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A Hamiltonian Decomposition of 5-star

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Abstract—Star graphs are Cayley graphs of symmetric groups of permutations, with transpositions as the generating sets. A star graph is a preferred interconnection network topology to a hypercube for its ability to connect a greater number of nodes with lower degree. However, an attractive property of the hypercube is that it has a Hamiltonian decomposition, i.e. its edges can be partitioned into disjoint Hamiltonian cycles, and therefore a simple routing can be found in the case of an edge failure. The existence of Hamiltonian cycles in Cayley graphs has been known for some time. So far, there are no published results on the much stronger condition of the existence of Hamiltonian decompositions. In this paper, we give a construction of a Hamiltonian decomposition of the star graph 5-star of degree 4, by defining an automorphism for 5-star and a Hamiltonian cycle which is edge-disjoint with its image under the automorphism.

Keywords—interconnection networks; paths and cycles; graphs and groups.

I. Introduction

NETWORKED computer systems have basic requirements such as fast communication and fault tolerance, which are met by an appropriate choice of interconnection topology. The star graph has been proposed as an interconnection network topology that is better than the hypercube for its ability to connect a greater number of nodes with lower degree [1]. On the other hand, an attractive property of the hypercube is that its edges can be partitioned into disjoint Hamiltonian cycles [2]. The presence of edge-disjoint Hamiltonian cycles is desirable for interconnection networks for various reasons. Fault tolerance is easier to achieve as a simple routing can be found in the case of an edge failure. Efficiency can also be improved. An example is the case of all-to-all broadcasting in multiport systems, where a node can send to or receive from all its neighbours in unit time, as messages can be broken down into smaller messages and sent along edge-disjoint Hamiltonian cycles. Edge-disjoint Hamiltonian cycles have been investigated in various interconnection topologies, for example in deBruijn networks [3] and tori [4]. They have also been studied in star graphs and lower bounds for the number of pairwise edge-disjoint Hamiltonian cycles have been given in [5]. However, there has been no significant progress on the optimum case of edge-disjoint cycles, where all the edges in the network topology are partitioned into Hamiltonian cycles, beyond the case of the hypercube.

Star graphs are Cayley graphs of symmetric groups of permutations of finitely many elements and certain restricted sets of transpositions as the generating sets. Properties such as the existence of Hamiltonian paths and cycles [6], Hamiltonian laceability [7] and indeed Hamiltonian decomposability [8], [9] have been considered for various classes of Cayley graphs. To date, it is known that a Cayley graph over a symmetric group and any generating set of transpositions has a Hamiltonian cycle [10]. Thus, this early result demonstrates that star graphs of any degree have a Hamiltonian cycle. To our knowledge, there are no published results on Hamiltonian decompositions of star graphs. The Hamiltonian decompositions of Cayley graphs of degree 4 in [8] concern Cayley graphs over abelian groups. In this paper, we give an a construction of a Hamiltonian decomposition of star graph 5-star of degree 4, by defining a graph automorphism on 5-star and a Hamiltonian cycle that has an edge-disjoint image under the automorphism. As 5-star is of degree 4, this gives a Hamiltonian decomposition.

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II. Preliminaries

We give the basic definitions of star graphs, Hamiltonian cycles and automorphisms.

Definition 1: The *n-star* graph St_n is the simple undirected regular graph of degree *n-1* whose vertices $V(St_n)$ are sequences of n elements $\{a_1, \ldots, a_n\}$

$$V(St_n) = \{a_{\rho(1)} \ldots a_{\rho(n)} : \rho \text{ is a permutation of } \{1, \ldots, n\}\}$$

and whose edges $E(St_n)$ correspond to swapping the positions of the first element with one of the other *n-1* elements, i.e. $e \in E(St_n)$ is of the form:

$$\begin{aligned} e \;=\; & (a_{\rho(1)} \ldots a_{\rho(i-1)} a_{\rho(i)} a_{\rho(i+1)} \ldots a_{\rho(n)}, \\ & a_{\rho(i)} \ldots a_{\rho(i-1)} a_{\rho(1)} a_{\rho(i+1)} \ldots a_{\rho(n)}) \end{aligned} \quad (1)$$

We define the *distance* between two distinct elements to be:

$$\delta(a_i, a_j) \;=\; min\{|i-j|, n-|i-j|\}$$

Clearly $\delta(a_i, a_j) = \delta(a_j, a_i)$. The *length* of the edge e above, $\lambda(e)$, is defined to be $\delta(a_{\rho(1)}, a_{\rho(i)})$.

