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A Study on Linking Upward Substitution and Fuzzy Demands in the Newsboy-Type Problem

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***Abstract*—**This paper investigates the effect of product substitution in the single-period 'newsboy-type' problem in a fuzzy environment. It is supposed that the single-period problem operates under uncertainty in customer demand, which is described by imprecise terms and modelled by fuzzy sets. To perform this analysis, we consider the fuzzy model for two-item with upward substitution. This upward substitutability is reasonable when the products can be stored according to certain attribute levels such as quality, brand or package size. We show that the explicit consideration of this substitution opportunity increase the average expected profit. Computational study is performed to observe the benefits of product's substitution.

***Keywords*—** Fuzzy demand, Newsboy, Single-period problem, Substitution.

I. Introduction

THE classical, single-period newsboy problem is a well-known operations research model and many extensions to it have been proposed in the last decade [1]. In commercial management there are various types of single-period problem (SPP), namely, the stocking of spare parts, perishable items, style goods and special seasonal products, etc., which have a wide relevance in business. An important aspect of the SPP is the effect of product substitution. In this paper, we extend the SPP to a case in which substitution leads to increase the profit under imprecise demand information.

Interest on the analysis of SPP in fuzzy environment has already been taken into account in the literature [2-4]. Recently, Dutta et al. [5] introduced the concept of fuzzy random demand in SPP in situations where fuzzy perception and random behavior both appear simultaneously into the demand level. In [6], they considered the model with reordering strategy under fuzzy demand, in which the reorder is to be made during the mid-season after the early-season demand has been observed. Ji and Shao [7] studied the model with fuzzy demands and price discount policy in bi-level context, in which there is a manufacturer in the upper level and several retailers in the lower level. However, none has studied the newsboy problem with substitution in fuzzy environment. Though, there is a growing interest to investigate the effect of product substitution in SPP with different contexts [8-11], but all these cited articles developed the newsboy-type problem with substitution in a probabilistic framework. However, there are many situations in real world that utilize knowledge-based information to describe the uncertainty. In this paper, an effort has been made to study the benefits of product's substitution for two-item newsboy problem with imprecise customer demand. This extension may be regarded as an amalgamation of product substitution and fuzzy perception into the single-period framework.

To perform this analysis, though several ways of substitution arise in practice, for simplicity, here we study the model with upward substitution only. That is, there is a possibility that a product with surplus inventory can be used as a substitute for other out of stock product, but not vice-versa. This one-way substitutability is reasonable when the products can be stored according to certain attribute levels such as quality, brand or package size. For example, suppose a retailer stores two grades of a product, a brand name product with higher rate and a low graded product with lower price. There is a chance for brand name customers to purchase the lower brand product if the brand name product is out of stock. In contrast, the customers for low brand product would prefer to satisfy their unmet demand at a different store because of their inability to purchase the brand name product. In developing the model, customer demands are taken as fuzzy in nature, which yields fuzzy profit functions. We use the interval-valued expectation of a fuzzy number proposed by Dubois and Prade [12] to rank fuzzy numbers such that the expected resultant profit can be obtained. In the computational study, we show that the explicit consideration of this substitution opportunity makes the decision-maker very much richer.

II. Modelling of the Problem with Upward Substitution in Fuzzy Environment

Consider a single-period 'newsboy-type' problem. Suppose, a retailer sells two grades of products with different brand, one with high brand (Product 1) and the other with low brand (Product 2). Without loss of generality, the demands for the products are subjectively believed to be independent fuzzy numbers.

The following notations are used to describe the model:

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i (=1, 2) the product's index,

c_i purchase cost for one unit of product i,

p_i selling price for one unit of product i,

Q_i order quantity for product i,

$\tilde{D}_i$ fuzzy demand for product i,

τ probability that unmet demand for product 1 is substitutable by product 2.

Let the demand of a product i is characterized by the triangular fuzzy number $\tilde{D}_i = (\underline{D}_i, D_i, \bar{D}_i)$ with the membership function $\mu_{\tilde{D}_i}$, where

$$\mu_{\tilde{D}_i}(x) = \begin{cases} L_i(x) = \left(\dfrac{x - \underline{D}_i}{D_i - \underline{D}_i}\right), & \underline{D}_i \le x \le D_i, \\ R_i(x) = \left(\dfrac{\bar{D}_i - x}{\bar{D}_i - D_i}\right), & D_i \le x \le \bar{D}_i, \\ 0, & \text{otherwise}. \end{cases} \quad (1)$$

