

Some Clopen sets in the Uniform Topology on BCI-algebras

A. Hasankhani, H. saadat, M.M. Zahedi

Abstract— In this paper some properties of the uniformity topology on a BCI-algebras are discussed.

Keywords—(Fuzzy) ideal, (Fuzzy) subalgebra, Uniformity, clopen sets.

I. INTRODUCTION

In 1966, K. Iseki introduced the concept of BCI-algebra [4]. In 1965, L.A. Zadeh [6] defined the concept of a fuzzy set, as a function from a non-empty set to $[0,1]$. In [1], B. Ahmad, apply this notion to BCI-algebra. In this paper we will discuss some properties of the uniform topology on a BCI-algebra.

II. PRELIMINARIES

Definition 2.1. By a BCI-algebra we mean an algebra $(X;*,0)$ of type (2,0) satisfying the axioms:

$$\text{BCI-1) } ((x * y) * (x * z)) * (z * y) = 0,$$

$$\text{BCI-2) } (x * (x * y)) * y = 0,$$

$$\text{BCI-3) } x * x = 0,$$

$$\text{BCI-4) } x * y = y * x = 0 \text{ implies } x = y,$$

$$\text{BCI-5) } x * 0 = 0 \text{ implies } x = 0.$$

For all x, y and z in X .

From now on $X = (X;*,0)$ is a BCI-algebra.

Definition 2.2 [3]. A subset B of X is called:

i) an ideal if for any x, y in X .

$$(1) 0 \in B$$

$$(2) x * y, y \in B \text{ imply } x \in B.$$

ii) a subalgebra if for any x, y in B , $x * y \in B$.

Definition 2.3 [1]. A fuzzy subset μ of X is called:

i) a fuzzy ideal of X if for any $x, y \in X$, we have

$$(1) \mu(0) \geq \mu(x), \text{ for all } x \text{ in } X,$$

$$(2) \mu(x) \geq \min\{\mu(x * y), \mu(y)\}.$$

ii) a fuzzy subalgebra of X if for any $x, y \in X$

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}.$$

Notation 2.4. The set of all (non-zero fuzzy) ideal of X is denoted by $(FI(X))I(X)$.

Note that by non-zero fuzzy set of X we mean, there is $x \in X$ such that $\mu(x) > 0$.

Lemma 2.5. (i) $A \in I(X)$ iff $\chi_A \in FI(X)$, where χ_A is the characteristic function of A .

(ii) If $\mu, \eta \in FI(X)$, then $\mu \cap \eta \in FI(X)$, where $\mu \cap \eta$ is a fuzzy subset of X which is defined by $\mu \cap \eta(x) = \min\{\mu(x), \eta(x)\}$, for all $x \in X$.

(iii) If $\mu \in FI(X)$, then

$$\mu(x * y) \geq \min\{\mu(x * z), \mu(z * y)\}, \quad \forall x, y, z \in X.$$

(iv) If $\mu \in FI(X)$, then $\mu(0) > 0$.

(v) A is a subalgebra of X if and only if χ_A is a fuzzy subalgebra.

Proof. The proofs of (i), (ii), (iv) and (v), are easy, and the proof of (iii) follows from BCI-1.

Remark 2.6. (i). By BCI-5, $\{0\} \in I(X)$ and hence $\chi_{\{0\}} \in FI(X)$.

(ii) For all $x \in X$, $A \in I(X)$, $\chi_A(x * x) = 1$, by BCI-3.

Definition 2.7 [2]. A BCI-algebra X is called medial if

$$(x * y) * (z * u) = (x * z) * (y * u), \quad \forall x, y, z, u \in X.$$

Definition 2.8 [1]. A BCI-algebra X is called quasi right alternate if

$$x * (y * y) = (x * y) * y, \quad \forall x, y \in X.$$

Definition 2.8. Let $\mu \in FI(X)$. We define the relation \sim_μ on X as follows:

A. Hasankhani and M.M. Zahedia are with Dept. of Math., Shahid Bahonar University of Kerman, Kerman, Iran (e-mails: abhasan@mail.uk.ac.ir, Zahedi_mm@mail.uk.ac.ir).

H. saadatb is with Islamic Azad University Science and Research Computes Kerman, Kerman, Iran (e-mail: Saadat@iauk.ac.ir).

$x \sim_{\mu} y$ if and only if $\min\{\mu(x * y), \mu(y * x)\} > 0$.

Proposition 2.9. The relation \sim_{μ} is an equivalence relation on X .

Notations. Let X be a non-empty set and U, V be subsets of $X \times X$. We let

- i) $U \circ V = \{(x, y) \in X \times X \mid \exists z \in X \text{ such that } (x, z) \in V \text{ and } (z, y) \in U\}$;
- ii) $U^{-1} = \{(x, y) \in X \times X \mid (y, x) \in U\}$;
- iii) $\Delta = \{(x, x) \in X \times X \mid x \in X\}$.

Definition 2.10 [5]. By a uniformity on X we shall mean a non-empty collection K of subsets of $X \times X$ which satisfies the following conditions:

(U₁) $\Delta \subseteq U$, for any $U \in K$;

(U₂) If $U \in K$, then $U^{-1} \in K$;

(U₃) If $U \in K$, then there exist a $V \in K$, such that $V \circ V \subseteq U$;

(U₄) If $U, V \in K$, then $U \cap V \in K$;

(U₅) If $U \in K$, and $U \subseteq V \subseteq X \times X$, then $V \in K$.