Definition 2: A *Hamiltonian cycle* in St_n is a pair of sequences $(\underline{v}, \underline{e})$ of vertices $\underline{v} = v_1 \ldots v_{n!+1}$ and edges $\underline{e} = e_1 \ldots e_{n!}$ such that:

(i) $e_i \;=\; (v_i, v_{i+1}) \in E(St_n)$ $(1 \leq i \leq n!)$,
(ii) $\{v_1, \ldots, v_{n!+1}\} \;=\; V(St_n)$,
(iii) $v_1 = v_{n!+1}$.

Thus, a Hamiltonian cycle follows a path along edges visiting each vertex exactly once before returning to the first vertex. A *Hamiltonian decomposition* of St_{2k+1} where $k \geq 1$ consists of k Hamiltonian cycles that are edge-disjoint, i.e. no two Hamiltonian cycles have a common edge.

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Definition 3: Let (V, E) be a graph, where V is a set of vertices and $E \subseteq V \times V$ a set of edges. Then, a mapping $\Phi : V \to V$ is an *automorphism* iff:

(i) Φ is bijective
(ii) for all $v_1, v_2 \in V$,
$(v_1, v_2) \in E$ implies $(\Phi(v_1), \Phi(v_2)) \in E$

III. An automorphism

The following lemma gives a basic class of graph automorphisms preserving Hamiltonian cycles.

Lemma 4: Let $\phi : \{a_1, \ldots, a_n\} \to \{a_1, \ldots, a_n\}$ be a bijection. Then:

(i) $\Phi : V(St_n) \to V(St_n)$, given by $\Phi(a_{\rho(1)} \ldots a_{\rho(n)}) = \phi(a_{\rho(1)}) \ldots \phi(a_{\rho(n)})$, is an automorphism of the graph St_n
(ii) if $\underline{v}=v_1 \ldots v_{n!+1}$, $\underline{e}=(v_1, v_2) \ldots (v_{n!}, v_{n!+1})$ and $(\underline{v}, \underline{e})$ is a Hamiltonian cycle in St_n, then $\Phi_H(\underline{v}, \underline{e}) = (\Phi(v_1)...\Phi(v_{n!+1}), (\Phi(v_1), \Phi(v_2))...(\Phi(v_{n!}), \Phi(v_{n!+1})))$ is also a Hamiltonian cycle.

Proof: We check that Φ is an automorphism. If $v_1 \neq v_2 \in V(St_n)$, say $v_1 = (a_{\rho^1(1)} \ldots a_{\rho^1(n)})$, $v_2 = (a_{\rho^2(1)} \ldots a_{\rho^2(n)})$, where $\rho^1(i) \neq \rho^2(i)$ for some $1 \leq i \leq n$, then $\phi(a_{\rho^1(i)}) \neq \phi(a_{\rho^2(i)})$ as ϕ is injective, and so $\Phi(v_1)$ and $\Phi(v_2)$ differ on their respective i-th elements $\phi(a_{\rho^1(i)})$ and $\phi(a_{\rho^2(i)})$. Thus, Φ is injective. It is surjective as, given $b_1 \ldots b_n \in V(St_n)$, by surjectivity of ϕ we can choose $a_{\rho(1)}, \ldots, a_{\rho(n)}$ such that $\phi(a_{\rho(1)}) = b_1, \ldots, \phi(a_{\rho(n)}) = b_n$ and therefore $\Phi(\phi(a_{\rho(1)}) \ldots \phi(a_{\rho(n)})) = b_1 \ldots b_n$. To show that Definition 3(ii) holds, let $(v_1, v_2) \in E(St_n)$. By Definition 1, for some permutation ρ and $1 \leq i \leq n!$,

$$e = (a_{\rho(1)} \ldots a_{\rho(i-1)} a_{\rho(i)} a_{\rho(i+1)} \ldots a_{\rho(n)}, \quad a_{\rho(i)} \ldots a_{\rho(i-1)} a_{\rho(1)} a_{\rho(i+1)} \ldots a_{\rho(n)})$$

