Periodic solutions for a two-prey one-predator system on time scales

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Abstract—In this paper, using the Gaines and Mawhin's continuation theorem of coincidence degree theory on time scales, the existence of periodic solutions for a two-prey one-predator system is studied. Some sufficient conditions for the existence of positive periodic solutions are obtained. The results provide unified existence theorems of periodic solution for the continuous differential equations and discrete difference equations.

Keywords—Time Scales; Competitive system; Periodic solution; Coincidence degree; Topological degree.

I. Introduction

The theory of calculus on time scales[1,2] was initiated by Stefan Hilger in his PhD in 1998[3] in order to unify continuous and discrete analysis, and it has a tremendous potential applications and has recently considerable attention[4-9] since his foundational work. In 2009, Baek[10] investigated the species extinction and permanence of the following two-prey one predator system with seasonal effects.

$$
\begin{cases}
\dot{x}_1(t) = x_1(t)\Big[a_1 + \gamma_1 \sin(\theta_1 t) - b_1 x_1(t) - c_1 x_2(t) \\
\qquad - \frac{\sigma_1 y(t)}{1+d_1 x_1(t)+e_1 x_2(t)+\mu_1 y(t)}\Big], \\
\dot{x}_2(t) = x_2(t)\Big[a_2 + \gamma_2 \sin(\theta_2 t) - b_2 x_2(t) - c_2 x_1(t) \\
\qquad - \frac{\sigma_2 y(t)}{1+d_2 x_1(t)+e_2 x_2(t)+\mu_2 y(t)}\Big], \\
\dot{y}(t) = y(t)\Big[-a_3 + \frac{\sigma_3 x_1(t)}{1+d_1 x_1(t)+e_1 x_2(t)+\mu_1 y(t)} \\
\qquad + \frac{\sigma_4 x_2(t)}{1+d_2 x_1(t)+e_2 x_2(t)+\mu_2 y(t)}\Big],
\end{cases}
\tag{1}
$$

where $x_1(t), x_2(t)$ denote the population densities of the two prey and the predator at time t, respectively. The constant $a_i(i=1,2)$ is the intrinsic growth rates of the prey population, $b_i(i=1,2)$ are the coefficients of intra-specific competition, $c_i(i=1,2)$ denote the parameters representing competitive effects between the two prey,$\sigma_i(i=1,2)$ are the per-capita rates of the predation of the predator, $d_i(i=1,2)$ and $e_i(i=1,2)$ represent the half-saturation constants, the constant a_3 is the death rate of the predator, the terms $\mu_i(i=1,2)$ scale the impact of the predator interference, $\sigma_i(i=3,4)$ are the rates of the conversing prey into predator.

It is well known that any biological or environmental parameters are naturally subject to fluctuation in time. It is necessary and important to consider models with periodic ecological parameters. Thus the assumption of periodicity of the parameters is a way of incorporating the periodicity of

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the environment. Based on the viewpoint, we modify (1) as follows

$$
\begin{cases}
\dot{x}_1(t) = x_1(t)\Big[a_1(t) + \gamma_1(t) \sin(\theta_1(t)t) - b_1(t)x_1(t) \\
\qquad -c_1(t)x_2(t) - \frac{\sigma_1(t)y(t)}{1+d_1 x_1(t)+e_1 x_2(t)+\mu_1 y(t)}\Big], \\
\dot{x}_2(t) = x_2(t)\Big[a_2(t) + \gamma_2(t) \sin(\theta_2(t)t) - b_2(t)x_2(t) \\
\qquad -c_2(t)x_1(t) - \frac{\sigma_2(t)y(t)}{1+d_2 x_1(t)+e_2 x_2(t)+\mu_2 y(t)}\Big], \\
\dot{y}(t) = y(t)\Big[-a_3(t) + \frac{\sigma_3(t)x_1(t)}{1+d_1 x_1(t)+e_1 x_2(t)+\mu_1 y(t)} \\
\qquad + \frac{\sigma_4(t)x_2(t)}{1+d_2 x_1(t)+e_2 x_2(t)+\mu_2 y(t)}\Big],
\end{cases}
\tag{2}
$$

The principle object of this article is to consider the model

$$
\begin{cases}
u_1^\Delta(t) = a_1(t) + \gamma_1(t) \sin(\theta_1(t)t) - b_1(t)e^{u_1(t)} \\
\qquad - c_1(t)e^{u_2(t)} - \frac{\sigma_1(t)e^{u_3(t)}}{1+d_1 e^{u_1(t)}+e_1 e^{u_2(t)}+\mu_1 e^{u_3(t)}}, \\
u_2^\Delta(t) = a_2(t) + \gamma_2(t) \sin(\theta_2(t)t) - b_2(t)e^{u_2(t)} \\
\qquad - c_2(t)e^{u_1(t)} - \frac{\sigma_2(t)e^{u_3(t)}}{1+d_2 e^{u_1(t)}+e_2 e^{u_2(t)}+\mu_2 e^{u_3(t)}}, \\
u_3^\Delta(t) = -a_3(t) + \frac{\sigma_3(t)e^{u_1(t)}}{1+d_1 e^{u_1(t)}+e_1 e^{u_2(t)}+\mu_1 e^{u_3(t)}} \\
\qquad + \frac{\sigma_4(t)e^{u_2(t)}}{1+d_2 e^{u_1(t)}+e_2 e^{u_2(t)}+\mu_2 e^{u_3(t)}}.
\end{cases}
\tag{3}
$$

