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A New Nonlinear PID Controller and its Parameter Design

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***Abstract*—**A new nonlinear PID controller and its stability analysis are presented in this paper. A nonlinear function is deduced from the similarities between the control effort and the electric-field effect of a capacitor. The conventional linear PID controller can be modified into a nonlinear one by this function. To analyze the stability of the nonlinear PID controlled system, an idea of energy equivalence is adapted to avoid the conservativeness which is usually arisen from some traditional theorems and Criterions. The energy equivalence is naturally related with the conceptions of Passivity and T-Passivity. As a result, an engineering guideline for the parameter design of the nonlinear PID controller is obtained. An inverted pendulum system is tested to verify the nonlinear PID control scheme.

***Keywords*—**Nonlinear PID controller, stability, gain equivalence, dissipative, T-Passivity.

I. INTRODUCTION

THE conventional linear PID controller is combined by the following three terms linearly, the control error, the integral of the error, and the derivative of the error. Many researches and practices show that it is helpful to the control results when the three terms are constructed in some kind of nonlinear function forms.

There are considerable papers present different ways to design nonlinear PID controller. Among them, those methods that modify linear PID controller using some kind of special nonlinear function are of more attractive to the engineering application [1]–[3]. Though their nonlinearities are limited and can't approach any arbitrary nonlinear curve as Fuzzy PID and Neural Network PID do [4], they are simple and effective. They can usually achieve satisfying control results with little parameter adjustment.

As far as the stability of nonlinear PID controller is concerned, it is mainly relied on some theorems and criterions deduced from Lyapunov Stability Theorem [5], for instance, Popov Criterion, Absolute Stability Theorem and so on. It is well known that Lyapunov Stability Theorem is of a sufficient condition, so those theorems and criterions are too conservative to use. In fact, the parameter design of nonlinear PID controller is usually referred to the linear PID controller in practice. The engineering experience indicates that if there is a linear PID controller is available, then the equivalent gain of nonlinear PID controller should be almost equal to the gain of the linear PID controller, whereas, instability will be aroused. This experience is helpful to analyze the stability of nonlinear PID controller by comparing its equivalent gain with the linear one.

This paper is organized as follows: a new nonlinear PID controller is introduced in section 2. After some brief explanation, The concepts of Passivity and T-Passivity [6],[7] is employed to carry out the stability analysis about the nonlinear PID controller, and a guideline for its parameter design is obtained as a result, as shown in section 3. As an example, a test to the inverted pendulum system is arranged in section 4. Finally, some remarks and reviews are given in section 5.

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II. A NONLINEAR PID CONTROLLER

Suppose that there exists a capacitor whose two plates are both positive, and another positive charge is placed between the two plates. Under the repelling force from the two positive plates, the charge will move to the location where that electric field intensity is balanced.

This repelling force is much similar to the control effort. Consider a capacitor shown as figure 1, where the locations of the two plates in horizontal axis are $+e_0$ and $-e_0$, and the voltage held on the plates are $+U_0$ and $-U_0$ respectively. The total repelling force at an arbitrary point A (located at e in axis) is combined with the force supplied by each single plate. It could be formulated as follows.

$$\begin{aligned} U_{total} &= U_{+U_0} + U_{-U_0} \\ &= F(L_1, L_2)U_0 + F(L_2, L_1)(-U_0) \end{aligned} \tag{1}$$

Usually, the function $F(*,*)$ in equation (1) takes as hyperbolic-like form:

$$\begin{cases} F(L_1, L_2) = L_1^{-m} / (L_1^{-m} + L_2^{-m}) \\ F(L_2, L_1) = L_2^{-m} / (L_2^{-m} + L_1^{-m}) \end{cases} \tag{2}$$

By the way, the function $F(*,*)$ could also take other function form. The hyperbolic-like form is cited directly from the formula of actual electric field intensity.