Then,

$$\Phi(v_1) = \phi(a_{\rho(1)})...\phi(a_{\rho(i-1)})\phi(a_{\rho(i)})\phi(a_{\rho(i+1)})...\phi(a_{\rho(n)})$$

$$\Phi(v_2) = \phi(a_{\rho(i)})...\phi(a_{\rho(i-1)})\phi(a_{\rho(1)})\phi(a_{\rho(i+1)})...\phi(a_{\rho(n)})$$

As $\phi : \{a_1, \ldots, a_n\} \to \{a_1, \ldots, a_n\}$ is a bijection, there is a permutation σ of $\{1, \ldots, n\}$ such that:

$$\phi(a_j) = a_{\sigma(j)} \text{ for } 1 \leq j \leq n$$

Therefore,

$$\Phi(v_1) = a_{\sigma\rho(1)} \ldots a_{\sigma\rho(i-1)} a_{\sigma\rho(i)} a_{\sigma\rho(i+1)} \ldots a_{\sigma\rho(n)}$$

$$\Phi(v_2) = a_{\sigma\rho(i)} \ldots a_{\sigma\rho(i-1)} a_{\sigma\rho(1)} a_{\sigma\rho(i+1)} \ldots a_{\sigma\rho(n)}$$

and so $(\Phi(v_1), \Phi(v_2))$ satisfies (1) with $\sigma\rho$ in place of ρ and thus $(\Phi(v_1), \Phi(v_2)) \in E$. This completes the check of (i) of this lemma, that Φ is an automorphism.

To prove (ii) of this lemma, we check that (i), (ii) and (iii) of Definition 2 are satisfied. Put $(\underline{v}', \underline{e}') = \Phi_H(\underline{v}, \underline{e})$ so that:

$$\underline{v}' = \Phi(v_1) \ldots \Phi(v_{n!+1})$$

$$\underline{e}' = (\Phi(v_1), \Phi(v_2)) \ldots (\Phi(v_{n!}), \Phi(v_{n!+1}))$$

For (i) of Definition 2, let $(\Phi(v_i), \Phi(v_{i+1})) \in e'$ where $1 \leq i \leq n!$. As $(\underline{v}, \underline{e})$ is a Hamiltonian cycle in St_n, we have that (v_i, v_{i+1}) is an edge in $E(St_n)$. By (i) of this lemma, Φ is an automorphism and so $(\Phi(v_i), \Phi(v_{i+1}))$ is an edge in $E(St_n)$ as required. For Definition 2(ii), as Φ is an automorphism, it maps the set of all vertices $\{v_1, \ldots, v_{n!+1}\}$ onto itself. Thus $V(St_n) = \{v_1, \ldots, v_{n!+1}\} = \{\Phi(v_1), \ldots, \Phi(v_{n!+1})\}$. For Definition 2(iii), we note that as $(\underline{v}, \underline{e})$ is a Hamiltonian cycle in St_n, $v_1 = v_{n!+1}$ and therefore $\Phi(v_1) = \Phi(v_{n!+1})$. ■

The automorphism of interest to us for the graph St_5, denoted Φ_5, corresponds to the bijection ϕ_5 on the five elements $a_1 = a$, $a_2 = b$, $a_3 = c$, $a_4 = d$ and $a_5 = e$ given by:

$$\phi_5(a) = c, \ \phi_5(b) = a, \ \phi_5(c) = d, \ \phi_5(d) = b, \ \phi_5(e) = e$$

An important property of Φ_5, that we shall make use of, is that edges are mapped to edges of a different length.

Lemma 5: If (v_1, v_2) in an edge in St_5, then $\lambda(v_1, v_2) \neq \lambda(\Phi_5(v_1), \Phi_5(v_2))$.

Proof: The length $\lambda(v_1, v_2)$ of an edge (v_1, v_2) is the distance between the two symbols $a_i, a_j \in \{a, b, c, d, e\}$ swapped at that edge. The length of the corresponding edge under Φ_5 is the distance between $\phi_5(a_i)$ and $\phi_5(a_j)$. There are 10 pairs of symbols $\{a_i, a_j\}$ to consider:

