International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:2, No:4, 2008

On Diffusion Approximation of Discrete Markov Dynamical Systems

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Abstract**—**The paper is devoted to stochastic analysis of finite dimensional difference equation with dependent on ergodic Markov chain increments, which are proportional to small parameter ε. A point-form solution of this difference equation may be represented as vertexes of a time-dependent continuous broken line given on the segment [0,1] with ε-dependent scaling of intervals between vertexes. Tending ε to zero one may apply stochastic averaging and diffusion approximation procedures and construct continuous approximation of the initial stochastic iterations as an ordinary or stochastic Ito differential equation. The paper proves that for sufficiently small ε these equations may be successfully applied not only to approximate finite number of iterations but also for asymptotic analysis of iterations, when number of iterations tends to infinity.

Keywords**—**Markov dynamical system, diffusion approximation, equilibrium stochastic stability.

I. Introduction

THE aim of this paper is to propose an asymptotic methods for analysis of random iteration procedure in $\mathbb{R}^d$ given in a form of difference equation

$$x_{t+1} = x_t + \varepsilon f_1(x_t, y_t) + \varepsilon^2 f_2(x_t, y_t), \tag{1}$$

where right part depends on small positive parameter ε and ergodic homogeneous Feller Markov process y_t [6] on probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ with invariant measure $\mu(dy)$ and transition probability $p(y, dz)$ given on compact metric space $\mathbb{Y}$. We will assume that mappings $f_1 : \mathbb{R}^d \times \mathbb{Y} \to \mathbb{R}^d$ and $f_2 : \mathbb{R}^d \times \mathbb{Y} \to \mathbb{R}^d$ are continuous on $y \in \mathbb{Y}$, $f_1(x, y)$ has two bounded continuous x-derivatives $Df_1(x, y)$ and $D^2 f_1(x, y)$, and $f_2(x, y)$ has bounded continuous x-derivative $Df_2(x, y)$. Starting at $t = 0$ with given x_0, y_0 and applying iteration (1) one can generate vector $\{x_t, 0 \le t \le N\}$ for any N. But it is very complicated problem to find distribution of this vector for sufficiently large number N and therefore to find an approximation of the above distribution one should employ the limit theorems of contemporary probability theory (see [10], [14],[15] and references there). For that one can construct the broken line in $\mathbb{R}^d$ with vertexes in the points $\{x_t\}$ by formula

$$s \in [t\varepsilon^2, (t+1)\varepsilon^2] : \ X_\varepsilon(s) = (x_{t+1} - x_t)(s\varepsilon^{-2} - t) + x_t \tag{2}$$

for all $t \in [0, N(\varepsilon^{-2})]$, where $N(\alpha)$ is integer part of number α. Applying limit theorem from [15] to distributions

$$\mathbf{P}_\varepsilon \sim \{X_\varepsilon(s), 0 \le s \le 1\}$$

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we will construct limit distribution $\mathbf{P} \sim \{X(s), 0 \le s \le 1\}$ defined by stochastic Ito equation

$$dX(t) = a(X(s))ds + \sum_{k=1}^{d} \sigma_k(X(s))dW_k(s) \tag{3}$$

with initial condition $X(0) = x_0$, where vector-functions $a(x)$ and $\sigma_k(x),\ k = 1, 2, ..., d$ are defined based on averaging by measure μ of functions $f_j(x, y),\ j = 1, 2$ and its derivatives, and $\{W_k, k = 1, 2, ..., d\}$ are independent standard Wiener processes. The finite dimensional distribution of the solution of this equation $\{X(t\varepsilon^2), t = 0, 1, ..., N\}$ one can use to approximate distribution of solution of difference equation (1) $\{x_t, t = 0, 1, ..., N\}$ for any finite N. It should be mentioned that for analysis of (3) there are comprehensive facilities of contemporary stochastic analysis and mathematical physics. Besides we will prove that for sufficiently small ε to solve equilibrium asymptotical stability problem for (1) one can employ the second Lyapunov method derived for stochastic differential equations (3) in [11].