If d_i be the actual realization of fuzzy demand $\tilde{D}_i$, for each Q_i, the parameter d_i can assume any value between $\underline{D}_i$ and $\bar{D}_i$. Of course, there may arise two situations, viz., over-stock($d_i \le Q_i$) or under-stock ($d_i > Q_i$). Assume that the substitution policy is upward, i.e., unmet demand of product 1 (*P*-I say) can be supplied by the excess item of product 2 (*P*-II), if any, but not vice-versa. Therefore the profit function can be formulated as follows:

When *P*-I is excess, without regarding product substitution, the expected profit function can be expressed as the sum of two independent newsboy problem and is given by

$$\begin{aligned} E[P] &= E[p_1 \min\{d_1, Q_1\} - c_1 Q_1] + E[p_2 \min\{d_2, Q_2\} - c_2 Q_2] \\ &= E[P_1] + E[P_2] \end{aligned} \quad (2)$$

On the other hand, when *P*-I is short and *P*-II is used to satisfy a portion τ of unmet demand for *P*-I, then the expected profit function is expressed as

$$\begin{aligned} E[P^S] &= E[p_1 \min\{d_1, Q_1\} - c_1 Q_1] + E[p_2 \min\{d_2, Q_2\} - c_2 Q_2 \\ &\quad + p_2 \max\{0, \min\{\tau(d_1 - Q_1), (Q_2 - d_2)\}\}] \\ &= E[P_1] + E[P_2^S] \end{aligned} \quad (3)$$

In $E[P_2^S]$, since $d_1 - Q_1$ is the unsatisfied demand of *P*-I and $Q_2 - d_2$ is the excess items of *P*-II, $p_2 \max\{0, \min\{\tau(d_1 - Q_1), (Q_2 - d_2)\}\}$ represents the extra revenue from *P*-II.

In SPP, the objective functions are the expected values of the resultant profits. Hence the retailer's problem is to (i) determine the optimal order quantities Q_i for each product i that maximizes the associated expected profit $E(\tilde{P}_i)$; (ii) analyze the expected profit function $E[P_2^S]$ if substitution is taken place and then (iii) calculate the benefit of product substitution, if any.

A. Determination of optimal order quantities

If there is no such opportunity for substitution, then Eq. (3) reduces to the sum of two independent newsboy problems (as in Eq. (2)). In this case (when $\tau = 0$), the profit functions for the individuals are given by

$$P_i(Q_i) = p_i \min\{d_i,\ Q_i\} - c_i\ Q_i\ ,\quad i = 1, 2\ .$$

When the customer demand $\tilde{D}_i$ is characterized imprecisely, for any feasible order quantity Q_i, it procures either a fuzzy over-stock profit $\tilde{P}_{i,OS}(Q_i)$ or a fuzzy under-stock profit $\tilde{P}_{i,US}(Q_i)$ and hence the resultant profit function for each product i is given by

$$\tilde{P}_i(Q_i) = (p_i \tilde{D}_{i,OS} - c_i Q_i) \cup (p_i - c_i) Q_i\ ;\quad i = 1,\ 2 \quad (4)$$

Since $\tilde{P}_i(Q_i)$ is a fuzzy quantity, using its several $\alpha-$level sets, we first find out the expected resultant profit and then the optimal order quantity (Q_i^0, say) is determined by maximizing the expected value of this fuzzy profit function (4).

Let $P_i(Q_i, \alpha) = [P_{i,L}(Q_i, \alpha), P_{i,R}(Q_i, \alpha)]$ be the $\alpha-$level set of $\tilde{P}_i(Q_i)$. According to Liou and Wang [13], since $P_{i,L}(\alpha)$ and $P_{i,R}(\alpha)$ are continuous on the closed interval [0,1], they are also integrable in [0,1]. Therefore, as in [12], the left and right probabilistic-mean values of $\tilde{P}_i$ are computed by

$$E_*[\tilde{P}_i(Q_i)] = \int_0^1 P_{i,L}(Q_i, \alpha) d\alpha\ ,\quad E^*[\tilde{P}_i(Q_i)] = \int_0^1 P_{i,R}(Q_i, \alpha) d\alpha$$

where $P_{i,L}(Q_i, \alpha)$ and $P_{i,R}(Q_i, \alpha)$ are the left and right endpoints of $P_i(Q_i, \alpha)$, respectively.