(i) $\delta(a, b) = 1$, $\delta(\phi_5(a), \phi_5(b)) = \delta(c, a) = 2$,
(ii) $\delta(a, c) = 2$, $\delta(\phi_5(a), \phi_5(c)) = \delta(c, d) = 1$,
(iii) $\delta(a, d) = 2$, $\delta(\phi_5(a), \phi_5(d)) = \delta(c, b) = 1$,
(iv) $\delta(a, e) = 1$, $\delta(\phi_5(a), \phi_5(e)) = \delta(c, e) = 2$,
(v) $\delta(b, c) = 1$, $\delta(\phi_5(b), \phi_5(c)) = \delta(a, d) = 2$,
(vi) $\delta(b, d) = 2$, $\delta(\phi_5(b), \phi_5(d)) = \delta(a, b) = 1$,
(vii) $\delta(b, e) = 2$, $\delta(\phi_5(b), \phi_5(e)) = \delta(a, e) = 1$,
(viii) $\delta(c, d) = 1$, $\delta(\phi_5(c), \phi_5(d)) = \delta(d, b) = 2$,
(ix) $\delta(c, e) = 2$, $\delta(\phi_5(c), \phi_5(e)) = \delta(d, e) = 1$,
(x) $\delta(d, e) = 1$, $\delta(\phi_5(d), \phi_5(e)) = \delta(b, e) = 2$.

■

IV. Construction of a Hamiltonian cycle

A Hamiltonian cycle for St_5 is constructed by partitioning the vertices of St_5 into 6 pairwise disjoint cycles $C_1, \ldots, C_6$, and then producing a 7th cycle C_7 that meets each of the other cycles at exactly two vertices and a common edge. It is clear that the union of the edges in the 7 cycles, excluding edges that C_7 has in common with any of the other 6 cycles, is then a Hamiltonian cycle; we denote it by C. Below, in Lemma 7, we define a cycle C_1 from which 5 further cycles C_2, C_3, C_4, C_5 and C_6 are generated by the 5 length-preserving automorphisms of the following lemma:

Lemma 6: The 5 maps given by:

$\Psi_2(a_{\rho(1)}a_{\rho(2)}a_{\rho(3)}a_{\rho(4)}a_{\rho(5)}) = a_{\rho(1)}a_{\rho(2)}a_{\rho(5)}a_{\rho(3)}a_{\rho(4)}$
$\Psi_3(a_{\rho(1)}a_{\rho(2)}a_{\rho(3)}a_{\rho(4)}a_{\rho(5)}) = a_{\rho(1)}a_{\rho(2)}a_{\rho(4)}a_{\rho(5)}a_{\rho(3)}$
$\Psi_4(a_{\rho(1)}a_{\rho(2)}a_{\rho(3)}a_{\rho(4)}a_{\rho(5)}) = a_{\rho(1)}a_{\rho(2)}a_{\rho(5)}a_{\rho(4)}a_{\rho(3)}$

$\Psi_5(a_{\rho(1)}a_{\rho(2)}a_{\rho(3)}a_{\rho(4)}a_{\rho(5)}) = a_{\rho(1)}a_{\rho(2)}a_{\rho(3)}a_{\rho(5)}a_{\rho(4)}$
$\Psi_6(a_{\rho(1)}a_{\rho(2)}a_{\rho(3)}a_{\rho(4)}a_{\rho(5)}) = a_{\rho(1)}a_{\rho(2)}a_{\rho(4)}a_{\rho(3)}a_{\rho(5)}$
where $a_{\rho(1)}a_{\rho(2)}a_{\rho(3)}a_{\rho(4)}a_{\rho(5)}$ is any given point in St_5 with corresponding permutation ρ, are automorphisms of St_5 which preserve cycles and lengths of edges, i.e.

$$\lambda(v_1, v_2) = \lambda(\Psi_i(v_1), \Psi_i(v_2)) \ \ (v_1, v_2 \in V(St_5),\ 2 \leq i \leq 6)$$

(The 5 maps correspond to the 5 possible alternative orders of the last 3 positions in a vertex.)
Proof: We check that the lemma holds for Ψ_3 - a very similar check can be performed for Ψ_2, Ψ_4, Ψ_5 and Ψ_6. Suppose that $a_{\rho(1)}a_{\rho(2)}a_{\rho(3)}a_{\rho(4)}a_{\rho(5)}$ and $a_{\rho'(1)}a_{\rho'(2)}a_{\rho'(3)}a_{\rho'(4)}a_{\rho'(5)} \in St_5$ differ, i.e. ρ and ρ' differ. Then, clearly,

$$\Psi_3(a_{\rho(1)}a_{\rho(2)}a_{\rho(3)}a_{\rho(4)}a_{\rho(5)}) = a_{\rho(1)}a_{\rho(2)}a_{\rho(4)}a_{\rho(5)}a_{\rho(3)} \neq a_{\rho'(1)}a_{\rho'(2)}a_{\rho'(4)}a_{\rho'(5)}a_{\rho'(3)} = \Psi_3(a_{\rho'(1)}a_{\rho'(2)}a_{\rho'(3)}a_{\rho'(4)}a_{\rho'(5)})$$

