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Reasoning With Non-Binary Logics

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Abstract—Students in high education are presented with new terms and concepts in nearly every lecture they attend. Many of them prefer Web-based self-tests for evaluation of their concepts understanding since they can use those tests independently of tutors' working hours and thus avoid the necessity of being in a particular place at a particular time. There is a large number of multiple-choice tests in almost every subject designed to contribute to higher level learning or discover misconceptions. Every single test provides immediate feedback to a student about the outcome of that test. In some cases a supporting system displays an overall score in case a test is taken several times by a student. What we still find missing is how to secure delivering of personalized feedback to a user while taking into consideration the user's progress. The present work is motivated to throw some light on that question.

Keywords—Clustering, rough sets, many valued logic, predictions

I. Introduction

Automated evaluation of students concepts' understanding has been a subject of interest to researchers from various fields. One part of the research focuses on the cognitive site [15], [26] while the part is considering modelling and technical implementations [22], [16], [19]. At the same time researches from both sides consider which types of rules should be used in the decision making process resulting giving feedback to students.

Historically viewed, Boolean logic [13] and [33] has been the most common basis in automated decision making process. Boolean logic operates with two values 0 and 1. This implies serious restrictions while describing non-binary occurrences, [14], [20]. Fuzzy logic [7], [12], [17], rough sets theory [27], [28], [29], [30], [31], grey theory, [11], [18] [32], many valued logics, [23] and formal concept analysis, [24] are among the well known attempts to describe continues processes and situations where several degrees of truth are required.

Students in high education are presented with new terms and concepts in nearly every lecture they attend. Many of them prefer Web-based self-tests for evaluation of their concepts understanding since they can use those tests independently of tutors' working hours and thus avoid the necessity of being in a particular place at a particular time. There is a large number of multiple-choice tests in almost every subject designed to contribute to higher level learning or discover misconceptions. Every single test provides immediate feedback to a student about the outcome of that test. In some cases a supporting system displays an overall score in case a test is taken several times by a student. What we still find missing is how to secure delivering of personalized feedback to a user while taking into consideration the user's progress. The present work is motivated to throw some light on that question.

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The rest of the paper is organized as follows. Related work and supporting theory may be found in Section II. The decision process is presented in Section III. Conclusions and future work can be found in Section IV.

II. Preliminaries

Rough Sets were originally introduced in [27]. The presented approach provides exact mathematical formulation of the concept of approximative (rough) equality of sets in a given approximation space. An *approximation space* is a pair $A = (U, R)$, where U is a set called universe, and $R \subset U \times U$ is an indiscernibility relation.

Equivalence classes of R are called *elementary sets* (atoms) in A. The equivalence class of R determined by an element $x \in U$ is denoted by $R(x)$. Equivalence classes of R are called *granules* generated by R. The following definitions are often used while describing a rough set $X, X \subset U$:

- the *R-upper approximation* of X, $R^\star(x) := \bigcup_{x\in U}\{R(x) : R(x) \cap X \neq \emptyset\}$
- the *R-lower approximation* of X, $R_\star(x) := \bigcup_{x\in U}\{R(x) : R(x) \subseteq X\}$
- the *R-boundary region* of X, $RN_R(X) := R^\star(X) - R_\star(X)$

In the rough set theory [28], objects are described by either physical observations or measurements. Consider an information system $\mathcal{A} = (U, A)$ where information about an object $x \in U$ is given by means of some attributes from A, i.e., an object x can be identified with the so-called signature of $x : Inf(x) = a(x) : a \in A$.

- The R-positive region of X with respect to the relation R is $POS_R(X) = \underline{R}X$
- The R-negative region of X with respect to the relation R is the set $NEG_R(X) = U - \overline{R}X$
- The R-boundary region of X with respect to the relation R is the set $BN_R(X) = \overline{R}X - \underline{R}X$

Based on the knowledge R, we can say that

- the elements of $POS_R(X)$ certainly belong to X,
- the elements of $NEG_R(X)$ certainly do not belong to X,
- we cannot tell if the elements of $BN_R(X)$ belong to X or not, Fig. 1.