Remark 1.1. Let $x_1(t) = e^{u_1(t)}, x_2(t) = e^{u_2(t)}, y(t) = e^{u_3(t)}$. If $\mathbb{T} = \mathbb{R}$, then (1.3) reduces to the model (2). If $\mathbb{T} = \mathbb{Z}$, then (3) is reformulated as

$$
\begin{cases}
x_1(t+1) = x_1(t) \exp\Big\{a_1(t) + \gamma_1(t) \sin(\theta_1(t)t) - b_1(t)x_1(t) \\
\qquad - c_1(t)x_2(t) - \frac{\sigma_1(t)y(t)}{1+d_1 x_1(t)+e_1 x_2(t)+\mu_1 y(t)}\Big\}, \\
x_2(t+1) = x_2(t) \exp\Big\{a_2(t) + \gamma_2(t) \sin(\theta_2(t)t) - b_2(t)x_2(t) \\
\qquad - c_2(t)x_1(t) - \frac{\sigma_2(t)y(t)}{1+d_2 x_1(t)+e_2 x_2(t)+\mu_2 y(t)}\Big\}, \\
y(t+1) = y(t) \exp\Big\{ - a_3(t) + \frac{\sigma_3(t)x_1(t)}{1+d_1 x_1(t)+e_1 x_2(t)+\mu_1 y(t)} \\
\qquad + \frac{\sigma_4(t)x_2(t)}{1+d_2 x_1(t)+e_2 x_2(t)+\mu_2 y(t)}\Big\}.
\end{cases}
\tag{4}
$$

In order to obtain the main results of our paper, throughout this paper, we assume
(H1) $a_i(t), \sigma_j(t), b_l(t), c_l(t), \gamma_l(t)(i = 1,2,3; j = 1,2,3,4; l = 1,2)$ are positive continuous ω-periodic functions for model (3).

The remainder of the paper is organized as follows: in Section 2, we present some preliminary definitions, notations and some basic knowledge for dynamic system on time scales. In Section 3, a sufficient condition for the existence of positive solutions of system (3) is obtained.

II. PRELIMINARY RESULTS ON TIME SCALES

In order to make an easy and convenient reading of this paper, we present some definitions and notations on time scales which can be found in the literatures[1,12].

Definition 2.1. *A time scale is an arbitrary nonempty closed subset $\mathbb{T}$ of $\mathbb{R}$, the real numbers. The set $\mathbb{T}$ inherits the standard topology of $\mathbb{R}$.*

Definition 2.2. *The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, the backward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+ = [0, \infty)$ are defined, respectively, by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$, $\mu(t) = \sigma(t) - t$ for $t \in \mathbb{T}$. If $\sigma(t) = t$, then t is called right-dense (otherwise: right-scattered), and if $\rho(t) = t$, then t is called left-dense (otherwise: left-scattered).*

Definition 2.3. *A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sides limits exists(finite) at left-dense points in $\mathbb{T}$. The set rd-continuous functions is shown by $C_{rd}^1 = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.*

Definition 2.4. *For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{R}$, we define $f^\Delta(t)$, the delta-derivative of f at t, to be the number(provided it exists) with the property that, given any $\varepsilon > 0$, there is a neighborhood U of t in $\mathbb{T}$ such that*

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \le \varepsilon|\sigma(t) - s| \; for \; all \; s \in U.$$

Thus f is said to be delta-differentiable if its delta-derivative exists. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are delta-differentiable and whose delta-derivative are rd-continuous functions is denoted by $C_{rd} = C_{rd}^1(\mathbb{T}) = C_{rd}^1(\mathbb{T}, \mathbb{R})$.

Definition 2.5. *A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\Delta(t) = f(t)$, for all $t \in \mathbb{T}$. Then we write $\int_r^s f(t)\Delta t := F(s) - F(r)$ for all $s, t \in \mathbb{T}$.*

For the usual time scales $\mathbb{T} = \mathbb{R}$, rd-continuous coincides with the usual continuity in calculus. Moreover, every rd-continuous function on $\mathbb{T}$ has a delta-antiderivative[9]. For more information about the above definitions and their related concepts, one can see [1,11-12].

III. EXISTENCE OF PERIODIC SOLUTIONS

For convenience and simplicity in the following discussion, we always use the notations below throughout the paper. Let $\mathbb{T}$ be ω-periodic, that is, $t \in \mathbb{T}$ implies $t + \omega \in \mathbb{T}$, $\kappa = \min\{\mathbb{R}^+ \cap \mathbb{T}\}$, $I_\omega = [\kappa, \kappa + \omega] \cap \mathbb{T}$, $\bar{g} = \frac{1}{\omega}\int_{I_\omega} g(s)\Delta s = \frac{1}{\omega}\int_\kappa^{\kappa+\omega} g(s)\Delta s$, where $g \in C_{rd}(\mathbb{T})$ is an ω-periodic real function, i.e., $g(t + \omega) = g(t)$ for all $t \in \mathbb{T}$.