Thus, Ψ_3 is injective. If $a_{\rho(1)}a_{\rho(2)}a_{\rho(3)}a_{\rho(4)}a_{\rho(5)}$ is any vertex in St_n, then $\Psi_3(a_{\rho(1)}a_{\rho(2)}a_{\rho(5)}a_{\rho(3)}a_{\rho(4)}) = a_{\rho(1)}a_{\rho(2)}a_{\rho(3)}a_{\rho(4)}a_{\rho(5)}$. Thus, Ψ_3 is surjective. To show that Ψ_3 is an automorphism, suppose that $v_1 = a_{\rho(1)} \ldots a_{\rho(i)} \ldots$, $v_2 = a_{\rho(i)} \ldots a_{\rho(1)} \ldots$, so that $(v_1, v_2) \in E(St_5)$, where $i = 2, 3, 4$ or 5. In the case $i = 2$,

$$(v_1, v_2) = (a_{\rho(1)}a_{\rho(2)}a_{\rho(3)}a_{\rho(4)}a_{\rho(5)}, a_{\rho(2)}a_{\rho(1)}a_{\rho(3)}a_{\rho(4)}a_{\rho(5)}),$$
$$(\Psi_3(v_1), \Psi_3(v_2)) = (a_{\rho(1)}a_{\rho(2)}a_{\rho(4)}a_{\rho(5)}a_{\rho(3)}, a_{\rho(2)}a_{\rho(1)}a_{\rho(4)}a_{\rho(5)}a_{\rho(3)}) \in E(St_5)$$

In the case $i = 3$,

$$(v_1, v_2) = (a_{\rho(1)}a_{\rho(2)}a_{\rho(3)}a_{\rho(4)}a_{\rho(5)}, a_{\rho(3)}a_{\rho(2)}a_{\rho(1)}a_{\rho(4)}a_{\rho(5)}),$$
$$(\Psi_3(v_1), \Psi_3(v_2)) = (a_{\rho(1)}a_{\rho(2)}a_{\rho(4)}a_{\rho(5)}a_{\rho(3)}, a_{\rho(3)}a_{\rho(2)}a_{\rho(4)}a_{\rho(5)}a_{\rho(1)}) \in E(St_5)$$

In the case $i = 4$,

$$(v_1, v_2) = (a_{\rho(1)}a_{\rho(2)}a_{\rho(3)}a_{\rho(4)}a_{\rho(5)}, a_{\rho(4)}a_{\rho(2)}a_{\rho(3)}a_{\rho(1)}a_{\rho(5)}),$$
$$(\Psi_3(v_1), \Psi_3(v_2)) = (a_{\rho(1)}a_{\rho(2)}a_{\rho(4)}a_{\rho(5)}a_{\rho(3)}, a_{\rho(4)}a_{\rho(2)}a_{\rho(1)}a_{\rho(5)}a_{\rho(3)}) \in E(St_5)$$

In the case $i = 5$,

$$(v_1, v_2) = (a_{\rho(1)}a_{\rho(2)}a_{\rho(3)}a_{\rho(4)}a_{\rho(5)}, a_{\rho(5)}a_{\rho(2)}a_{\rho(3)}a_{\rho(4)}a_{\rho(1)}),$$
$$(\Psi_3(v_1), \Psi_3(v_2)) = (a_{\rho(1)}a_{\rho(2)}a_{\rho(4)}a_{\rho(5)}a_{\rho(3)}, a_{\rho(5)}a_{\rho(2)}a_{\rho(4)}a_{\rho(1)}a_{\rho(3)}) \in E(St_5)$$