II. Related Work

The problem of asymptotic analysis of dynamical systems under small random perturbations has been discussed in many mathematical and engineering papers. Apparently, A.V. Skorokhod was the first mathematician, which has proved that the probabilistic limit theorems may be successfully used to approximate distributions of solutions of random dynamical systems by the solutions of stochastic differential equations on any finite time interval (see bibliography in [14],[15], and [10]). The above result at once has met with wide application in engineering and economical papers (see [5], [3], [1], [7], [12] and references there). It should be mentioned that in spite of the fact that the above result has been developed for the analysis of equations on a finite time interval, the averaging and diffusion approximation procedures have been applied in many applications for asymptotic stability analysis of possible stationary solutions, that is, for analysis of differential equations as $t \to \infty$. To prove the validity of this approach for random dynamical systems with continuous trajectories the researchers had to use not only a special type of limit theorem (see for example [4] and [2]) but also a stochastic version of the Second Lyapunov method developed for stochastic Ito differential equations in [11]. But most of dynamical systems of the recent Economics (see, for example, [8], [9], [3], [7], [12] and review there) require an extension of the above ”smooth” models to allow the phase motion to have a jump type discontinuity. Some of results permissive to resolve this problem have been developed by author in [16] and [17] for dynamical systems with switching in Markov time moments. Proposal paper is devoted to similar approach to discrete Markov dynamical systems. This problem is very important in contemporary financial econometrics for analysis of ARCH type stationary iterative procedures (see, for example, [7] and [12]).

III. Probabilistic limit theorems and equilibrium stochastic stability

Let $p(y, dz)$ is transition probability of Markov chain y_t and $\mathcal{P}$ is Markov operator

$$(\mathcal{P}v)(y) := \int_{\mathbb{Y}} v(z)p(y, dz)$$

defined on the space $\mathbb{C}(\mathbb{Y})$ of bounded continuous functions. We will assume that the spectrum $\sigma(\mathcal{P})$ has the simple eigenvalue 1, $\sigma(\mathcal{P})\diagdown\{1\} \subset \{z \in \mathbb{C} : |z| < \rho < 1\}$, and probability distribution $\{\mu(dy)\}$ is the solution of the equation $\mathcal{P}^*\mu = \mu$, where $\mathcal{P}^*$ is conjugate operator. Averaging procedure by the above invariant measure of any dependent on Markov process vector or matrix will be denoted with overline. Under these conditions one can extend [6] the potential of the above Markov process and to define the linear continuous operator by equality

$$(\Pi v)(y) := \sum_{k=0}^{\infty}(\mathcal{P}^k v)(y) \tag{4}$$

on the space $\bar{\mathbb{C}}(\mathbb{Y})$ of continuous functions $v \in \mathbb{C}(\mathbb{Y})$ with zero average $\bar{v} := \int_{\mathbb{Y}} v(y)\mu(dy)$. This means that the equation $\mathcal{P}g - g = -v$ with $v \in \bar{\mathbb{C}}(\mathbb{Y})$ has unique solution (4) in $\bar{\mathbb{C}}(\mathbb{Y})$. Using the above Markov chain one can define on the segment $[0, 1]$ step processes

$$s \in [t\varepsilon^2, (t+1)\varepsilon^2) : Y_\varepsilon(s) := y_t \tag{5}$$

If $\mathfrak{F}^t \subset \mathfrak{F},\ t \geq 0$ is minimal filtration for stationary process y_t then for any $t \geq 0$ and $s \in [t\varepsilon^2, (t+1)\varepsilon^2)$ random vectors $X_\varepsilon(s)$ and $Y_\varepsilon(s)$ are $\mathfrak{F}^t$-measurable. To avoid cumbersome formulae we will denote conditional expectation $\mathbf{E}\{\xi/\mathfrak{F}^t\}|_{x_t=x, y_t=y}$ in abridged form $\mathbf{E}^t_{x,y}\{\xi\}$.