In [1] rough sets are described via three-valued logic. The value t corresponds to positive region of a set, the value f - to the negative region, and the undefined value u - to the border of the set. Due to the properties of the above regions in rough set theory, the logic's semantics is based on a non-deterministic matrix (Nmatrix).

Definition 1: [2] A non-deterministic matrix (Nmatrix) for a propositional language L is a tuple $\mathcal{M} = (\mathcal{T}, \mathcal{D}, \mathcal{O})$, where:

- $\mathcal{T}$ is a non-empty set of truth values.

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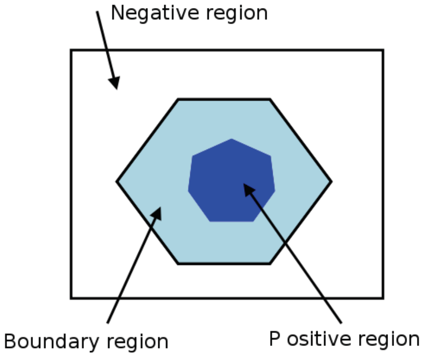

Fig. 1. The *R*-positive region, the *R*-negative region of *X* and the *R*-boundary region of *X*

TABLE I
Negation in the predicate language L_{RS}

$\simeq$	f	u	t
	t	u	f

- $\emptyset \subset D \subseteq \mathcal{T}$ is the set of designated values.
- For every *n*-ary connective '$\diamond$' of *L*, O includes a corresponding *n*-ary function $\tilde{\diamond}$ from $\mathcal{T}^n$ to $2^{\mathcal{T}} - \emptyset$.

The semantics the predicate language L_{RS} is given by the Nmatrix $\mathcal{M}_{RS} = (\mathcal{T}, \mathcal{D}, O)$, where $\mathcal{T} = \{f, u, t\}, \mathcal{D} = \{t\}$, and $-, \cup, \cap$ are interpreted as set-theoretic operations on rough sets. Their semantics is given in Table I, Table II, Table III, and Table IV where f, u and t stand for the appropriate singleton sets.

The implication is defined as $A \to B = \neg A \vee B$, for further details see Table V.

A. Non Binary Logics

Three-valued logic is often viewed as an extension to two-valued logic where apart from the two truth values 'true' and 'false' one operates with another truth value called 'unknown'. The last truth value is sometimes understood as 'undefined', or 'neither'. Among the widely applied in practise three-valued logics are Łukasiewicz's and Kleene's [6], [25].

The semantic characterization of a four-valued logic for expressing practical deductive processes is presented in [4] and [5]. In most information systems the management of databases is not considered to include neither explicit nor hidden inconsistencies. In real life situation information often come from different contradicting sources. Thus different sources can provide inconsistent data while deductive reasoning may result in hidden inconsistencies. The idea in Belnap's approach is to develop a logic that is not that dependable of inconsistencies. The Belnap's logic has four truth values 'T, F, Both, None'. The meaning of these values can be described as follows:

- an atomic sentence is stated to be true only (T),
- an atomic sentence is stated to be false only (F),
- an atomic sentence is stated to be both true and false, for instance, by different sources, or in different points of time (Both), and
- an atomic sentences status is unknown. That is, neither true, nor false (None).

The information about the truth-value of a sentence can have values from None to Both. Extensions of Belnap's logic are discussed in [21].