Thus, Ψ_3 is an automorphism and therefore also preserves cycles by an argument similar to that in Lemma 4(ii). We note from the above cases that, given an edge of the form:

$$(a_{\rho(1)} \ldots a_{\rho(i)} \ldots, a_{\rho(i)} \ldots a_{\rho(1)} \ldots) \in E(St_5), \qquad (2)$$

we have that

$$(\Psi_3(a_{\rho(1)} \ldots a_{\rho(i)} \ldots), \Psi_3(a_{\rho(i)} \ldots a_{\rho(1)} \ldots))$$

is still of the form (2), albeit $a_{\rho(i)}$ occurs in a different position in $\Psi_3(a_{\rho(1)} \ldots a_{\rho(i)} \ldots)$ and $a_{\rho(1)}$ occurs in a different position in $\Psi_3(a_{\rho(i)} \ldots a_{\rho(1)} \ldots)$. It follows, by the definition of the length of edges, that Ψ_3 preserves lengths of edges. ■

Lemma 7: The six cycles $C_1, C_2, C_3, C_4, C_5, C_6$ formed by starting at vertices *abcde*, *abdec*, *abced*, *abedc*, *abecd*, *abdce* respectively, and progressing along edges of length 1 until cycles are completed, partition St_5 into 6 disjoint cycles.
Proof: We list the 20 vertices of C_1:

$$abcde, bacde, cabde, dabce, eabcd,$$
$$aebcd, beacd, ceabd, deabc, edabc,$$
$$adebc, bdeac, cdeab, dceab, ecdab,$$
$$acdeb, bcdea, cbdea, dbcea, ebcda$$

As, we have that

$$\Psi_2(abcde) = abecd, \Psi_3(abcde) = abdec, \Psi_4(abcde) = abedc,$$
$$\Psi_5(abcde) = abced, \text{ and } \Psi_6(abcde) = abdce,$$

it follows by Lemma 6 that the other cycles also contain 20 vertices. The only vertex with ab in the first two positions in C_1 is $abcde$. By Lemma 6, as Ψ_2, Ψ_3, Ψ_4, Ψ_5 and Ψ_6 only reorder the last 3 elements of a vertex, the only vertices with ab in the first two positions in C_2, C_3, C_4, C_5, and C_6 are *abecd*, *abdec*, *abedc*, *abced*, and *abdce* respectively. Thus, each of *abcde*, *abecd*, *abdec*, *abedc*, *abced*, *abdce* can only occur in one of the cycles, and it follows that the C_i's are pairwise disjoint and account for the 6×20 vertices of St_5. ■

Lemma 8: The cycle C_7 given by:

$$bacde, abcde, cbade, dbace, bdace, adbce,$$
$$cdbae, dcbae, bcdae, acdbe, cadbe, dacbe, bacde \qquad (3)$$

meets each C_i $(1 \leq i \leq 6)$ at exactly two vertices and a common edge.
Proof: We have that:

$$(bacde, abcde) \in E(C_1)$$

by Lemma 7,

$$(cbade, dbace) = (\Psi_2(cbdea), \Psi_2(dbcea)) \in E(C_2)$$

by Lemmas 7 and 6,

$$(bdace, adbce) = (\Psi_3(bdeac), \Psi_3(adebc)) \in E(C_3)$$

by Lemmas 7 and 6,

$$(cdbae, dcbae) = (\Psi_4(cdeab), \Psi_4(dceab)) \in E(C_4)$$

by Lemmas 7 and 6,

$$(bcdae, acdbe) = (\Psi_5(bcdea), \Psi_5(acdeb)) \in E(C_5)$$

by Lemmas 7 and 6, and

$$(cadbe, dacbe) = (\Psi_6(cabde), \Psi_6(dabce)) \in E(C_6)$$

by Lemmas 7 and 6. ■

V. Edge-disjoint Hamiltonian cycles

The Hamiltonian cycle defined by means of $C_1, \ldots C_7$ in Section IV, produces a Hamiltonian cycle when mapped by the automorphism Φ_5 of Section III. It remains to show that the 2 Hamiltonian cycles are edge disjoint.

Lemma 9:

(i) For the set of vertices $V(C_7)$ of C_7, we have that $\Phi_5(V(C_7)) = V(C_7)$.

(ii) For the set of edges $E(C_7)$ of C_7, we have that $\Phi_5(E(C_7) \cap \bigcup_{i=1}^{6} E(C_i)) = E(C_7) - \bigcup_{i=1}^{6} E(C_i)$.