In order to explore the existence of positive periodic solutions of (3) and for the reader's convenience, we shall first summarize below a few concepts and results without proof, borrowing from [13].

Let X, Y be normed vector spaces, $L : \text{Dom}L \subset X \to Y$ is a linear mapping, $N : X \to Y$ is a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim\text{Ker}L = \text{codim}\text{Im}L < +\infty$ and $\text{Im}L$ is closed in Y. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\text{Im}P = \text{Ker}L, \text{Im}L = \text{Ker}Q = \text{Im}(I - Q)$, It follows that $L \mid \text{Dom}L \cap \text{Ker}P : (I - P)X \to \text{Im}L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L−compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact. Since $\text{Im}Q$ is isomorphic to $\text{Ker}L$, there exist isomorphisms $J : \text{Im}Q \to \text{Ker}L$.

Lemma 3.1. ([13]Continuation Theorem) *Let L be a Fredholm mapping of index zero and let N be L−compact on $\bar{\Omega}$. Suppose*
(a) For each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;
(b) $QNx \ne 0$ for each $x \in \text{Ker}L \cap \partial\Omega$, and $\deg\{JQN, \Omega \cap \text{Ker}L, 0\} \ne 0$;
Then the equation $Lx = Nx$ has at least one solution lying in $\text{Dom}L \cap \bar{\Omega}$.

Lemma 3.2. [3] *Let $t_1, t_2 \in I_\omega$ and $t \in \mathbb{T}$. If $g : \mathbb{T} \to \mathbb{R}$ is ω-periodic, then*

$$g(t) \le g(t_1) + \int_\kappa^{\kappa+\omega} |g^\Delta(s)|\Delta s,$$

and

$$g(t) \ge g(t_2) - \int_\kappa^{\kappa+\omega} |g^\Delta(s)|\Delta s.$$

Lemma 3.3. *If the condition (H1) holds, then the following equations*

$$\begin{cases} \bar{a}_1 - \bar{b}_1 e^{u_1} = 0, \\ \bar{a}_2 - \bar{b}_2 e^{u_2} = 0, \\ -\bar{a}_3 + \frac{\bar{\sigma}_3 e^{u_1}}{1 + d_1 e^{u_1} + e_1 e^{u_2} + \mu_1 e^{u_3(t)}} = 0 \end{cases} \quad (5)$$

has a unique solution $(u_1^*, u_2^*, u_3^*)^T$.

The proofs of Lemma 3.3 are trivial, so we omitted the details here.

Theorem 3.1. *Let $S_1, \overline{S}_1, S_2$ and $\overline{S}_2$ be defined by (15),(28),(19) and (32), respectively. In addition to (H1), suppose that*

$$(H2) \; \bar{a}_3 > \max\left\{\bar{\sigma}_3 e^{S_1}, \bar{\sigma}_4 e^{\overline{S}_1}\right\}, \bar{\sigma}_3 e^{\overline{S}_2} > \bar{a}_3\left(1 + d_1 e^{\overline{S}_2} + e_1 e^{\overline{S}_1}\right)$$

hold, then (3) has at least one ω-periodic solution.

Proof. Define $X = Z = \{(u_1, u_2, u_3)^T \in C(\mathbb{T}, \mathbb{R}^3) | u_i \in C_{rd}, u_i(t + \omega) = u_i(t), i = 1, 2, 3\}$,

$$\|(u_1, u_2, u_3)^T\| = \sum_{i=1}^3 \max_{t \in I_\omega} |u_i(t)|, (u_1, u_2, u_3)^T \in X(\text{or } Z).$$

$$\text{Dom}L = \{x = (u_1, u_2, u_3)^T \in X | u_i \in C_{rd}, i = 1, 2, 3\}.$$

It is easy to see that X and Z are both Banach spaces if they are endowed with the above norm $\|.\|$. For $(u_1, u_2, u_3)^T \in X$,

we define

$$N \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix}, L \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}(t) = \begin{pmatrix} u_1^\Delta(t) \\ u_2^\Delta(t) \\ u_3^\Delta(t) \end{pmatrix},$$

$$P \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}(t) = Q \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}(t) = \begin{pmatrix} \frac{1}{\omega}\int_\kappa^{\kappa+\omega} u_1(t)\Delta t \\ \frac{1}{\omega}\int_\kappa^{\kappa+\omega} u_2(t)\Delta t \\ \frac{1}{\omega}\int_\kappa^{\kappa+\omega} u_3(t)\Delta t \end{pmatrix},$$