TABLE II
The $\tilde{\cup}$ operation in the predicate language L_{RS}

$\tilde{\cup}$	f	u	t
f	f	u	t
u	u	{u, t}	t
t	t	t	t

TABLE III
The $\tilde{\cap}$ operation in the predicate language L_{RS}

$\tilde{\cap}$	f	u	t
f	f	f	f
u	f	{f, u}	u
t	f	u	t

TABLE IV
The $\tilde{\neg}$ operation in the predicate language L_{RS}

$\tilde{\neg}$	f	u	t
	t	u	f

TABLE V
The $\tilde{\to}$ operation in the predicate language L_{RS}

$\tilde{\to}$	f	u	t
f	t	t	t
u	u	{u, t}	t
t	f	u	t

B. Fuzzy Functions

Fuzzy reasoning methods are often applied in intelligent systems, decision making and fuzzy control. Some of them present a reasoning result as a real number, while others use fuzzy sets. Fuzzy reasoning methods involving various fuzzy implications and compositions are discussed by many authors, f. ex. [3], and [8].

The included in this subsection definitions of fuzzy sets and fuzzy functions are taken from [35].

Definition 2: Let X be a space of points (objects), and $x \in X$ being a generic element. A fuzzy set (class) A in X is characterized by a membership (characteristic) function $f_A(x)$ which associates with each point in X a real number in the interval $[0, 1]$.

The value of $f_A(x)$ represents the "grade of membership" of x in A. This in contrast to the classical set theory where a membership function takes one of the two values 1 and 0, an element belongs the set or it does not.

C. Grey Theory

Grey theory is an effective method used to solve uncertainty problems with discrete data and incomplete information. The theory includes five major parts: grey prediction, grey relational analysis, grey decision, grey programming and grey control, [10], [11], and [18]. A quantitative approach for assessing the qualitative nature of organizational visions is presented in [32].

Definition 3: A grey system is defined as a system containing uncertain information presented by a grey number and grey variables.

Definition 4: Let X be the universal set. Then a grey set G of X is defined by its two mappings $\overline{\mu}_G(x)$ and $\underline{\mu}_G(x)$.

$$\begin{cases} \overline{\mu}_G(x) : x \to [0,1] \\ \underline{\mu}_G(x) : x \to [0,1] \end{cases}$$

$\overline{\mu}_G(x) \geq \underline{\mu}_G(x), x \in X, X = R$, $\overline{\mu}_G(x)$ and $\underline{\mu}_G(x)$ are the upper and lower membership functions in G respectively.

When $\overline{\mu}_G(x) = \underline{\mu}_G(x)$, the grey set G becomes a fuzzy set. It shows that grey theory considers the condition of the fuzziness and can deal flexibly with the fuzziness situation.

The grey number can be defined as a number with uncertain information. For example, the ratings of attributes are described by the linguistic variables; there will be a numerical interval expressing it. This numerical interval will contain uncertain information. A grey number is often written as $\otimes G$, $(\otimes G = G|_{\underline{\mu}}^{\overline{\mu}})$.

Definition 5: Lower-limit, upper-limit, and interval grey numbers.

$\otimes G = [\underline{G}, \infty]$ - if only the lower limit of G can be possibly estimated and G is defined as a lower-limit grey number.

$\otimes G = [-\infty, \overline{G}]$ - if only the upper limit of G can be possibly estimated and G is defined as a upper-limit grey number.

$\otimes G = [\underline{G}, \overline{G}]$ - the lower and upper limits of G can be estimated and G is defined as an interval grey number.

Grey number operation is an operation defined on sets of intervals, rather than real numbers. The length of a grey number $\otimes G$ is defined as

$$L(\otimes G) = [\overline{G} - \underline{G}].$$

Definition 6: [24] For two grey numbers $\otimes G_1 = [\underline{G}_1, \overline{G}_1]$ and $\otimes G_2 = [\underline{G}_2, \overline{G}_2]$, the possibility degree of $\otimes G_1 \leq \otimes G_2$ can be expressed as follows:

$$P\{\otimes G_1 \leq \otimes G_2\} = \frac{\max(0, L^\star - \max(0, \overline{G}_1 - \underline{G}_2))}{L^\star}$$

where $L^\star = L(\otimes G_1) + L(\otimes G_2)$.