The parameter m in (2) could be any positive number (the parameter m plays an important role in functional nonlinearity). The symbol L_1 and L_2 express the distances between point A and each of the two plates. More in detail, they can be written as:

$$\begin{cases} L_1 = |e_0 - e| \\ L_2 = |-e_0 - e| = |e_0 + e| \end{cases} \tag{3}$$

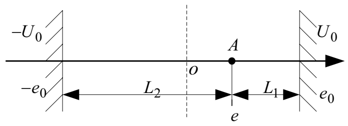

Fig.1 Sketch map of a capacitor structure

Refreshing the equation (1) with (2) and (3), a more explicit expression could be obtained:

$$U_{total} = \frac{|e_0 + e|^m - |e_0 - e|^m}{|e_0 + e|^m + |e_0 - e|^m} U_0 \tag{4}$$

If the location e is regarded as control error and the repelling force U_{total} as control output, the equation (4) re-expressed as a function of the control error:

$$u(e) = \frac{(e_0 + e)^m - (e_0 - e)^m}{(e_0 + e)^m + (e_0 - e)^m} U_0 \qquad |e| \le e_0 \tag{5}$$

This function (5) could be regarded as the proportional term of the nonlinear PID controller. Similarly, if the two other terms, the integral term and the derivative term, are structured in the same form as function (5), a nonlinear PID controller is achieved.

$$u_{PID} = u(e) + u(\int e) + u(\dot{e}) \tag{6}$$

where the term $u(\int e)$ is the integral term and $u(\dot{e})$ is the derivative term.

Remark 2.1 As far as the engineering simplicity, the parameters U_0 and m usually take some fixed numbers in actual application. Considering the function (5), the variable e is associated with the control error and the other three parameters e_0 , U_0 and m , are exploited to dominate its nonlinearity. The parameter e_0 determines its range, U_0 determines its magnitude and m determines its nonlinear curvature. Figure 2 shows their contributions to the nonlinearity of function (5):

If $m = 1$, then function (5) is linear in the extent $|e| \le e_0$.

If $0 < m < 1$, then the function is of small slope near the center and large slope near the two plates. This function style is helpful to stability but is disadvantageous to noise rejection.

if $m > 1$, then the function has large slope near the center and small slope near the two plates. It is contrary to the case above. It can reject noise better but is easy to occur limit cycle.

Those properties are compliant with the need of the variable gain controller.

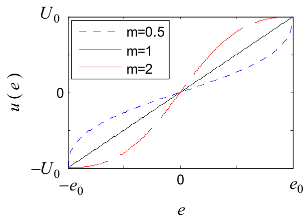

Fig.2 Relations between the features of the Nonlinear function and its parameters

Remark 2.2 The balanced electric field intensity is zero in figure 1. If the two plates are translated along the axis, then the balanced location will move along the axis correspondingly. This brings on a more general expression:

$$u(e) = \frac{(e_0 + \Delta + e)^m - (e_0 - \Delta - e)^m}{(e_0 + \Delta + e)^m + (e_0 - \Delta - e)^m} U_0 \qquad |e| \le e_0 \tag{7}$$

where Δ is the translation. The desired value of constraint function (7) could be any nonzero value.

III. I/O Stability Analysis

As mentioned in section 1, the stability analysis of nonlinear PID controller is usually carried out by Absolute Stability Theorem or Popov Criterion and so on. Their results tend to be conservative. For example, suppose there exists a plant whose transfer function is $1/(s^3 + s^2 + s)$, the upper boundary of proportional gain is 1, and the relevant poles of closed loop are $-1, \pm j$. Concern the nonlinear proportional controller in section 2 as follows:

$$u(e) = 1.2\frac{(1+e)^{0.5} - (1-e)^{0.5}}{(1+e)^{0.5} + (1-e)^{0.5}}$$

$$\text{If } |e| > 1 \qquad \text{then } u(e) = 1.2 sign(e)$$

Its gain curve is shown in figure 3 shows. Clearly, the nonlinear P controller violates the Absolute Stability Theorem partly, but it can still control the plant stably.

From the point of energy balance, the controlled system will still hold stabilization as long as the excessive energy produced in area A can be used up in area B (as shown in figure 3). Based on this point, some concepts of Passivity and T-Passivity is adapted to analyze the stability of the nonlinear PID controller proposed in section 2.