where

$$\begin{aligned} f_1(t) &= a_1(t) + \gamma_1(t)\sin(\theta_1(t)t) - b_1(t)e^{u_1(t)} - c_1(t)e^{u_2(t)} \\ &\quad - \frac{\sigma_1(t)e^{u_3(t)}}{1 + d_1 e^{u_1(t)} + e_1 e^{u_2(t)} + \mu_1 e^{u_3(t)}}, \\ f_2(t) &= a_2(t) + \gamma_2(t)\sin(\theta_2(t)t) - b_2(t)e^{u_2(t)} - c_2(t)e^{u_1(t)} \\ &\quad - \frac{\sigma_2(t)e^{u_3(t)}}{1 + d_2 e^{u_1(t)} + e_2 e^{u_2(t)} + \mu_2 e^{u_3(t)}}, \\ f_3(t) &= -a_3(t) + \frac{\sigma_3(t)e^{u_1(t)}}{1 + d_1 e^{u_1(t)} + e_1 e^{u_2(t)} + \mu_1 e^{u_3(t)}} \\ &\quad + \frac{\sigma_4(t)e^{u_2(t)}}{1 + d_2 e^{u_1(t)} + e_2 e^{u_2(t)} + \mu_2 e^{u_3(t)}}. \end{aligned}$$

Then

$$\begin{aligned} \mathrm{Ker} L &= \{(u_1, u_2, u_3)^T \in X | (u_1(t), u_2(t), u_3(t))^T \\ &= (h_1, h_2, h_3)^T \in \mathbb{R}^3 \text{ for } t \in \mathbb{T}\}, \\ \mathrm{Im} L &= \{(u_1, u_2, u_3)^T \in X | \int_\kappa^{\kappa+\omega} u_i(t)\Delta t = 0, \\ &\quad (i = 1, 2, 3) \text{ for } t \in \mathbb{T}\}. \end{aligned}$$

Then dim $\mathrm{Ker} L = 3 = \mathrm{codim}\, \mathrm{Im} L$. Since $\mathrm{Im} L$ is closed in Z, L is a Fredholm mapping of index zero, it is easy to show that P and Q are continuous projections and $\mathrm{Im} P = \mathrm{Ker} L$, $\mathrm{Im} L = \mathrm{Ker} Q = \mathrm{Im}(I - Q)$. Clearly, QN and $K_p(I - Q)N$ are continuous. It can be shown that N is L-compact on $\bar\Omega$ for every open bounded set, $\Omega \subset X$.

Now we are at the point to search for an appropriate open, bounded subset Ω for the application of the continuation theorem. Corresponding to the operator equation $L(u_1, u_2, u_3)^T = \lambda N(u_1, u_2, u_3)^T, \lambda \in (0, 1)$, we have

$$\begin{cases} u_1^\Delta(t) = \lambda f_1(t), \\ u_2^\Delta(t) = \lambda f_2(t), \\ u_3^\Delta(t) = \lambda f_2(t). \end{cases} \tag{6}$$

Suppose that $x(t) = (u_1(t), u_2(t), u_3(t))^T \in X$ is an arbitrary solution of system (6) for a certain $\lambda \in (0, 1)$, Integrating (6) over the set I_ω, we obtain

$$\begin{cases} \bar a_1\omega + \int_\kappa^{\kappa+\omega} \gamma_1(t)\sin(\theta_1(t)t)\Delta t = \int_\kappa^{\kappa+\omega} b_1(t)e^{u_1(t)}\Delta t \\ + \int_\kappa^{\kappa+\omega} c_1(t)e^{u_2(t)}\Delta t + \int_\kappa^{\kappa+\omega} \frac{\sigma_1(t)e^{u_3(t)}}{1+d_1 e^{u_1(t)}+e_1 e^{u_2(t)}+\mu_1 e^{u_3(t)}}\Delta t, \\ \bar a_2\omega + \int_\kappa^{\kappa+\omega} \gamma_2(t)\sin(\theta_2(t)t)\Delta t = \int_\kappa^{\kappa+\omega} b_2(t)e^{u_2(t)}\Delta t \\ + \int_\kappa^{\kappa+\omega} c_2(t)e^{u_1(t)}\Delta t + \int_\kappa^{\kappa+\omega} \frac{\sigma_2(t)e^{u_3(t)}}{1+d_2 e^{u_1(t)}+e_2 e^{u_2(t)}+\mu_2 e^{u_3(t)}}\Delta t, \\ \bar a_3\omega = \int_\kappa^{\kappa+\omega} \frac{\sigma_3(t)e^{u_1(t)}}{1+d_1 e^{u_1(t)}+e_1 e^{u_2(t)}+\mu_1 e^{y(t)}}\Delta t \\ + \int_\kappa^{\kappa+\omega} \frac{\sigma_4(t)e^{u_2(t)}}{1+d_2 e^{u_1(t)}+e_2 e^{u_2(t)}+\mu_2 e^{u_3(t)}}\Delta t \end{cases} \tag{7}$$

Since $(u_1, u_2, u_3)^T \in X$, there exists $\xi_i, \eta_i \in [\kappa, \kappa + \omega], i = 1, 2, 3$ such that

$$u_i(\xi_i) = \min_{t\in[\kappa,\kappa+\omega]}\{u_i(t)\}, u_i(\eta_i) = \max_{t\in[\kappa,\kappa+\omega]}\{u_i(t)\}.$$