Proof: The vertices of (3) are mapped to

$$acdbe, cadbe, dacbe, bacde, abcde, cbade,$$

$$dbace, bdace, adbce, cdbae, dcbae, bcdae, acdbe \tag{4}$$

respectively, by Φ_5. We see that (4) is just a reordering of the edges in (3) and thus (i) holds. For (ii), consider the edges of $E(C_7) \cap \bigcup_{i=1}^{6} E(C_i)$ in Lemma 8 shown underlined below:

$$\underline{bacde, abcde}, \underline{cbade, dbace}, \underline{bdace, adbce},$$

$$\underline{cdbae, dcbae}, \underline{bcdae, acdbe}, \underline{cadbe, dacbe},$$

We have:

$$(\Phi_5(bacde), \Phi_5(abcde)) =$$

$$(acdbe, cadbe) \in E(C_7) - \bigcup_{i=1}^{6} E(C_i)$$

$$(\Phi_5(cbade), \Phi_5(dbace)) =$$

$$(dacbe, bacde) \in E(C_7) - \bigcup_{i=1}^{6} E(C_i)$$

$$(\Phi_5(bdace), \Phi_5(adbce)) =$$

$$(abcde, cbade) \in E(C_7) - \bigcup_{i=1}^{6} E(C_i)$$

$$(\Phi_5(cdbae), \Phi_5(dcbae)) =$$

$$(dbace, bdace) \in E(C_7) - \bigcup_{i=1}^{6} E(C_i)$$

$$(\Phi_5(bcdae), \Phi_5(acdbe)) =$$

$$(adbce, cdbae) \in E(C_7) - \bigcup_{i=1}^{6} E(C_i)$$

$$(\Phi_5(cadbe), \Phi_5(dacbe)) =$$

$$(dcbae, bcdae) \in E(C_7) - \bigcup_{i=1}^{6} E(C_i)$$

■

Theorem 10: The Hamiltonian cycles C and $\Phi_5(C)$ are edge-disjoint.

Proof: Let $v \in V(C) - V(C_7)$, $v \in V(C_i)$ say, where $1 \le i \le 6$. Then, there exist $u_1, u_2 \in V(C_i)$ such that the edges incident at v in C, (u_1, v) and (v, u_2), belong to C_i and so, by the definition of C_i in Lemma 7,

$$\lambda(u_1, v) = \lambda(v, u_2) = 1 \tag{5}$$

Consider the edges (v_1, v) and (v_2, v) incident at v in $\Phi_5(C)$. As $v \notin V(C_7)$, by Lemma 9(i) $v = \Phi_5(v')$ for some $v' \in V(C) - V(C_7)$, say $v' \in C_j$ where $1 \le j \le 6$. Then, there exist edges (v_1', v'), (v', v_2') in C_j such that $\Phi_5(v_1') = v_1$ and $\Phi_5(v_2') = v_2$. By the definition of C_j in Lemma 7,

$$\lambda(v_1', v') = \lambda(v', v_2') = 1 \tag{6}$$

By (6) and Lemma 5,

$$\lambda(\Phi_5(v_1'), \Phi_5(v')) = \lambda(\Phi_5(v'), \Phi_5(v_2')) = 2$$

i.e.

$$\lambda(v_1, v) = \lambda(v, v_2) = 2 \tag{7}$$

By (5) and (7), different edges are incident at the vertex $v \in C - C_7$ in C and $\Phi_5(C)$. Hence, we have shown that an edge in C, which has a vertex not in C_7, cannot belong to $\Phi_5(C)$. It follows that an edge common to both C and $\Phi_5(C)$ must be an edge in C_7. So, let e be an edge of C belonging to C_7. For $1 \le i \le 6$, e cannot be an edge in C_i as C does not contain edges common to C_7 and C_i. By Lemma 9(ii), Φ_5 maps an edge in C_7 and some C_i, where $1 \le i \le 6$ to e. Thus, Φ_5 maps an edge not in C to e. Therefore $e \notin \Phi_5(C)$. We conclude that C and $\Phi_5(C)$ have no common edges. ■

VI. Conclusions

We have given a Hamiltonian decomposition of 5-star, based on a graph automorphism relating the two Hamiltonian cycles. Our further work will investigate properties of similar automorphisms in higher degree star graphs to determine whether they can be used to establish or refute the existence of Hamiltonian decompositions there.

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