Definition 3.1 Given an affine nonlinear system as follows [6],[7]:

$$\begin{cases} \dot{x} = f(x,t) + g(x,t)u \\ y = h(x,t) \end{cases} \tag{8}$$

where $u, y \in R^p$ are the input and the output respectively. The system (8) is called strictly T-Passivity, if there exists a positive and decrescent function $V(x,t)$ called 'storage function', a strictly increasing function $\gamma(.)$ which passes through the origin, and a constant $T > 0$, s.t.

$$V(x(t+T),t+T) - V(x(t),t) \le \int_t^{t+T} y^T(\tau)u(\tau)d\tau - \gamma(\|x(t)\|) \qquad \forall t \tag{9}$$

The system is called T-Passivity if $\gamma(.) = 0$.

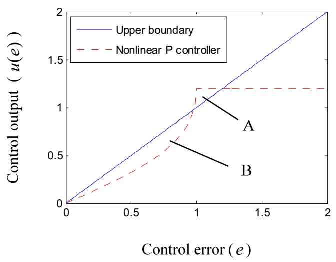

Fig.3 The gain curve of the nonlinear P controller

Comparing the definition of Passivity [8],[9], T-Passivity pays less restrictiveness on terminated time of the integral, so if a system is Passivity then it must be T-Passivity.

Lemma 3.2 [8], [9] The available storage function, S_a, is finite for all $x \in X$ if and only if Σ is dissipative.

A proposition about the equivalent gain of nonlinear PID controller can be deduced out based on the conclusions above.

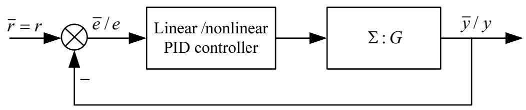

Fig.4 Feedback configuration

Proposition 3.3 Suppose there exists a linear PID controller (whose gain operator is $\bar{K}$) can control a positive real plant Σ ($G \ge 0$) I/O stably, and a nonlinear PID controller whose gain operator expressed as $K(e)$. If the following inequality is held at the extent of control error $[e_t, e_{t+T}]$

$$\int_{e_t}^{e_{t+T}} (K(e) - \bar{K})de \ge 0 \qquad \forall e_t \in [e_0, e_{t+T}] \tag{10}$$

then the nonlinear PID controller also could control the same plant stably. Where e is the control error, e_0 is the initial control error, e_{t+T} is the terminated control error.

Proof. Referring to the feedback configuration shown in figure 4, the closed loop with linear PID controller is I/O stable. So its input and output $\bar{r}, \bar{y}$ must be finite. The available storage function is also finite:

$$S_a(x) = \sup_{\substack{x \to \\ t_1 \ge 0}} - \int_0^{t_1} w(\bar{r}(t), \bar{y}(t))dt$$

where $w(\bar{r}(t), \bar{y}(t))$ is the supply rate. According to Lemma 3.2 and Definition 3.1, the closed loop system with linear PID controller is dissipative, and furthermore it must be T-Passivity. So the T-Passivity inequality can be listed as follows:

$$V(\bar{x}(t+T),t+T) - V(\bar{x}(t),t) \le \int_t^{t+T} \bar{y}^T(\tau)\bar{r}(\tau)d\tau - \gamma(\|\bar{x}(t)\|) \qquad \forall t$$

where $\bar{x}(t+T)$ is the terminated state.

Concerning the nonlinear PID controlled system is of same feedback configuration as shown in figure 4. It is reasonable to set its states as $x(t) = \bar{x}(t), x(t+T) = \bar{x}(t+T)$, because our attention is focused on the gain equivalence of the two kind of controllers. The T-Passivity inequality related to the nonlinear PID controlled system is:

$$V(x(t+T),t+T) - V(x(t),t) \le \int_t^{t+T} y^T(\tau)r(\tau)d\tau - \gamma(\|x(t)\|) \qquad \forall t \tag{11}$$