A. Derivation of diffusion approximation formula

In this subsection we will assume that $\bar{f}_1(x) \equiv 0$. Using the solution $x_t, t \in \mathbb{N}$ of difference equation (1) with initial condition $x_0 = x$ and Markov process y_t one can define the broken lines by formulae (2) and step process (5) for all $t \in [0, N(\varepsilon^{-2})]$. Not so difficult to be certain of Markov properties for the pare $\{X_\varepsilon(s), Y_\varepsilon(s), 0 \leq s \leq 1\}$. Therefore under assumption that $\varepsilon \to 0$ one can apply the Skorokhod limit theorems from [14] and [15] for sequences of Markov processes and look for diffusion approximation of $\{X_\varepsilon(s), 0 \leq s \leq 1\}$ if the latter exists. Much as it has been done in [16] for jump type Markov processes in our case for any arbitrary twice continuous differentiable on x function $v(x)$ one has to look for Lyapunov function in a form of decomposition

$$v^\varepsilon(x, y) := v(x) + \varepsilon[((\Pi f_1)(x, y), \triangledown)v](x, y) + \varepsilon^2\hat{v}(x, y) \tag{6}$$

with some smooth function $\hat{v}(x, y)$. Here and further $\triangledown v(x)$ is gradient and $(\cdot,\cdot)$ is scalar product in $\mathbb{R}^d$. Now one should compute derivative

$$(\mathbf{L}(\varepsilon)v^\varepsilon)(x, y) := \lim_{\delta\downarrow 0}\frac{1}{\delta}\mathbf{E}^t_{x,y}\{v^\varepsilon(X^\varepsilon(s+\delta), Y^\varepsilon(s+\delta)) - v^\varepsilon(X^\varepsilon(s), Y^\varepsilon(s))\} = \frac{1}{\varepsilon^2}\mathbf{E}^t_{x,y}\{v^\varepsilon(x_{t+1}, y_{t+1}) - v^\varepsilon(x, y) + o(\varepsilon^2)\} \tag{7}$$

for all $x \in \mathbb{R}^d, y \in \mathbb{Y}, t \geq 0$ and $s \in [t\varepsilon^2, (t+1)\varepsilon^2)$, and chose in (6) function $\hat{v}(x, y)$ in such a way as to exist limit

$$\lim_{\varepsilon\to 0}(\mathbf{L}(\varepsilon)v^\varepsilon)(x, y) = (\mathbf{L}v)(x) \tag{8}$$

As it will be shown later right side of the above equation has a form of diffusion operator applied to function $v(x)$:

$$(\mathbf{L}v)(x) = \{(a(x), \triangledown) + (\sigma(x)\triangledown, \triangledown)\}v(x) \tag{9}$$

with vector $a(x)$ and positive defined symmetric matrix $\sigma(x)$. To derive the above formula one has to present operator $\mathbf{L}(\varepsilon)$ accurate within $0(\varepsilon)$

$$\mathbf{L}(\varepsilon) = \tfrac{1}{\varepsilon^2}(\mathcal{P} - I) + \tfrac{1}{\varepsilon}(f_1(x, y), \triangledown)\mathcal{P} + (f_2(x, y), \triangledown)\mathcal{P} + \tfrac{1}{2}(f_1(x, y), \triangledown)^2\mathcal{P} + 0(\varepsilon) \tag{10}$$

to employ (10) to (6) and to decompose resulting function by powers of ε accurate within $0(\varepsilon)$:

$$(\mathbf{L}(\varepsilon)v^\varepsilon)(x, y) = \frac{1}{\varepsilon^2}(\mathcal{P} - I)v(x) + \frac{1}{\varepsilon}[(f_1(x, y), \triangledown)v(x) + (\mathcal{P} - I)((\Pi f_1)(x, y), \triangledown)v(x)] + (f_2(x, y), \triangledown)v(x) + \frac{1}{2}(f_1(x, y), \triangledown)^2 v(x)] + (f_1(x, y), \triangledown)\mathcal{P}[(f_1(x, y), \triangledown)v(x)] + (\mathcal{P} - I)\hat{v}(x, y) + 0(\varepsilon)$$