It follows from (6) and (7) that

$$\int_\kappa^{\kappa+\omega} |u_1^\Delta(t)|\Delta t < 2(\bar a_1 + \bar\gamma_1)\omega, \tag{8}$$

$$\int_\kappa^{\kappa+\omega} |u_2^\Delta t|\Delta(t) < 2(\bar a_2 + \bar\gamma_2)\omega, \tag{9}$$

$$\int_\kappa^{\kappa+\omega} |u_3^\Delta t|\Delta(t) < 2\bar a_3\omega. \tag{10}$$

From the first equation of (7), it follows that

$$\begin{aligned} (\bar a_1 + \bar\gamma_1)\omega &> \int_\kappa^{\kappa+\omega} b_1(t)e^{u_1(t)}\Delta t \\ &\geq \int_\kappa^{\kappa+\omega} b_1(t)e^{u_1(\xi_1)}\Delta t = \bar b_1\omega e^{u_1(\xi_1)} \end{aligned}$$

and

$$\begin{aligned} (\bar a_1 + \bar\gamma_1)\omega &> \int_\kappa^{\kappa+\omega} c_1(t)e^{u_2(t)}\Delta t \\ &\geq \int_\kappa^{\kappa+\omega} c_1(t)e^{u_2(\xi_2)}\Delta t = \bar c_1\omega e^{u_2(\xi_1)}. \end{aligned}$$

Then

$$u_1(\xi_1) < \ln\left[\frac{\bar a_1 + \bar\gamma_1}{\bar b_1}\right] := m_1, u_2(\xi_2) < \ln\left[\frac{\bar a_1 + \bar\gamma_1}{\bar c_1}\right] := m_2. \tag{11}$$

In the sequel, we consider two cases.

(i) If $u_1(\eta_1) \geq u_2(\eta_2)$, then it follows from the third equation of (7) that

$$\begin{aligned} \bar a_3\omega &\leq \int_\kappa^{\kappa+\omega} \sigma_3(t)e^{u_1(t)}\Delta t + \int_\kappa^{\kappa+\omega} \sigma_4(t)e^{u_2(t)}\Delta t \\ &\leq \int_\kappa^{\kappa+\omega} \sigma_3(t)e^{u_1(\eta_1)}\Delta t + \int_\kappa^{\kappa+\omega} \sigma_4(t)e^{u_2(\eta_2)}\Delta t \\ &\leq \int_\kappa^{\kappa+\omega} \sigma_3(t)e^{u_1(\eta_1)}\Delta t + \int_\kappa^{\kappa+\omega} \sigma_4(t)e^{u_1(\eta_1)}\Delta t \\ &= (\bar\sigma_3 + \bar\sigma_4)\omega e^{u_1(\eta_1)} \end{aligned}$$

which leads to

$$u_1(\eta_1) > \ln\left[\frac{\bar a_3}{\bar\sigma_3 + \bar\sigma_4}\right] := M_1. \tag{12}$$

Based on (8), (11) and (12), using the Lemma 3.2, we get

$$\begin{aligned} u_1(t) &\leq u_1(\xi_1) + \int_\kappa^{\kappa+\omega} |u_1^\Delta(t)|\Delta t \\ &\leq m_1 + 2(\bar a_1 + \bar\gamma_1)\omega =: B_1, \end{aligned} \tag{13}$$

$$\begin{aligned} u_1(t) &\geq u_1(\eta_1) - \int_\kappa^{\kappa+\omega} |u_1^\Delta(t)|\Delta t \\ &\geq M_1 - 2(\bar a_1 + \bar\gamma_1)\omega =: B_2. \end{aligned} \tag{14}$$

Thus

$$\max_{t\in I_\omega} |x_1(t)| \leq \max\{|B_1|, |B_2|\} := S_1. \tag{15}$$

From the third equation of (7), it follows that

$$
\begin{aligned}
\bar{a}_3\omega &\leq \int_\kappa^{\kappa+\omega} \sigma_3(t)e^{u_1(t)}\Delta t + \int_\kappa^{\kappa+\omega} \sigma_4(t)e^{u_2(t)}\Delta t \\
&\leq \int_\kappa^{\kappa+\omega} \sigma_3(t)e^{S_1}\Delta t + \int_\kappa^{\kappa+\omega} \sigma_4(t)e^{u_2(\eta_2)}\Delta t \\
&\leq \bar{\sigma}_3\omega e^{S_1} + \int_\kappa^{\kappa+\omega} \sigma_4(t)e^{u_2(\eta_2)}\Delta t \\
&= \bar{\sigma}_3\omega e^{S_1} + \bar{\sigma}_4\omega e^{u_2(\eta_2)}.
\end{aligned}
$$

Then

$$
u_2(\eta_2) \geq \ln\left[\frac{\bar{a}_3 - \bar{\sigma}_3 e^{S_1}}{\bar{\sigma}_4}\right] := M_2. \tag{16}
$$

From (9),(11) and (16) and using the Lemma 3.2, we obtain

$$
\begin{aligned}
u_2(t) &\leq u_2(\xi_2) + \int_\kappa^{\kappa+\omega} |u_2^\Delta(t)|\Delta t \\
&\leq m_2 + 2(\bar{a}_2 + \bar{\gamma}_2)\omega =: B_3, \tag{17}
\end{aligned}
$$

$$
\begin{aligned}
u_2(t) &\geq u_2(\eta_2) - \int_\kappa^{\kappa+\omega} |u_2^\Delta(t)|\Delta t \\
&\geq M_2 - 2(\bar{a}_2 + \bar{\gamma}_2)\omega =: B_4. \tag{18}
\end{aligned}
$$