where $r(t) = e(t) + y(t) = \bar{r}(t) = \bar{e}(t) + \bar{y}(t)$ is the input of the nonlinear PID controlled system, $y(t) = \bar{y}(t) + \Delta y(t)$ is its output. Expands the inequality (11):

$$\begin{aligned} V(x(t+T),t+T) - V(x(t),t) &= V(\bar{x}(t+T),t+T) - V(\bar{x}(t),t) \\ &\le \int_t^{t+T} (\bar{y}^T(\tau) + \Delta y^T(\tau))r(\tau)d\tau - \gamma(\|\bar{x}(t)\|) \\ &= \left[\int_t^{t+T} \bar{y}^T(\tau)\bar{r}(\tau)d\tau - \gamma(\|\bar{x}(t)\|) \right] + \int_t^{t+T} \Delta y^T(\tau)r(\tau)d\tau \end{aligned} \tag{12}$$

In expression (12), if the term $\int_t^{t+T} \Delta y^T(\tau)r(\tau)d\tau < 0$, then the nonlinear PID controlled system will be "more dissipative" than the linear PID controlled system. To expand this term, it is assumed both the linear and nonlinear PID controlled systems are SISO, then the control errors $e(t), \bar{e}(t)$ and the control outputs $y(t), \bar{y}(t)$ are all scalar.

$$\int_t^{t+T} \Delta y^T(\tau)r(\tau)d\tau = \int_t^{t+T} (y^T(\tau) - \bar{y}^T(\tau))r(\tau)d\tau \tag{13}$$

$$= \int_t^{t+T} [y^T(\tau)(\bar{e}(\tau) + \bar{y}(\tau)) - \bar{y}^T(\tau)(e(\tau) + y(\tau))]d\tau$$

$$= \int_{t}^{t+T} [y^T(\tau)\bar{e}(\tau) - \bar{y}^T(\tau)e(\tau)]d\tau$$

$$= \int_{t}^{t+T} [e^T(\tau)K(e)G\bar{e}(\tau) - \bar{e}^T(\tau)\bar{K}Ge(\tau)]d\tau$$

$$= \int_{t}^{t+T} \{e(\tau)[K(e) - \bar{K}]G\bar{e}(\tau)\}d\tau \le 0$$

Referring to the figure 3, the control output $u(e)$ is a function of the control error e. To compare the gains of the linear and nonlinear PID controllers, it is assumed further that the control error $e(t)$ and $\bar{e}(t)$ are equal. Then it goes that:

$$\int_{t}^{t+T} \{e(\tau)[K(e) - \bar{K}]G\bar{e}(\tau)\}d\tau \le 0 \Leftrightarrow \int_{t}^{t+T} [K(e) - \bar{K}]d\tau \le 0 \qquad \forall t$$

By integral transform, the inequality above could be re-written as:

$$\int_{e_t}^{e_{t+T}} (K(e) - \bar{K})de \ge 0 \qquad \forall e_t \in [e_0, e_{t+T}] \tag{14}$$

This means if (14) is held then the nonlinear PID controlled system will be "more dissipative" than linear PID controlled system. ∎

Remark 3.4 There is another way to achieve conclusion (14). From (13), next equation is hold:

$$\int_{t}^{t+T} \Delta y^T(\tau)r(\tau)d\tau = \int_{t}^{t+T} (y^T(\tau) - \bar{y}^T(\tau))r(\tau)d\tau$$

$$= \int_{t}^{t+T} r(\tau)(\phi(s) - \bar{\phi}(s))r(\tau)d\tau \le 0$$

where $\phi(s), \bar{\phi}(s)$ are respective closed loop transfer functions. According to the feedback configuration shown in figure 4, the closed loop transfer function can be expressed as $\phi(s) = KG/(1+KG) = k/(1/G+K)$. Its derivative about K is $\phi'(s)\big|_K = (1/G)/(1/G+K)^2$. From the premise $G \ge 0$, $\phi(s)$ is monotonic about K. So, the condition $\int_{t}^{t+T} (\phi(s) - \bar{\phi}(s))d\tau \le 0$ is equivalent with $\int_{t}^{t+T} [K(e) - \bar{K}]d\tau \le 0$.