Therefore using obvious equalities $(\mathcal{P} - I)\Pi = -I$, $(\mathcal{P} - I)v(x) = 0$ and formula (8) one can write equation

$$\mathbf{L}v(x) = (f_2(x, y), \triangledown)v(x) + \frac{1}{2}(f_1(x, y), \triangledown)^2 v(x) + (f_1(x, y), \triangledown)[(\mathcal{P}\Pi f_1(x, y), \triangledown)v(x)] + (\mathcal{P} - I)\hat{v}(x, y)$$

with unknown function $\hat{v}(x, y)$. As it has been mentioned at the beginning of this subsection the above equation relative to $\hat{v}(x, y)$ has solution

$$\hat{v}(x, y) = \Pi\{(f_2(x, y), \triangledown)v(x) + \frac{1}{2}(f_1(x, y), \triangledown)^2 v(x) + (f_1(x, y), \triangledown)[(\mathcal{P}\Pi f_1(x, y), \triangledown)v(x)] - \mathbf{L}v(x)\} \tag{11}$$

if and only if

$$\mathbf{L}v(x) = \{\overline{(f_2, \triangledown)} + \frac{1}{2}\overline{(f_1, \triangledown)^2} + \overline{(f_1, \triangledown)[(\mathcal{P}\Pi f_1, \triangledown)}\}v(x) \tag{12}$$

where overline denotes averaging by measure μ. This equation one can write in a form (9) using notations

$$\begin{aligned} a &= \overline{f}_2 + \overline{[\mathcal{P}\Pi D f_1]^T f_1} \\ \sigma &= \overline{f_1 f_1^T} + \overline{f_1\mathcal{P}\Pi f_1^T} + \overline{(\mathcal{P}\Pi f_1)f_1^T} \end{aligned} \tag{13}$$

where D is Frechet derivative by x and upper index T denotes transposition. To write this equation in a form (3) one has to find d dependent on x vectors σ_k defined by equation

$$\sum_{k=1}^{d} \sigma_k(x)\sigma_k^T(x) = \sigma(x)$$

As it has been mentioned in [10] this equation has solution for any positive defined matrix $\sigma(x)$.

B. Averaging and normalized deviations

Let us remind of assumption $\bar{f}_1(x) \equiv 0$ which permits in previous subsection to derive formulae (12) and (13). Otherwise one may not divide segment $[0, 1]$ by intervals of length ε^2 because $\Pi f_1(x, y)$ does not exist and therefore there are singularity in the definition of operator (7) as $\varepsilon \to 0$. To apply a diffusion approximation in this case one has to find solution of averaged equation

$$\bar{x}_{t+1} = \bar{x}_t + \varepsilon \bar{f}_1(\bar{x}_t) \tag{14}$$

and to derive an asymptotic formula for so called *normalized deviations*

$$z_t := \frac{x_t - \bar{x}_t}{\sqrt{\varepsilon}} \tag{15}$$

Substituting $x_t = \sqrt{\varepsilon} z_t + \bar{x}_t$ in (1)

$$z_{t+1} = z_t + \delta g_1(\bar{x}_t, y_t) + \delta^2[Df_1(\bar{x}_t, y_t)]z_t + o(\delta^2), \tag{16}$$

where $\delta = \sqrt{\varepsilon}, g_1(x, y) = f_1(x, y) - \bar{f}_1(x)$, one can apply to system (14)-(16) approach of previous subsection. The sequence (15) gives rise to random processes

$$Z^\delta(s) = \frac{X^\delta(s) - \bar{X}^\delta(s)}{\delta}$$

where $X^\delta(s)$ and $\bar{X}^\delta(s)$ are defined in the same way like (2) for all $s \in [t\delta^2, (t+1)\delta^2)$ and any $t \in [0, N(\delta^{-2})]$. After substitution $Z^\delta(s)$ instead of $X_\varepsilon(s)$, $[Df_1(\bar{X}(s), Y^\delta(s))]Z^\delta(s)$ instead of $f_2(X_\varepsilon(s), Y_\varepsilon(s))$ and $g_1(\bar{X}(s), Y^\delta(s))$ instead of $f_1(X_\varepsilon(s), Y_\varepsilon(s))$ in corresponding formulae and vanishing δ one can approximate probability distribution $\mathbf{P}^Z_\delta$ of process $Z^\delta(s)$ by probability distribution $\mathbf{P}^Z$ of process Z satisfying stochastic differential equation equation