Hence, the expected value of fuzzy profit function $\tilde{P}_i(Q_i)$ is computed as

$$\begin{aligned} \bar{E}[\tilde{P}_i(Q_i)] &= \frac{1}{2}[E_*[\tilde{P}_i(Q_i)] + E^*[\tilde{P}_i(Q_i)]] \\ &= \int_0^1 0.5(P_{i,L}(Q_i, \alpha) + P_{i,R}(Q_i, \alpha)) d\alpha\ . \end{aligned} \quad (5)$$

Now, we analyze the following two cases according to the position of Q_i in $[\underline{D}_i, \bar{D}_i]$ as follows:

Case 1. When Q_i lies in the range of $\underline{D}_i$ and D_i, from fig. 1a, we obtain the respective $\alpha-$level sets of $\tilde{D}_i$ in over-stocking situation and under-stocking situation as

$$D_{i,OS}(\alpha) = \begin{cases} [D_{i,L}(\alpha), Q_i] & \alpha \le L_i(Q_i) \\ \phi & \alpha > L_i(Q_i) \end{cases},$$

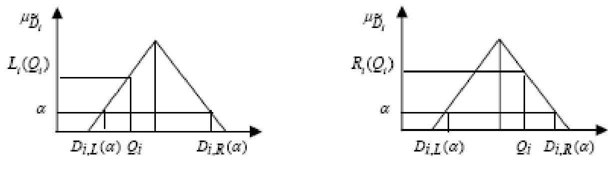
$$D_{i,US}(\alpha) = \begin{cases} [Q_i, D_{i,R}(\alpha)] & \alpha \le L_i(Q_i) \\ [D_{i,L}(\alpha), D_{i,R}(\alpha)] & \alpha > L_i(Q_i) \end{cases}.$$

Consequently, the left and right end points of $P_i(Q_i, \alpha)$ are derived as

$$P_{i,L}(Q_i,\alpha)=\begin{cases} p_i D_{i,L}(\alpha)-c_iQ_i, & 0\le\alpha\le L_i(Q_i)\\ (p_i-c_i)Q_i, & L_i(Q_i)\le\alpha\le 1\end{cases}$$

and

$$P_{i,R}(Q_i,\alpha)=(p_i-c_i)Q_i, \qquad 0\le\alpha\le 1.$$



a) When $\underline{D}_i \le Q_i \le D_i$ b) When $D_i \le Q_i \le \bar{D}_i$

Fig. 1 Alpha cut of $\tilde{D}_i$

Therefore, using (5) the expected profit for each product i is obtained as

$$\bar{E}[\tilde{P}_i(Q_i)]=\int_0^{L_i(Q_i)} 0.5((p_iD_{i,L}(\alpha)-c_iQ_i)+(p_i-c_i)Q_i)d\alpha$$
$$+\int_{L_i(Q_i)}^1 0.5((p_i-c_i)Q_i+(p_i-c_i)Q_i)d\alpha$$
$$=p_i\int_0^{L_i(Q_i)} 0.5D_{i,L}(\alpha)d\alpha+p_iQ_i(1-0.5L_i(Q_i))-c_iQ_i \qquad (6)$$

Case 2. When Q_i lies in the range of D_i and $\bar{D}_i$, from fig. 1b, we obtain the respective $\alpha-$level sets of $\tilde{D}_i$ in over-stocking situation and under-stocking situation as

$$D_{i,OS}(\alpha)=\begin{cases}[D_{i,L}(\alpha),Q_i] & \alpha\le R_i(Q_i)\\ [D_{i,L}(\alpha),D_{i,R}(\alpha)] & \alpha>R_i(Q_i)\end{cases},$$

$$D_{i,US}(\alpha)=\begin{cases}[Q_i,D_{i,R}(\alpha)] & \alpha\le R_i(Q_i)\\ \phi & \alpha>R_i(Q_i)\end{cases}.$$

In this case, using (5) the expected profit for each product i is obtained as

$$\bar{E}[\tilde{P}_i(Q_i)]=\int_0^{R_i(Q_i)} 0.5((p_iD_{i,L}(\alpha)-c_iQ_i)+(p_i-c_i)Q_i)d\alpha$$
$$+\int_{R_i(Q_i)}^1 0.5((p_iD_{i,L}(\alpha)-c_iQ_i)+(p_iD_{i,R}(\alpha)-c_iQ_i))d\alpha$$
$$=p_i\int_o^1 0.5D_{i,L}(\alpha)d\alpha+p_i\int_{R_i(Q_i)}^1 0.5D_{i,R}(\alpha)d\alpha$$
$$+0.5p_iQ_iR_i(Q_i)-c_iQ_i \qquad (7)$$

Having discussed these two situations, we may now introduce the following proposition.