Remark 3.5 The conclusion (14) builds up a guideline for the parameter design of nonlinear PID controller. That is, when the area enveloped by the control output curve of the nonlinear PID controller and the horizontal axis is less than the relating area of the linear PID controller from any state to the ending state, the nonlinear PID control system is I/O stable as long as the linear PID control system is stable.

IV. Examples

An inverted pendulum system is taken to evaluate the control result. The pole placement control scheme is applied to obtain an available linear controller at first. Based on this linear controller, the nonlinear PID controller is obtained by some simple modification with function (5). Their control effects are compared under the same disturbance.

TABLE I
THE PARAMETERS OF THE INVERTED PENDULUM

The name of Parameter	Value
mass of cart	1.096Kg
mass of pendulum	0.109Kg
length of pendulum	0.5m
sampling time	0.005Sec

The parameters of the inverted pendulum system are list as table I.

A. The Pole Placement Control Scheme

When the expected close loop poles are set as: $s_{1,2} = -10$, $s_{3,4} = -2 \pm j2\sqrt{3}$, the pole placement controller to the inverted pendulum with the parameters listed as table 1 is achieved as follow:

$$u = 54.4218x + 24.4898\dot{x} - 93.739\theta - 16.1633\dot{\theta} \tag{15}$$

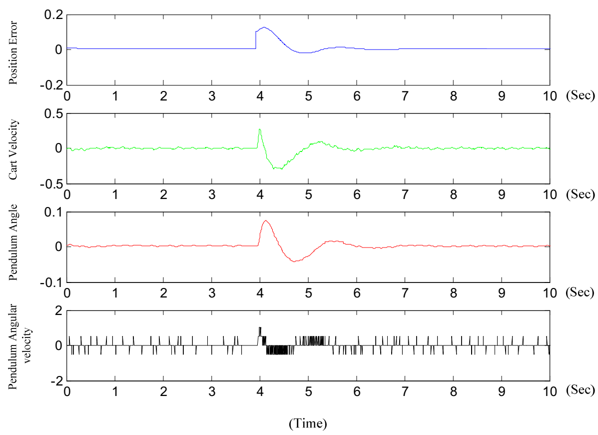

Fig. 5 The control effect of the pole placement control scheme under a position shift

where x is the position error of the cart and $\dot{x}$ is its velocity, θ is the vertical angle of the pendulum and $\dot{\theta}$ is the corresponding angular velocity. The control result is shown in figure 5. There is a position shift to evaluate the capability of controller for resisting disturbance. The response performance of cart's position under the disturbance is listed as bellow.

The maximal value of x is 0.125, and its transition time is about 3.9 seconds.

B. The Nonlinear PD Control Scheme

The controller (15) deduced from the pole placement control scheme is of the linear PD control structure. It can be modified

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into a nonlinear PD controller according to the nonlinear function (5).

$$u = 54.4218X + 24.4898\dot{X} - 93.739\theta - 16.1633\dot{\theta} \tag{16}$$

where:

$$\begin{cases} X = \dfrac{(0.2+x)^{0.7} - (0.2-x)^{0.7}}{(0.2+x)^{0.7} + (0.2-x)^{0.7}} 0.24 \\ \dot{X} = \dfrac{(0.3+\dot{x})^{1.1} - (0.3-\dot{x})^{1.1}}{(0.3+\dot{x})^{1.1} + (0.3-\dot{x})^{1.1}} 0.27 \end{cases} \tag{17}$$

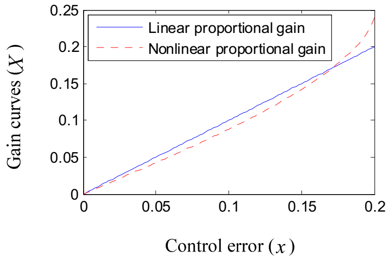

Fig. 6(a) The nonlinear modification about the cart's position

For simplicity, the angle θ and its angular velocity $\dot{\theta}$ keep linearly. The gain curves about X and $\dot{X}$ are shown in figure 6, where, the figure 6(a) is about cart's position error and the figure 6(b) is about its velocity.