$$dZ(s) = D\bar{f}_1(\bar{X}(s))Z(s)ds + \sum_{k=1}^{d} \sigma_k(\bar{X}(s))dW_k(s)$$

with initial condition $\bar{X}(0) = x_0$, where $\{W_k(s), k = 1, 2, ..., d\}$ are independent standard Wiener processes, and vectors $\{\sigma_k, k = 1, 2, ..., d\}$ satisfy an equality

$$\sum_{k=1}^{d} \sigma_k(x)\sigma_k^T(x) = [\overline{g_1 g_1^T} + \overline{g_1 \mathcal{P}\Pi g_1^T} + \overline{(\mathcal{P}\Pi g_1)g_1^T}](x)$$

Deterministic function $\bar{X}(s)$ one can find as the solution of ordinary differential equation

$$d\bar{X}(s) = \bar{f}_1(\bar{X}(s))ds$$

Roughly speaking for sufficiently small ε one can approximate distribution of the sequence $\{x_t, 0 \geq t \geq N(\varepsilon^{-1})\}$ by distribution of sequence $\{X(t\varepsilon)+\sqrt{\varepsilon}Z(t\varepsilon), 0 \geq t \geq N(\varepsilon^{-1})\}$.

C. Equilibrium asymptotic stability

As it has been mentioned in the Section 2 some of application iterative procedures analysis require asymptotic analysis of equation (1) as $t \to \infty$. For example discussing diffusion approximation approach to GARCH time series authors of papers [12] and [7] indicate this problem in view of the approximation and asymptotic stability analysis of stationary conditional variance. In previous section we have derived an approximate distribution of sequence $\{x_t, 0 \leq t \leq N\}$ for any finite integer number N by distribution of solution of stochastic differential equation $\{X(s), 0 \leq s \leq 1\}$ but for the above mentioned asymptotic analysis as $t \to \infty$ one has to deal with equation (3) with unrestrictedly large s. Besides there is a problem of legality results which are based on the diffusion approximation as $s \to \infty$. This subsection is devoted to the above problem.

Let point $x = 0$ be an equilibrium of iteration procedure (1), i.s. $f_1(0, y) \equiv 0$ and $f_2(0, y) \equiv 0$. If for any $\eta > 0$ there exists such a neighborhood $U_r := \{x \in \mathbb{R}^d : |x| < r\}$ that any starting in U_η solution x_t of (1) does not leave U_r and tends to zero as $t \to \infty$ with probability greater than $1-\eta$ the above equilibrium is called *asymptotic stochastically stable*. As it has been shown in [13] for equilibrium stability analysis one can employ the second Lyapunov method with Lyapunov operator defined by formula

$$(\mathcal{L}v)(x, y) := \mathbf{E}^0_{x,y}\{v(x_1, y_1)\} - v(x, y)$$

and Lyapunov functions satisfying inequality

$$|x|^p < v(x, y) < c|x|^p$$

with some positive p a $c \geq 1$. If there exists such a Lyapunov function $v(x, y)$ that

$$(\mathcal{L}v)(x, y) < -\gamma|x|^p$$

with $\gamma \in (0, 1)$ then [13] equilibrium is asymptotic stochastically stable and $\mathbf{E}_{x,y}\{|x_t|\} \leq M|x|^p \exp\{-\rho t\}$ with some positive constants M and ρ. Besides under smoothness assumptions of the Section 1 on vectors $f_1(x, y)$ and $f_2(x, y)$, this equilibrium is asymptotic stochastically stable if and only if [13] the same property has the trivial solution of its linear approximation

$$\tilde{x}_{t+1} = \tilde{x}_t + \varepsilon \tilde{f}_1(\tilde{x}_t, y_t) + \varepsilon^2 \tilde{f}_2(\tilde{x}_t, y_t) \tag{17}$$

where $\tilde{f}_j(x, y) = (Df_j)(0, y)x, \ j = 1, 2$. Therefore for asymptotic analysis of (1) as $t \to \infty$ one can apply formulae (9) with (6), (11), (12), and (13) substituting linear on $x \in \mathbb{R}^d$ functions $\tilde{f}_j(x, y)$ instead of $f_j(x, y), j = 1, 2$ and rewriting equation (3) in a form of linear stochastic Ito equation

$$d\tilde{X}(s) = A\tilde{X}(s))ds + \sum_{k=1}^{d} B_k\tilde{X}(s)dW_k(s) \tag{18}$$