Proposition 1. *There exists an optimal solution* Q_i^* *of problem* (4) *subject to the condition* $2c_i-p_i\ge 0$ *or* <0 *according to the position of* Q_i *in* $[\underline{D}_i,D_i]$ *or* $[D_i,\bar{D}_i]$, *respectively.*

Proof. To derive the optimality conditions for individual items, let us first consider that Q_i^* lies between $\underline{D}_i$ and D_i.

In this case, the interval-valued expectation $\bar{E}[\tilde{P}_i(Q_i)]$ is given in equation (6).

The optimal Q_i^* is determined by $\dfrac{d\bar{E}[\tilde{P}_i(Q_i)]}{dQ_i}=0$, which yields

$$L_i(Q_i^*)=\frac{2(p_i-c_i)}{p_i}.$$

Since $\dfrac{d^2\bar{E}[\tilde{P}_i(Q_i)]}{dQ_i^{\,2}}=-0.5p_iL_i'(Q_i)<0$ and the left-shape function $L_i(\cdot)$ lies between 0 and 1, we have the necessary condition as $2c_i-p_i\ge 0$.

Similarly, if $2c_i-p_i<0$, then the optimal Q_i^* lies between D_i and $\bar{D}_i$ and is given by $R_i(Q_i^0)=\dfrac{2c_i}{p_i}$ using (7).

Therefore, to obtain the expected resultant profit $E[P^S]$, the next step is to compute the expected profit $\bar{E}[\tilde{P}_2^S]$ earned against *P*-II when it is being used for substitution.

B. Expected resultant profit from product 2 after substitution

Consider the situation with upward substitution. That is, the excess item of *P*-II is being used to satisfy a portion τ of unmet demand for *P*-I, when it is out of stock. In this case

$$E[P_2^S]=E[p_2\min\{d_2,Q_2\}-c_2Q_2$$
$$+p_2\max\{0,\min\{\tau(d_1-Q_1),(Q_2-d_2)\}\}]$$

Since demands are fuzzy, the associated fuzzy profit function can be rewritten as the union of over stocking profit and under stocking profit and is given by

$$\tilde{P}_2^S=[(p_2\tilde{D}_{2,OS}-c_2Q_2)$$
$$+p_2\min\{\tau(\tilde{D}_{1,US}-Q_1),(Q_2-\tilde{D}_{2,OS})\}]\cup[(p_2-c_2)Q_2] \qquad (8)$$

In order to calculate the expected value of $\tilde{P}_2^S$ let us evoke (8) as

$$\tilde{P}_2^S=\tilde{P}_{2,OS}^S\cup\tilde{P}_{2,US}^S$$

where

$$\tilde{P}_{2,OS}^S=(p_2\tilde{D}_{2,OS}-c_2Q_2)+p_2\min\{\tau(\tilde{D}_{1,US}-Q_1),(Q_2-\tilde{D}_{2,OS})\} \qquad (9)$$

and

$$\tilde{P}_{2,US}^S=(p_2-c_2)Q_2 \qquad (10)$$

The term $\tilde{P}_{2,OS}^S$ indicates the event that demand for *P*-I is above the order quantity Q_1, whereas demand for *P*-II is below its order quantity Q_2 and therefore there is a chance for extra revenue form *P*-II by substitution. In the following, we will calculate the possible over-stock gain from *P*-II after substitution.

If $\tau(\tilde{D}_{1,US}-Q_1)\le(Q_2-\tilde{D}_{2,OS})$ (i.e., likely unmet demand of *P*-I less than the excess item of *P*-II) then (9) reduces to

$$\tilde{P}^S_{2,OS}=(p_2\tilde{D}_{2,OS}-c_2Q_2)+p_2\tau(\tilde{D}_{1,US}-Q_1) \tag{11}$$