It follows from (17) and (18) that

$$
\max_{t\in I_\omega} |u_2(t)| \leq \max\{|B_3|, |B_4|\} := S_2. \tag{19}
$$

In view of the third equation of (7), we get

$$
\begin{aligned}
\bar{a}_3\omega &> \int_\kappa^{\kappa+\omega} \frac{\sigma_3(t)e^{u_1(t)}}{1 + d_1 e^{u_1(t)} + e_1 e^{u_2(t)} + \mu_1 e^{u_3(t)}}\Delta t \\
&\geq \int_\kappa^{\kappa+\omega} \frac{\sigma_3(t)e^{-S_1}}{1 + d_1 e^{S_1} + e_1 e^{S_2} + \mu_1 e^{u_3(\eta_3)}}\Delta t \\
&= \frac{\bar{\sigma}_3\omega e^{-S_1}}{1 + d_1 e^{S_1} + e_1 e^{S_2} + \mu_1 e^{u_3(\eta_3)}}.
\end{aligned}
$$

Then

$$
u_3(\eta_3) > \ln\left[\frac{\bar{\sigma}_3 e^{-S_1} - \bar{a}_3(1 + d_1 e^{S_1} + e_1 e^{S_2})}{\bar{a}_3\mu_1}\right] := M_3. \tag{20}
$$

According to the third equation of (7), we also have

$$
\begin{aligned}
\bar{a}_3\omega &< \int_\kappa^{\kappa+\omega} \frac{\sigma_3(t)e^{u_1(t)}}{\mu_1 e^{u_3(t)}}\Delta t + \int_\kappa^{\kappa+\omega} \frac{\sigma_4(t)e^{u_2(t)}}{\mu_2 e^{u_3(t)}}\Delta t \\
&\leq \frac{\bar{\sigma}_3 e^{S_1}\omega}{\mu_1 e^{u_3(\xi_3)}} + \frac{\bar{\sigma}_4 e^{S_2}\omega}{\mu_2 e^{u_3(\xi_3)}}.
\end{aligned}
$$

Hence

$$
u_3(\xi_3) \leq \ln\left[\frac{\bar{\sigma}_3\mu_2 e^{S_1} + \bar{\sigma}_4\mu_1 e^{S_2}}{\bar{a}_3\mu_1\mu_2}\right] := m_3. \tag{21}
$$

From (10),(20) and (21) and using the Lemma 3.2, we obtain

$$
\begin{aligned}
u_3(t) &\leq u_3(\xi_3) + \int_\kappa^{\kappa+\omega} |u_3^\Delta(t)|\Delta t \\
&\leq m_3 + 2\bar{a}_3\omega =: B_5, \tag{22}
\end{aligned}
$$

$$
\begin{aligned}
u_3(t) &\geq u_3(\eta_3) - \int_\kappa^{\kappa+\omega} |u_3^\Delta(t)|\Delta t \\
&\geq M_3 - 2\bar{a}_3\omega =: B_6. \tag{23}
\end{aligned}
$$

It follows from (22) and (23) that

$$
\max_{t\in I_\omega} |u_3(t)| \leq \max\{|B_5|, |B_6|\} := S_3. \tag{24}
$$

(ii) If $u_1(\eta_1) < u_2(\eta_2)$, then it follows from the third equation of (7) that

$$
\begin{aligned}
\bar{a}_3\omega &\leq \int_\kappa^{\kappa+\omega} \sigma_3(t)e^{u_1(t)}\Delta t + \int_\kappa^{\kappa+\omega} \sigma_4(t)e^{u_2(t)}\Delta t \\
&\leq \int_\kappa^{\kappa+\omega} \sigma_3(t)e^{u_1(\eta_1)}\Delta t + \int_\kappa^{\kappa+\omega} \sigma_4(t)e^{u_2(\eta_2)}\Delta t \\
&\leq \int_\kappa^{\kappa+\omega} \sigma_3(t)e^{u_2(\eta_2)}\Delta t + \int_\kappa^{\kappa+\omega} \sigma_4(t)e^{u_2(\eta_2)}\Delta t \\
&= (\bar{\sigma}_3 + \bar{\sigma}_4)\omega e^{u_2(\eta_2)}
\end{aligned}
$$

which leads to

$$
u_2(\eta_2) > \ln\left[\frac{\bar{a}_3}{\bar{\sigma}_3 + \bar{\sigma}_4}\right] := \overline{M}_1. \tag{25}
$$

Based on (9), (11) and (25), using the Lemma 3.2, we get

$$
\begin{aligned}
u_2(t) &\leq u_2(\xi_2) + \int_\kappa^{\kappa+\omega} |u_2^\Delta(t)|\Delta t \\
&\leq m_2 + 2(\bar{a}_2 + \bar{\gamma}_2)\omega =: \overline{B}_1, \tag{26}
\end{aligned}
$$

$$
\begin{aligned}
u_2(t) &\geq u_2(\eta_2) - \int_\kappa^{\kappa+\omega} |u_2^\Delta(t)|\Delta t \\
&\geq \overline{M}_1 - 2(\bar{a}_2 + \bar{\gamma}_2)\omega =: \overline{B}_2. \tag{27}
\end{aligned}
$$