The same result like mentioned above for Markov iterations (17) one can find in [11] for stochastic differential equation (18): trivial solution of (18) is asymptotic stochastically stable if and only if there exists such twice continuous differentiable Lyapunov function $V(x)$ that

$$|x|^p \leq V(x) \leq h_1|x|^p, \quad \mathbf{L}\,V(x) \leq -h_2|x|^p \tag{19}$$

and $||D^l \nabla v(x)|| \le h_3|x|^{p-l-1}$, $l = 1, 2, 3$ for some $p > 0$, positive constants h_j, $j = 1, 2, 3$. and any $x \in \mathbb{R}^d$. Now for analysis of asymptotic behaviour of linear iteration (17) one can apply the second Lyapunov method with function

$$V^\varepsilon(x, y) := V(x) + \varepsilon[((\Pi\tilde{f}_1)(x, y), \nabla)V](x, y) + \varepsilon^2\hat{V}(x, y) \quad (20)$$

where $V(x)$ satisfies inequalities (19) and

$$\hat{V}(x, y) = \Pi\{(\tilde{f}_2(x, y), \nabla)V(x) + \frac{1}{2}(\tilde{f}_1(x, y), \nabla)^2V(x) + (\tilde{f}_1(x, y), \nabla)[(\mathcal{P}\Pi\tilde{f}_1(x, y), \nabla)V(x)]\} \quad (21)$$

Owing to linearity of functions $\tilde{f}_j(x, y), j = 1, 2$ and definition of $\mathbf{L}V(x)$ for all sufficiently small $\varepsilon > 0$ there exist such positive constants $h_j, j = \overline{4, 9}$ that the above defined functions satisfy inequalities

$$h_4|x|^p \le |\hat{V}(x, y)| \le h_5|x|^p,$$
$$h_6|x|^p \le |[((\Pi\tilde{f}_1)(x, y), \nabla)V](x, y)| \le h_7|x|^p$$
$$|V^\varepsilon(x, y) - V(x)| \le \varepsilon h_8|x|^p$$

and

$$|(\mathbf{L}(\varepsilon)V^\varepsilon)(x, y) - \mathbf{L}\,V(x)| < \varepsilon h_9|x|^p$$

Therefore if the trivial solution of diffusion approximation is asymptotically stable then there exists Lyapunov function satisfying (19) and for stability analysis of (17) one can use function (20):

$$\begin{aligned}(\mathcal{L}V^\varepsilon)(x, y) &= \varepsilon^2(\mathbf{L}(\varepsilon)V^\varepsilon)(x, y) \le \\ \le \varepsilon^2\mathbf{L}\,V(x) &\le \varepsilon^2(-h_2 + \varepsilon h_9)|x|^p\end{aligned}$$

This inequality convinces of asymptotical stochastic stability for trivial solution of difference equation (17) if ε is sufficiently small.

D. Example. Markov type GARCH model

In papers [7] and [12] the authors discuss a problem of diffusion approximation for very popular in contemporary econometrics GARCH (General AutoRegressive Conditional Heteroscedastic) process for conditional time series variance. The paper [12] deals with model given in a form of first order linear difference equation

$$\sigma^2_{t+1} = \omega_h + \sigma^2_t[\beta_h + h^{-1}\alpha_{h\,h}Z^2_t] \quad (22)$$

where h is small positive parameter, $\{_hZ_t, t \in \mathbb{Z}\}$ is sequence of i.i.d. random variables with zero mean, variance $\mathbf{E}\{_hZ^2_t\} = h$, and fourth moment $\mathbf{E}\{_hZ^4_t\} = 3h^2$. Under assumptions

$$1 - \alpha_h - \beta_h = h\theta + o(h), \omega_h = h\omega + o(h), \alpha_h = \frac{\sqrt{h}}{\sqrt{2}}\alpha + o(h)$$

author of paper [12] derives diffusion approximation equation in a form

$$d\sigma^2_t = (\omega - \theta\sigma^2_t)dt + \alpha\sigma^2_t dW(t) \quad (23)$$