Again, if $\tau(\tilde{D}_{1,US}-Q_1)>(Q_2-\tilde{D}_{2,OS})$ (i.e., all the excess quantity of *P*-II is demanded for unmet demand of *P*-I) then (9) reduces to

$$\tilde{P}^S_{2,OS}=(p_2-c_2)Q_2 \tag{12}$$

It is assumed that the demands for the products are independent, so the exact term $\min\{\tau(\tilde{D}_{1,US}-Q_1),(Q_2-\tilde{D}_{2,OS})\}$ can be realized only at the spot. In order to incorporate it into the optimization setting, the relation $\tau(\tilde{D}_{1,US}-Q_1)\le(Q_2-\tilde{D}_{2,OS})$ can be rewritten as

$$\tilde{D}_{2,OS}\le Q_2-\tau(\tilde{D}_{1,US}-Q_1) \tag{13}$$

For a given $\alpha-$level set of $\tilde{D}_{1,US}$, the right hand side of (13) reduces to

either

$$Q_2-\tau\times\begin{cases}([Q_1,D_{1,R}(\alpha)]-Q_1) & \text{for}\quad 0\le\alpha\le L_1(Q_1)\\ ([D_{1,L}(\alpha),D_{1,R}(\alpha)]-Q_1) & \text{for}\quad L_1(Q_1)\le\alpha\le 1\end{cases}$$

i.e.,

$$Q_2-\tau\times\begin{cases}[0,D_{1,R}(\alpha)-Q_1] & \text{for}\quad 0\le\alpha\le L_1(Q_1)\\ [D_{1,L}(\alpha)-Q_1,D_{1,R}(\alpha)-Q_1] & \text{for}\quad L_1(Q_1)\le\alpha\le 1\end{cases}$$

or

$$Q_2-\tau\times([Q_1,D_{1,R}(\alpha)]-Q_1) \qquad \text{for}\quad 0\le\alpha\le R_1(Q_1)$$

i.e.,

$$Q_2-\tau\times[0,D_{1,R}(\alpha)-Q_1] \qquad \text{for}\quad 0\le\alpha\le R_1(Q_1)$$

according to the position of Q_1 in $[\underline{D}_1,D_1]$ or $[D_1,\bar{D}_1]$, respectively.

Thus, for any realization d_1 of demand $\tilde{D}_{1,US}$, we may approximate $Q_2-\tau(\tilde{D}_{1,US}-Q_1)$ by defining

$$d_{2,0}=Q_2-\tau\bar{E}(\tilde{D}_{1,US}-Q_1), \tag{14}$$

where $\bar{E}(\tilde{D}_{1,US}-Q_1)$ $(=ES(Q_1)$, say) represents the expected number of shortages for *P*-I and is given by

either

$$ES(Q_1)=\int_0^{L_1(Q_1)}0.5(D_{1,R}(\alpha)-Q_1)d\alpha+\int_{L_1(Q_1)}^1 0.5(D_{1,L}(\alpha)+D_{1,R}(\alpha)-2Q_1)d\alpha$$

$$=\int_0^1 0.5D_{1,R}(\alpha)d\alpha+\int_{L_1(Q_1)}^1 0.5D_{1,L}(\alpha)d\alpha-Q_1(1-0.5L_1(Q_1))$$

or

$$ES(Q_1)=\int_0^{R_1(Q_1)}0.5(D_{1,R}(\alpha)-Q_1)d\alpha=\int_0^{R_1(Q_1)}0.5D_{1,R}(\alpha)d\alpha-0.5Q_1R_1(Q_1)$$

according to the position of Q_1 in $[\underline{D}_1,D_1]$ or $[D_1,\bar{D}_1]$, respectively.

Therefore, it is easy to identify that $\tilde{P}^S_{2,OS}$ will takes the form of (11) or (12) if and only if d_2 of $\tilde{D}_{2,OS}$ satisfy the relation $d_2\le$ or $>d_{2,0}$, respectively. That is, when *P*-I is short, profit from *P*-II can be computed by (11) or (12) or (10), according to the position of d_2 in $[\underline{D}_2,d_{2,0}]$ or $[d_{2,0},Q_2]$ or $[Q_2,\bar{D}_2]$, respectively.

In view of the above results we may now propose the following:

Proposition 2. *If d_2 be the actual realization of $\tilde{D}_2$, then*

(i) *For d_2 lying in the range of $\underline{D}_2$ and $d_{2,0}$ (i.e., $\underline{D}_2\le d_2\le d_{2,0}$) the profit function $\tilde{P}^S_{2,OS}$ takes the form* $\tilde{P}^S_{2,OS}=(p_2\tilde{D}_{2,OS}-c_2Q_2)+p_2\tau(\tilde{D}_{1,US}-Q_1)$

(ii) *Again, if d_2 lies between $d_{2,0}$ and Q_2 (i.e., $d_{2,0}<d_2\le Q_2$), then $\tilde{P}^S_{2,OS}$ reduces to* $(p_2-c_2)Q_2$

(iii) *On the other hand, if $d_2>Q_2$, then the under-stock profit $\tilde{P}^S_{2,US}$ is computed as* $\tilde{P}^S_{2,US}=(p_2-c_2)Q_2$.

