}[\frac{1}{\hat{z}-\hat{\zeta}}+\frac{1}{\hat{z}+\hat{\zeta}}+k_2(\hat{z}-\hat{\zeta},\omega)+k_2(\hat{z}+\hat{\zeta},\omega)]\times\hat{\tau}(\hat{\zeta},\omega)d\hat{\zeta}\}.\qquad \hat{z}\le\hat{l} \tag{51}$$

The forgoing representation also translates to

$$\frac{\tau_{z\theta}(a,z,\omega)}{\mu}=\frac{1}{\pi}\{f(\hat{z},\omega)+\sum_{k=1}^{N}W_kT(v_k,\omega)\times[\frac{1}{v_k-\hat{l}/\hat{z}}-\frac{1}{v_k+\hat{l}/\hat{z}}+\frac{\hat{l}}{v_k^{\,2}}\times(k_2(\hat{z}-\frac{\hat{l}}{v_k},\omega)+k_2(\hat{z}+\frac{\hat{l}}{v_k},\omega))]\} \tag{52}$$

V. Illustrative Results

The displacement Green's function for the problem in hand is determined by applying torsional shear stress, $\tau^*(z,t)$, on a ring on the wall of the cylindrical cavity at an arbitrary depth, s say,

$$\tau^*(z,t)=a\delta(z-s)e^{i\omega t},\qquad 0<s<l \tag{53}$$

where $\delta(x)$ denotes the Dirac-delta function. For the loading (53), one finds

$$f(\hat{z},\omega)=k_2(\hat{z}-\hat{s},\omega)+\frac{1}{\hat{z}-\hat{s}}+k_2(\hat{z}+\hat{s},\omega)+\frac{1}{\hat{z}+\hat{s}} \tag{54}$$

$$Q(\hat{z},\omega)=q_2(\hat{z}+\hat{s},\omega)+q_2(\hat{z}-\hat{s},\omega). \tag{55}$$

The solution presented by Pak and Abedzadeh [7] is used as a benchmark, to provide a comparison with the results in this paper for the static case. To this end, a cylindrical cavity with depth $\hat{l}=2$ in a half-space is considered. Fig. 2 shows the displacement of the wall for different s evaluated in this study and the same results reported by Pak and Abedzadeh [7], where an excellent agreement is discovered between two solutions.

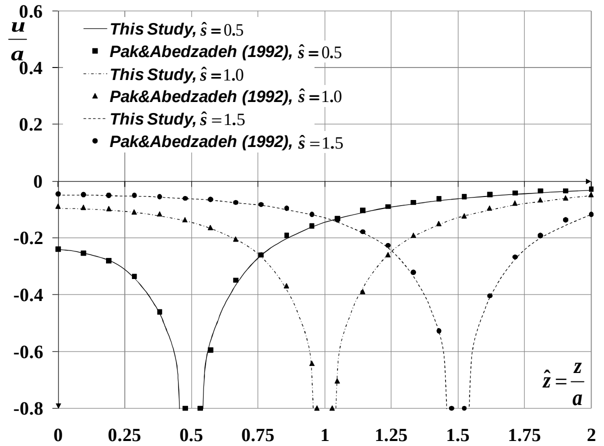

Fig. 2 Comparison of static displacements at $r=a$ along depth due to ring load in isotropic elastic half-space with Pak and Abedzadeh (1992) for $\hat{l}=2.0$, $\hat{s}=0.5,1.0$ and 1.5

The displacement, u, and the stress τ for different dimensionless frequency are illustrated in Figs. 3 and 4, respectively. It can be seen from the figures that, the responses of the half-space are decisively affected by the frequency of excitation. As frequency increase, both the real and imaginary parts show more oscillatory variation with the depth. As expected the dissipation of the displacement happens in both upward and downward direction. From the figures, the presence of the singularity in the displacement field due to the torsional ring load is apparent. In addition, it can be easily deduced from (51) that the shear distribution has a load induced singularity at $z=s$ of the order $(z-s)^{-1}$ and shape induced singularity at the bottom of the hole at $z=l$ of the order $|z-l|^{-\frac{1}{3}}$.

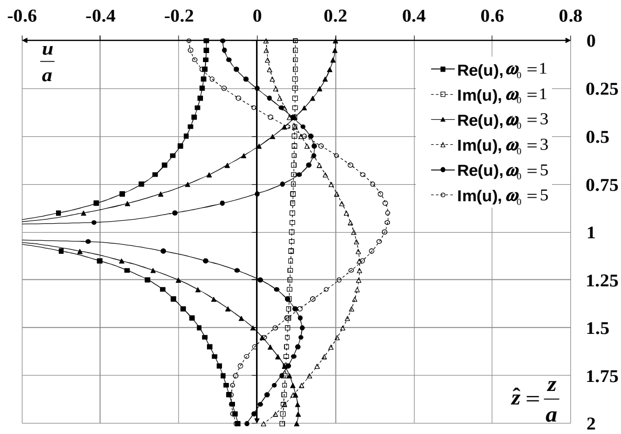

Fig. 3 Real and imaginary parts of displacement at $r=a$ along depth for isotropic half-space with different dimensionless frequency for $\hat{l}=2.0$ and $\hat{s}=1.0$

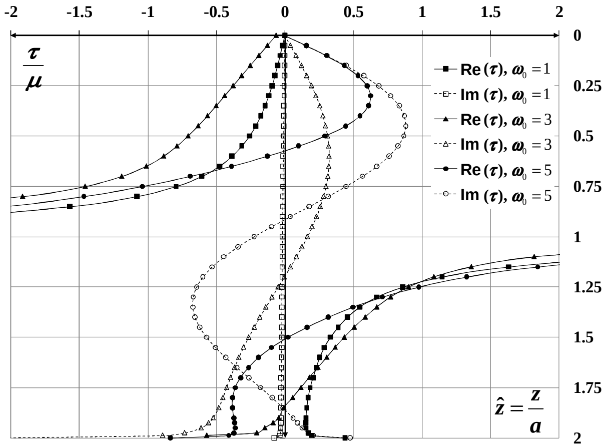

Fig. 4 Real and imaginary parts of shear stress at $r = a$ along depth for isotropic half-space with different dimensionless frequency for $\hat{l} = 2.0$ and $\hat{s} = 1.0$

VI. Conclusion

An isotropic half-space containing an open cylindrical cavity of finite length has been considered to be under the effect of a time-harmonic torsion force applied on the surface of the cavity. Applying cosine transforms, the boundary value problem for the fundamental solution has reduced to a generalized Cauchy singular integral equation. The obtained Cauchy integral equation has numerically been solved with the aid of both the Gauss-Jacobi procedure and collocation method. Integral representations for the stress and displacement have been obtained, and it has been shown that their degenerated form to the static problem is coincide with the solutions given by Pak and Abedzadeh (1992). The results are numerically evaluated and illustrated. Some singularities are observed in the illustrations in both the displacement and shear stress fields, which are either load induced or shape induced singularities.

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