To compare this result with our derived formulae one can denote

$$h = \varepsilon^2, x_t = \sigma^2_t, y_t = \frac{_hZ^2_t - h}{\sqrt{2}h}$$

and rewrite equation (22) in a form of difference equation (1) accurate within ε-items of second order

$$x_{t+1} = x_t + \varepsilon^2[\omega - \theta x_t] + \varepsilon\alpha y_t x_t \quad (24)$$

Let y_t be stationary Markov process with the same unconditional moments as $\frac{_hZ^2_t - h}{\sqrt{2}h}$, that is, $\mathbf{E}y_t = 0, \mathbf{E}y^2_t = 1$ and correlation function $C(k) = \mathbf{E}\{y_t y_{t+k}\}$ for $k \in \mathbb{N}$. Following our proposal method of diffusion approximation one should for this equation calculate parameters (13) with $f_1(x, y) = \alpha yx$, $f_2(x, y) = \omega - \theta x$. By definition

$$a(x) = \omega - \theta x + \alpha^2 x \sum_{l=1}^{\infty}\left\{\int_{\mathbb{Y}} \mathbf{E}^0_y\{yy_l\}\mu(dy)\right\} = = \omega + \left[\alpha^2\sum_{k=1}^{\infty}C(k) - \theta\right]x$$

$$\sigma^2(x) = \alpha^2x^2\int_{\mathbb{Y}} y^2\mu(dy) + 2\alpha^2x^2\sum_{k=1}^{\infty}C(k) = \alpha^2x^2\left[\mathrm{Var}\{y_t\} + 2\sum_{k=1}^{\infty}C(k)\right]$$

If $\{y_t, t \in \mathbb{Z}\}$ are independent random variables with zero mean and unit variance like it has been assumed in [12] we have derived equation (23) because $C(k) \equiv 0$. If $\kappa := \sum_{k=1}^{\infty} C(k) \ne 0$ one should apply diffusion approximation for GARCH(1,1)-process in a following form

$$d\sigma^2_t = (\omega + (\alpha^2\kappa - \theta)\sigma^2_t)dt + \alpha\sqrt{1 + 2\kappa}\sigma^2_t dW(t) \quad (25)$$

As it has been proved this equation one can use also for analysis of (22) as $t \to \infty$. According to [11] if

$$\alpha^2\kappa - \theta - \frac{\alpha^2(1 + 2\kappa)}{2} = -\theta - \frac{\alpha^2}{2} < 0$$

there exists stationary solution $\hat{s}^2_t$ of the above equation and deviations $z_t := s^2_t - \hat{s}^2_t$ of any other solution from this stationary process exponentially tend to zero as $t \to \infty$. In spite of the fact that process y_t has nonzero correlation this result no differs from similar result of the paper [12]. But to approximate stationary process for GARCH(1,1) with Markov process y_t instead of i.i.d. sequence one has to deal with stationary solution of equation (25) where $\kappa \ne 0$. As it has been derived by E.Wong [18] for linear stochastic Ito equation the stationary process defined by (25) has density function $f(x) = \frac{s^r x^{(r-1)}}{\Gamma(r)}e^{-sx}$ where $r = 1 + \frac{2(\theta - \alpha^2\kappa)}{\alpha^2(1+2\kappa)}$,$s = \frac{2\omega}{\alpha^2(1+2\kappa)}$ that is, correlation affects distinctly on the asymptotic approximation of stationary distribution.

IV. Conclusion

Stochastic averaging and diffusion approximation procedures may be successfully applied both to approximate finite number of Markov type iterations and for asymptotic analysis of iterations, when number of iterations tends to infinity. On application of the above probabilistic limit theorems to asymptotic analysis of stochastic iterative procedures a consideration must be given to possible correlation of perturbing processes. This affects not only a diffusion approximation formula but also a decision on iterations convergence and limit distribution.

ACKNOWLEDGMENT

The paper is supported by Riga Technical University (grant No. R7388) and Latvian Council of Science (grant No. Z05.1879).

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