Thus

$$
\max_{t\in I_\omega} |u_2(t)| \leq \max\{|\overline{B}_1|, |\overline{B}_2|\} := \overline{S}_1. \tag{28}
$$

From the third equation of (7), it follows that

$$
\begin{aligned}
\bar{a}_3\omega &\leq \int_\kappa^{\kappa+\omega} \sigma_3(t)e^{u_1(t)}\Delta t + \int_\kappa^{\kappa+\omega} \sigma_4(t)e^{u_2(t)}\Delta t \\
&\leq \int_\kappa^{\kappa+\omega} \sigma_3(t)e^{u_1(\eta_1)}\Delta t + \int_\kappa^{\kappa+\omega} \sigma_4(t)e^{\overline{S}_1}\Delta t \\
&\leq \int_\kappa^{\kappa+\omega} \sigma_3(t)e^{u_1(\eta_1)}\Delta t + \bar{\sigma}_4\omega e^{\overline{S}_1} \\
&= \bar{\sigma}_3\omega e^{u_1(\eta_1)} + \bar{\sigma}_4\omega e^{\overline{S}_1}.
\end{aligned}
$$

Then

$$
u_1(\eta_1) \geq \ln\left[\frac{\bar{a}_3 - \bar{\sigma}_4 e^{\overline{S}_1}}{\bar{\sigma}_3}\right] := \overline{M}_2. \tag{29}
$$

From (8),(11) and (29) and using the Lemma 3.2, we obtain

$$
\begin{aligned}
u_1(t) &\leq u_1(\xi_1) + \int_\kappa^{\kappa+\omega} |u_1^\Delta(t)|\Delta t \\
&\leq m_1 + 2(\bar{a}_1 + \bar{\gamma}_1)\omega =: \overline{B}_3 \tag{30}
\end{aligned}
$$

$$
\begin{aligned}
u_1(t) &\geq u_1(\eta_1) - \int_\kappa^{\kappa+\omega} |u_1^\Delta(t)|\Delta t \\
&\geq \overline{M}_2 - 2(\bar{a}_1 + \bar{\gamma}_1)\omega =: \overline{B}_4. \tag{31}
\end{aligned}
$$

It follows from (30) and (31) that

$$
\max_{t\in I_\omega} |u_1(t)| \leq \max\{|\overline{B}_3|, |\overline{B}_4|\} := \overline{S}_2. \tag{32}
$$

In view of the third equation of (7), we get

$$
\begin{aligned}
\bar{a}_3\omega \; &> \; \int_\kappa^{\kappa+\omega} \frac{\sigma_3(t)e^{u_1(t)}}{1+d_1 e^{u_1(t)}+e_1 e^{u_2(t)}+\mu_1 e^{u_3(t)}}\Delta t \\
&\geq \; \int_\kappa^{\kappa+\omega} \frac{\sigma_3(t)e^{-\overline{S}_2}}{1+d_1 e^{\overline{S}_2}+e_1 e^{\overline{S}_1}+\mu_1 e^{u_3(\eta_3)}}\Delta t \\
&= \; \frac{\bar{\sigma}_3\omega e^{-\overline{S}_2}}{1+d_1 e^{\overline{S}_2}+e_1 e^{\overline{S}_1}+\mu_1 e^{u_3(\eta_3)}}.
\end{aligned}
$$

Then

$$
u_3(\eta_3) > \ln\left[\frac{\bar{\sigma}_3 e^{-\overline{S}_2}-\bar{a}_3(1+d_1 e^{\overline{S}_2}+e_1 e^{\overline{S}_1})}{\bar{a}_3\mu_1}\right] := \overline{M}_3. \tag{33}
$$

According to the third equation of (7), we also have

$$
\begin{aligned}
\bar{a}_3\omega \; &< \; \int_\kappa^{\kappa+\omega} \frac{\sigma_3(t)e^{u_1(t)}}{\mu_1 e^{u_3(t)}}\Delta t + \int_\kappa^{\kappa+\omega} \frac{\sigma_4(t)e^{u_2(t)}}{\mu_2 e^{u_3(t)}}\Delta t \\
&\leq \; \frac{\bar{\sigma}_3 e^{\overline{S}_2}\omega}{\mu_1 e^{u_3(\xi_3)}} + \frac{\bar{\sigma}_4 e^{\overline{S}_1}\omega}{\mu_2 e^{u_3(\xi_3)}}.
\end{aligned}
$$

Hence

$$
u_3(\xi_3) \leq \ln\left[\frac{\bar{\sigma}_3\mu_2 e^{\overline{S}_2}+\bar{\sigma}_4\mu_1 e^{\overline{S}_1}}{\bar{a}_3\mu_1\mu_2}\right] := \bar{m}_3. \tag{34}
$$