Now, to predict the expected value of fuzzy profit function $\tilde{P}^S_2$ there are two cases to be considered according to the position of Q_2 in $[\underline{D}_2,\bar{D}_2]$.

Case 1. When Q_2 lies between $\underline{D}_2$ and D_2, using Eq. (14) and Proposition 2, we have

(a) When $d_{2,0}\le\underline{D}_2$, the $\beta-$level set of $\tilde{P}^S_2$ is derived as

$$P^S_{2,OS}(Q_2,\beta)=[(p_2-c_2)Q_2,\ (p_2-c_2)Q_2];\qquad\forall\beta\in[0,1]$$

and

$$P^S_{2,US}(Q_2,\beta)=[(p_2-c_2)Q_2,\ (p_2-c_2)Q_2];\qquad\forall\beta\in[0,1]$$

Therefore, the expected resultant profit from *P*-II after substitution is given by

$$\bar{E}[\tilde{P}^S_2]=\int_0^1 0.5((p_2-c_2)Q_2+(p_2-c_2)Q_2)d\beta=(p_2-c_2)Q_2\,. \tag{15}$$

(b) When $\underline{D}_2<d_{2,0}\le Q_2$, the $\beta-$level set of $\tilde{P}^S_2$ is derived as

$$\begin{cases}P^S_{2,OS}(Q_2,\beta)=[p_2D_{2,OS,L}(\beta)-c_2Q_2+p_2\tau ES(Q_1),\ (p_2-c_2)Q_2]\\ P^S_{2,US}(Q_2,\beta)=[(p_2-c_2)Q_2,\ (p_2-c_2)Q_2]\end{cases};$$

for $0\le\beta\le L_2(d_{2,0})$

$$\begin{cases} P^S_{2,OS}(Q_2,\beta)=[(p_2-c_2)Q_2,\,(p_2-c_2)Q_2] \\ P^S_{2,US}(Q_2,\beta)=[(p_2-c_2)Q_2,\,(p_2-c_2)Q_2] \end{cases};$$

for $L_2(d_{2,0}) \le \beta \le L_2(Q_2)$

$$\begin{cases} P^S_{2,OS}(Q_2,\beta)=\phi \\ P^S_{2,US}(Q_2,\beta)=[(p_2-c_2)Q_2,\,(p_2-c_2)Q_2] \end{cases};$$

for $L_2(Q_2) \le \beta \le 1$

Therefore,

$$P^S_{2,L}(Q_2,\beta)=\begin{cases} p_2D_{2,OS,L}(\beta)-c_2Q_2+p_2\tau ES(Q_1); & 0\le\beta\le L_2(d_{2,0}) \\ (p_2-c_2)Q_2; & L_2(d_{2,0})\le\beta\le L_2(Q_2) \\ (p_2-c_2)Q_2; & L_2(Q_2)\le\beta\le 1 \end{cases}$$

and

$$P^S_{2,R}(Q_2,\beta)=\ (p_2-c_2)Q_2 \qquad 0\le\beta\le 1.$$

Hence, the expected resultant profit $\tilde{P}^S_2$ from *P*-II after substitution is computed by

$$\begin{aligned}\bar{E}[\tilde{P}^S_2] &= \int_0^{L_2(d_{2,0})} 0.5(p_2D_{2,OS,L}(\beta)-c_2Q_2+p_2\tau ES(Q_1) \\ &\quad +(p_2-c_2)Q_2)d\beta+\int_{L_2(d_{2,0})}^{1} 0.5((p_2-c_2)Q_2+(p_2-c_2)Q_2)d\beta \\ &= p_2\int_0^{L_2(d_{2,0})} 0.5D_{2,OS,L}(\beta)d\beta+0.5p_2\tau L_2(d_{2,0})ES(Q_1) \\ &\quad +(1-0.5L_2(d_{2,0}))p_2Q_2-c_2Q_2 \end{aligned} \qquad (16)$$

Case 2. When Q_2 lies between D_2 and $\bar{D}_2$, using Eq. (14) and Proposition 2, we have

(a) When $d_{2,0}$ lying out side the left endpoint of $\tilde{D}_2$ (i.e., $d_{2,0}\le \underline{D}_2$), the expected resultant profit from *P*-II after substitution is the same as in (15).