D. Formal Concepts

Let P be a non-empty ordered set. If $sup\{x,y\}$ and $inf\{x,y\}$ exist for all $x, y \in P$, then P is called a *lattice* [9]. In a lattice illustrating partial ordering of knowledge values, the logical conjunction is identified with the meet operation and the logical disjunction with the join operation.

A *context* is a triple (G, M, I) where G and M are sets and $I \subset G \times M$. The elements of G and M are called *objects* and *attributes* respectively [9], [34].

For $A \subseteq G$ and $B \subseteq M$, define

$$A' = \{m \in M \mid (\forall g \in A) \quad gIm\},$$

$$B' = \{g \in G \mid (\forall m \in B) \quad gIm\}$$

where A' is the set of attributes common to all the objects in A and B' is the set of objects possessing the attributes in B.

A *concept* of the context (G, M, I) is defined to be a pair (A, B) where $A \subseteq G$, $B \subseteq M$, $A' = B$ and $B' = A$. The *extent* of the concept (A, B) is A while its *intent* is B. A subset A of G is the extent of some concept if and only if $A'' = A$ in which case the unique concept of the which A is an extent is (A, A'). The corresponding statement applies to those subsets $B \in M$ which is the intent of some concepts.

The set of all concepts of the context (G, M, I) is denoted by $\mathfrak{B}(G, M, I)$. $\langle \mathfrak{B}(G, M, I); \leq \rangle$ is a complete lattice and it is known as the *concept lattice* of the context (G, M, I).

III. The Approach

Student's understanding of a recently introduced term is evaluated automatedly using Web-based tests where a question is followed by a set of answers. A test contains one question and three answer alternatives

- correct (c),
- incorrect (w) or
- unanswered (u).

A test is randomly taken from set of tests related to the same term.

The three answer alternatives can naturally be placed in the three regions in rough set theory, provided the positive region corresponds to understanding, the negative region to lack of understanding, and the boundary region to lack of data (unanswered questions). The 3 valued logic connected to rough set theory, [1] is going to be applied for drawing conclusions when a test is taken several tymes by a particular student.

The 3 valued logic related to rough sets operates with rules defined in Table II and Table III while establishing the truth value of the result of two events. The rules in Table II are of more optimistic nature where the rules in Table III are more conservative. Applying one of them only is insufficient since:

- if we use only Table II and one of the outcomes is positive then the accumulative result will be always positive,
- if we use only Table III and one of the outcomes is negative then the accumulative result will be always negative.

Example 1: Results from trials and feedback for two cases are shown in Table VI and Table VII.

Therefore we suggest a combination of these rules. Note that our intention is not to introduce a new logic but to find a way to provide more adequate feedback to students.

The outcomes from the 1st and the 2nd trial are based on Table II. The idea is to apply rules that will reflect student's progress and at the same time give encouragement.

The conclusion after the 3rd trial is based on the outcomes of the the 2nd trial and the 3rd trial applying rules from Table III. In other words the rules from Table III will have a control function.

TABLE VI
Case 1

1st test	2st test	Feedback	3rd test	Feedback	Recommendation
w	c	possible understanding	w	lack of understanding	Theoretical explanations are suggested

TABLE VII
Case 2

1st test	2st test	Feedback	3rd test	Feedback	Recommendation
c	u	some understanding	u	cannot confirm understanding	Some examples are suggested

Outcomes from further trials are to be incorporated using the alternation between rules from Table II and Table III.

A small prototype was practically implemented and used in a subject at master level. The limited number of students gave an opportunity for receiving their personal views on the usefulness of such evaluation. As expected they prefer to work independently and to be able to test their understanding in a neutral environment. While being pleased with this new approach we still struggle with developing a pool of questions of reasonable size as well as good answer alternatives.

IV. Conclusion

This work was motivated by the need for providing sensible feedback to students about their current progress. Further work is needed to find out whether such reasoning can be used to give early indication about substantial misconceptions that can f. ex. cause student's inability to continue a particular study.

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