From (10),(33) and (34) and using the Lemma 3.2, we obtain

$$
\begin{aligned}
u_3(t) \; &\leq \; u_3(\xi_3) + \int_\kappa^{\kappa+\omega}|u_3^\Delta(t)|\Delta t \\
&\leq \; \bar{m}_3 + 2\bar{a}_3\omega =: \overline{B}_5,
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
u_3(t) \; &\geq \; u_3(\eta_3) - \int_\kappa^{\kappa+\omega}|u_3^\Delta(t)|\Delta t \\
&\geq \; \overline{M}_3 - 2\bar{a}_3\omega =: \overline{B}_6.
\end{aligned} \tag{36}
$$

It follows from (35) and (36) that

$$
\max_{t\in I_\omega}|u_3(t)| \leq \max\{|\overline{B}_5|,|\overline{B}_6|\} := \overline{S}_3. \tag{37}
$$

Obviously, $S_i, \overline{S}_i (i=1,2,3)$ are independent of the choice of $\lambda \in (0,1)$. Take $M = \max\{S_1,\overline{S}_2\} + \max\{S_2,\overline{S}_1\} + \max\{S_3,\overline{S}_3\} + S_0$, where S_0 is taken sufficiently large such that $S_0 \geq |m_1| + |m_2| + |m_3| + \max\{|M_1|,|\overline{M}_2|\} + \max\{|M_2|,|\overline{M}_1|\} + \max\{|M_3|,|\overline{M}_3|\}$.

Now we define $\Omega := \{(u_1,u_2,u_3)^T \in X : ||u|| < M\}$. It is clear that Ω verifies the requirement (a) of Lemma 3.1. If $(u_1,u_2,u_3)^T \in \partial\Omega \bigcap \text{Ker}L = \partial\Omega \bigcap \mathbb{R}^3$, then $(u_1,u_2,u_3)^T$ is a constant vector in $\mathbb{R}^3$ with $||(u_1,u_2,u_3)^T|| = |u_1|+|u_2|+|u_3| = M$. Then

$$
QN\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},
$$

where

$$
\begin{aligned}
P_1 \; =& \; \bar{a}_1 - \bar{b}_1 e^{u_1} - \bar{c}_1 e^{u_2} - \frac{\bar{\sigma}_1 e^{u_3}}{1+d_1 e^{u_1}+e_1 e^{u_2}+\mu_1 e^{u_3}} \\
& - \frac{1}{\omega}\int_\kappa^{\kappa+\omega}\gamma_1(t)\sin(\theta_1(t)t)\Delta t, \\
P_2 \; =& \; \bar{a}_2 - \bar{b}_2 e^{u_2} - \bar{c}_2 e^{u_1} - \frac{\bar{\sigma}_2 e^{u_3}}{1+d_2 e^{u_1}+e_2 e^{u_2}+\mu_2 e^{u_3}} \\
& - \frac{1}{\omega}\int_\kappa^{\kappa+\omega}\gamma_2(t)\sin(\theta_2(t)t)\Delta t, \\
P_3 \; =& \; -\bar{a}_3 + \frac{\bar{\sigma}_3 e^{u_1}}{1+d_1 e^{u_1}+e_1 e^{u_2}+\mu_1 e^{u_3(t)}} \\
& + \frac{\bar{\sigma}_4 e^{u_2}}{1+d_2 e^{u_1}+e_2 e^{u_2}+\mu_2 e^{u_3}}.
\end{aligned}
$$

Now let us consider homotopic $\phi(u_1,u_2,u_3,\mu) = \mu QNx + (1-\mu)Gx, \mu \in [0,1], u = (u_1,u_2,u_3)^T$, where

$$
Gx = \begin{pmatrix} \bar{a}_1 - \bar{b}_1 e^{u_1} \\ \bar{a}_2 - \bar{b}_2 e^{u_2} \\ -\bar{a}_3 + \frac{\bar{\sigma}_3 e^{u_1}}{1+d_1 e^{u_1}+e_1 e^{u_2}+\mu_1 e^{u_3(t)}} \end{pmatrix}.
$$

Letting J be the identity mapping. By direct calculation, we derive

$$
\begin{aligned}
&\deg\left[JQN(u_1,u_2,u_3)^T;\Omega\bigcap\text{Ker}L;0\right] \\
=& \; \deg\left[QN(u_1,u_2,u_3)^T;\Omega\bigcap\text{Ker}L;0\right] \\
=& \; \deg\left[\phi(u_1,u_2,u_3,1);\Omega\bigcap\text{Ker}L;0\right] \\
=& \; \deg\left[\phi(u_1,u_2,u_3,0);\Omega\bigcap\text{Ker}L;0\right] \\
=& \; \text{sign}\left[-(\bar{b}_1\bar{b}_2\mu_1\bar{\sigma}_3)e^{(2u_1^*+u_2^*+u_3^*)}\right] = -1 \neq 0,
\end{aligned}
$$

where $\deg(.,.,.,)$ is the Brower degree. Thus we have proved that Ω verifies all requirements of Lemma 3.1, then it follows that $Lx = Nx$ has at least one solution in $\text{Dom}L \cap \overline{\Omega}$. The proof is complete.

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