(b) When $d_{2,0}$ lying between $\underline{D}_2$ and D_2 (i.e., $\underline{D}_2 < d_{2,0} \le D_2$), the expected resultant profit from *P*-II after substitution is the same as in (16).

(c) When $D_2 < d_{2,0} \le Q_2$, the $\beta-$level set of $\tilde{P}^S_2$ is derived as

$$\begin{cases} P^S_{2,OS}(Q_2,\beta)=[p_2D_{2,OS,L}(\beta)-c_2Q_2+p_2\tau ES(Q_1),\,(p_2-c_2)Q_2] \\ P^S_{2,US}(Q_2,\beta)=[(p_2-c_2)Q_2,\,(p_2-c_2)Q_2] \end{cases};$$

for $0\le\beta\le R_2(Q_2)$

$$\begin{cases} P^S_{2,OS}(Q_2,\beta)=[p_2D_{2,OS,L}(\beta)-c_2Q_2+p_2\tau ES(Q_1),\,(p_2-c_2)Q_2] \\ P^S_{2,US}(Q_2,\beta)=\phi \end{cases};$$

for $R_2(Q_2)\le\beta\le R_2(d_{2,0})$

$$\begin{cases} P^S_{2,OS}(Q_2,\beta)=[p_2D_{2,OS,L}(\beta)-c_2Q_2+p_2\tau ES(Q_1), \\ \qquad\qquad p_2D_{2,OS,R}(\beta)-c_2Q_2+p_2\tau ES(Q_1)]; \\ P^S_{2,US}(Q_2,\beta)=\phi \end{cases}$$

for $R_2(d_{2,0})\le\beta\le 1$

Therefore,

$$P^S_{2,L}(Q_2,\beta)=p_2D_{2,OS,L}(\beta)-c_2Q_2+p_2\tau ES(Q_1); \qquad 0\le\beta\le 1$$

and

$$P^S_{2,R}(Q_2,\beta)=\begin{cases} (p_2-c_2)Q_2; & 0\le\beta\le R_2(Q_2) \\ (p_2-c_2)Q_2; & R_2(Q_2)\le\beta\le R_2(d_{2,0}) \\ p_2D_{2,OS,R}(\beta)-c_2Q_2+p_2\tau ES(Q_1); & R_2(d_{2,0})\le\beta\le 1 \end{cases}$$

Finally, the expected value of $\tilde{P}^S_2$ is computed as

$$\begin{aligned}\bar{E}[\tilde{P}^S_2] &= p_2\int_0^1 0.5D_{2,OS,L}(\beta)d\beta+p_2\int_{R_2(d_{2,0})}^{1} 0.5D_{2,OS,R}(\beta)d\beta \\ &\quad +0.5p_2\tau(1-0.5R_2(d_{2,0}))ES(Q_1)+0.5R_2(d_{2,0})p_2Q_2-c_2Q_2. \end{aligned} \qquad (17)$$

Therefore, either of the Eqs. (15)-(17) predict the expected profit earned from *P*-II if it is being used for substitution.

C. Benefit of product substitution

After analyzing the fuzzy model for a two-item newsboy problem with upward substitution as described above, we may now utilize $\bar{E}[\tilde{P}^S_2]$ to calculate the benefit of product substitution when ordering according to the independent newsboy problem is taken into account.

We then derive the following proposition.

Proposition 3. *Optimal pair* (Q_1^*,Q_2^*) *must satisfy the relation* $\bar{E}[\tilde{P}(Q_1^*,Q_2^*)] \ge \bar{E}[P^S(Q_1^*,Q_2^*)]$.

Proof: For a two-item newsboy problem with one-way substitution and imprecise customer demand, $\bar{E}[\tilde{P}]=\bar{E}[\tilde{P}_1]+\bar{E}[\tilde{P}_2]$ represents the total expected profit earned individually, whereas $\bar{E}[\tilde{P}^S]=\bar{E}[\tilde{P}_1]+\bar{E}[\tilde{P}^S_2]$ represents the total expected profit, if substitution is taken place. Therefore, it would be sufficient to show that $\bar{E}[\tilde{P}^S_2(Q_1^*,Q_2^*)] \ge \bar{E}[\tilde{P}_2(Q_2^*)]$. The expected profit $\bar{E}[\tilde{P}_2(Q_2^*)]$ can be found from (6) and (7), whereas $\bar{E}[\tilde{P}^S_2(Q_1^*,Q_2^*)]$ is computed from (15)-(17). It is observed from (3) that when $\lambda=0$, $\bar{E}[\tilde{P}^S_2]$ reduces to $\bar{E}[\tilde{P}_2]$. Therefore, by substituting the values of Q_1^* and Q_2^* into the Eqs. (15) or (16) or (17), it can be easily realized that $\bar{E}[\tilde{P}^S_2]$ (when $\lambda\ne 0$) will give a higher profit than $\bar{E}[\tilde{P}_2]$. Thus, we conclude that $\bar{E}[\tilde{P}(Q_1^*,Q_2^*)] \ge \bar{E}[P^S(Q_1^*,Q_2^*)]$, i.e., the decision-maker obtains a higher expected profit if there is a opportunity for substitution.