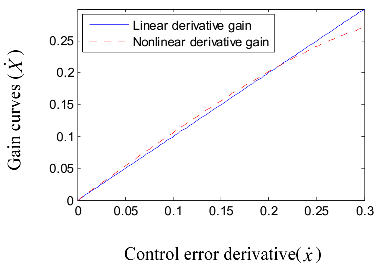

Fig. 6(b) The nonlinear modification about the cart's velocity

Under same position shift of the cart as the case A, the control effect of the nonlinear PD controller is shown in figure 7. It is clearly that the control effect is better than the linear controller's. The corresponding response performance of cart's position is:

The maximal value of x is 0.118, and its transition time is about 1.75 seconds.

V. Conclusions

From the idea of electric field between the two plates of a capacitor, a kind of nonlinear function is introduced in this paper. By modifying the conventional linear PID controller using this nonlinear function, a nonlinear PID controller is formed. This nonlinear PID controller is not only simple and effective but also easy to adjust its parameters in the light of the conception of capacitor. Simulated examples show the nonlinear PID controller is flexible to compromise the stability and the rigidity.

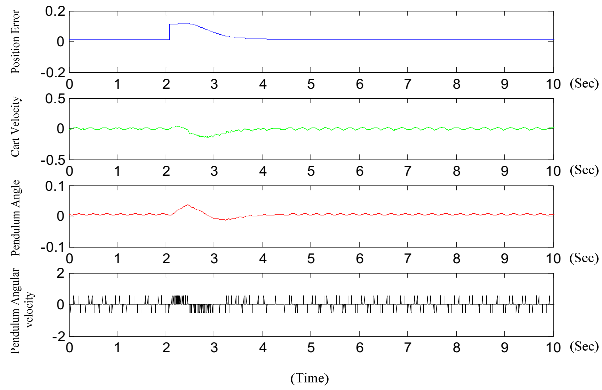

Fig. 7 The control effect of the nonlinear PDcontrol scheme under a same position shift

As far as parameters design of this nonlinear PID controller, a guideline of area equivalence between the nonlinear PID controller and the linear one is proposed. This guideline is compatible with designing steps of nonlinear PID controller in practice, which is usually completing a linear PID controller at first, then referring to the linear PID controller to adjust nonlinear PID controller till it works well. In addition, it should be mentioned that the guideline is also valid to other kinds of nonlinear PID controllers.

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References

[1] J.Q. Han, "Auto disturbances rejection control technique," *Frontier Science*, 2007, 1(1): 24 – 31(in Chinese).
[2] M. Margaliot, G. Langholz, "Hyperbolic optimal control and fuzzy control," *IEEE Transactions on Systems, Man, and Cybernetics, Part A: Systems and Humans*, 1999, 29(1): 1 – 10.
[3] A. Brian, A.W. Bruce, "Nonlinear PID control with partial state knowledge: Damping without derivatives," *The International Journal of Robotics Research*, 2000, 19(8): 715 – 731.
[4] Z.Q. Sun, "Intelligence control theory and technique," Beijing: Tsinghua University Press, 1997(in Chinese).
[5] W.B. Gao, "Introduction to Nonlinear Control Systems," Beijing: Science Press, 1988(in Chinese).
[6] G. Calcev, G. Campion, "T-passivity and related concepts," *European Control Conference*. Belgium: Brussels, 1997, paper WE-E-D1.

[7] D. Aeyels, “A new asymptotic stability criterion for nonlinear time-variant differential equations,” *IEEE Transactions on automatic control*. 1998, 43(7): 968-971.
[8] J.C. Willems, “Dissipative dynamical systems─Part I: general theory,” *Arch. Rational Mechanics and Analysis*. 1972, 45: 321-351.
[9] I.B. Christopher, I. Alberto, “Passivity,feedback equivalence, and the global stabilization of minimum phase nonlinear system,” *IEEE Transactions on automatic control*. 1991, 36(11):1228-1240.

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