To look into the advantage of product substitution, we compute the percentage increase in expected profit (PIEP) as

$$\text{PIEP}=100\times(\bar{E}_{sub}-\bar{E}_{non-sub})/\bar{E}_{non-sub},$$

where $\bar{E}_{sub}$ and $\bar{E}_{non-sub}$ represents the expected resultant profit for substitutable and non-substitutable case, respectively.

In the following, we represent the benefit of product substitution when ordering according to the traditional newsboy-type problem is taken into account.

III. Computational Study with Discussion

While development of inventory models is an effective and elegant tool towards the problem of maximizing expected profit, here we developed the newsboy problem in fuzzy environment, where the retailer has the opportunity for upward substitution. Equations (14)-(16) represent the predicted resultant profit earned from *P*-II if it is used for the unmet demand of *P*-I. Recall from the feasible solution of non-substitutable newsvendor problem here we calculate the benefit of expected profit when upwards substitution is taken place. Consider a two-item newsboy problem with triangular fuzzy demand. Suppose the demand for brand name item (*P*-I) is around 150 and for low graded item (*P*-II) is around 200. The associate parameters are given in Table I.

TABLE I
Parametric values of the model

Product	*P*-I	*P*-II
Purchase cost	10	6
Selling price	15	10
Demand	(400,500,600)	(600,700,800)

In this study, we (i) first explore the conditions of the parameters, which initially determine the order quantities Q_1^* and Q_2^* for each product individually and (ii) then compare the gains in profits between the substitutable case and non-substitutable case.

Since, $2c_1 - p_1 = 5 > 0$ and $2c_2 - p_2 = 2 > 0$, by using proposition 1, the optimal Q_1^* lies between 400 and 500 and is given by $L_1(Q_1^*) = 0.66$, whereas the optimal Q_2^* lies between 600 and 700 and is determined by $L_2(Q_2^*) = 0.8$. Table II shows the expected profit before substitution.

TABLE II
RESULTS BEFORE SUBSTITUTION

Expected resultant profit ($\bar{E}[\tilde{P}]$)	*P*-I $Q_1^*, \bar{E}[\tilde{P}_1]$	*P*-II $Q_2^*, \bar{E}[\tilde{P}_2]$
4726.67	466.66, 2166.67	680.00, 2560.00

In Table III, we worked out the results for substitution. The percentage increase in expected profit, which compares the benefit of products substitution, is also computed.

When $\tau = 0$, from Table III it is shown that the above fuzzy model with substitution reduces to the model without substitution and the results coincides with each other. The highest percentage gains in expected profit from the substitutable case are accrued when $\tau = 1$.

TABLE III
RESULTS AFTER SUBSTITUTION

τ	$d_{2,0}$	$\bar{E}[\tilde{P}_2^S]$	$\bar{E}[\tilde{P}^S]$	PIEP
0.0	680.00	2560.00	4726.67	0.00%
0.1	675.55	2577.29	4743.96	0.36%
0.2	671.11	2593.58	4760.25	0.71%
0.3	666.66	2608.89	4775.56	1.03%
0.4	662.22	2623.21	4789.88	1.33%
0.5	657.77	2636.55	4803.22	1.61%
0.6	653.33	2648.89	4815.56	1.88%
0.7	648.88	2660.25	4826.92	2.12%
0.8	644.44	2670.62	4837.29	2.34%
0.9	639.99	2680.01	4846.68	2.53%
1.0	635.55	2688.40	4855.07	2.71%

IV. Conclusion

In this paper, we have studied a single-period "newsboy-type" inventory model with imprecise customer demand in which the retailer has the opportunity for substitution. Finally, we conclude that the use of one-way substitution policy increases both the average expected profit and customer satisfaction. It can easily be used in modeling of unidirectional lateral transshipments in warehouses or products remanufacturing policies in fuzzy environment.

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