Theorem 2.11. Let $\mu \in FI(X)$ and

$$U_{\mu} = \{(x, y) \in X \times X \mid x \sim_{\mu} y\}.$$

If

$$K^* = \{U_{\mu} \mid \mu \in FI(X)\},$$

then K^* satisfies the conditions (U₁)-(U₄).

Theorem 2.12. Let $K = \{U \subseteq X \times X \mid U_{\mu} \subseteq U, \text{ for some } \mu \in FI(X)\}$.

Then K satisfies a uniformity on X and the pair (X, K) is a uniform structure.

Notation. Let $x \in X$, and $U \in K$, we define

$$U[x] := \{y \in X \mid (x, y) \in U\}.$$

Theorem 2.13. Let

$\mathbf{u} = \{G \subseteq X \mid \forall x \in G, \exists U \in K, U[x] \subseteq G\}$. The \mathbf{u} is a topology on X .

Remark 2.14. Note that for any x in X , $U[x]$ is an open neighborhood of x .

Definition 2.15. Let (X, K) be a uniform space. Then the topology \mathbf{u} is called the uniform topology on X induced by K .

III. MAIN RESULTS

Proposition 3.1. Every ideal I of X is a clopen set in (X, \mathbf{u}) .

Proof. Let I be an ideal of X . To prove that I is closed, we shall show that $I^c = \bigcup_{x \notin I} U_{\chi_I}[x]$. Indeed, assume $y \in I^c$, then from $y \in U_{\chi_I}[y]$ it follows that $y \in \bigcup_{x \notin I} U_{\chi_I}[x]$. Hence

$$I^c \subseteq \bigcup_{x \notin I} U_{\chi_I}[x]. \tag{1}$$

Conversely, let $y \in \bigcup_{x \notin I} U_{\chi_I}[x]$. Then there is $z \in I^c$ such that $y \in U_{\chi_I}[z]$. Hence $y * z$ and $z * y \in I$. Now we show that $y \notin I$. On the contrary, let $y \in I$. Then from $z * y \in I$, we get that $z \in I$, which is contradiction. Therefore

$$\bigcup_{x \notin I} U_{\chi_I}[x] \subseteq I^c \tag{2}$$

consequently from (1) and (2) we obtain that I is closed. To prove that I is open we show that

$$I = \bigcup_{x \in I} U_{\chi_I}[x]. \tag{3}$$

Clearly $y \in U_{\chi_I}[y], \forall y \in X$. Hence, $I \subseteq \bigcup_{x \in I} U_{\chi_I}[x]$.

On the other hand, let $y \in \bigcup_{x \in I} U_{\chi_I}[x]$, then there is $z \in I$ such that $y \in U_{\chi_I}[z]$. Thus $y * z \in I$ and $z * y \in I$. Now by BCI-2 we have

$$(y * (y * z)) * z = 0 \in I.$$

Since $z \in I$ and $z * y \in I$ we get that $y \in I$. Thus

$$\bigcup_{x \in I} U_{\chi_I}[x] \subseteq I.$$

Therefore (3) holds, and hence I is open.

Theorem 3.2. Each $U_{\mu}[x]$ is a clopen set for all $\mu \in FI(X)$.

Proof. Let $\mu \in FI(X), x \in X$. We want to show that $U_{\mu}[x]$ is a closed subset of X . Let $y \in (U_{\mu}[x])^c$. We claim that for the given element y we have

$$U_{\mu}[y] \subseteq (U_{\mu}[x])^c. \tag{4}$$

Let $z \in U_{\mu}[y]$, then $\mu(z * y) > 0$ and $\mu(y * z) > 0$. If $z \in U_{\mu}[x]$, then $\mu(x * z) > 0$ and $\mu(z * x) > 0$. By Lemma 2.5 (iii), $\mu(x * y) > 0$ and $\mu(y * x) > 0$. It follows that $y \in U_{\mu}[x]$, which is a contradiction. Hence

$z \in (U_\mu[x])^c$, and (4) holds. Therefore $(U_\mu[x])^c$ is open, that is $U_\mu[x]$ is closed.

Theorem 3.3 [1]. In a quasi right alternate BCI-algebra, fuzzy ideals and fuzzy subalgebra coincide.

Corollary 3.4. Let X be a quasi right alternate BCI-algebra, then

- i) Every subalgebra of X is clopen set in (X, u) .
- ii) If μ is a fuzzy subalgebra of X , then $U_\mu[x]$ is a

clopen set in (X, u) .

Proof. The proof follows from Theorems 3.12, 3.13 and Proposition 3.11.

Proposition 3.1. K is a discrete topology.

Proof. Let x be an arbitrary element of X . Then

$$\begin{aligned} \{x\} &= \{y \in X \mid y = x\} \\ &= \{y \in X \mid x * y = 0, y * x = 0\} \\ &= \{y \in X \mid \chi_{\{0\}}(x * y) > 0, \chi_{\{0\}}(y * x) > 0\} \\ &= U_{\chi_{\{0\}}}[x]. \end{aligned}$$

Now, the proof follows from Theorem 3.2.

Remark. Clearly $(X \times X, \otimes, (0,0))$ is a BCK-algebra, where

$$\begin{aligned} \otimes : (X \times X) \times (X \times X) &\rightarrow X \times X \\ ((x, y), (x', y')) &\mapsto (x * x', y * y'). \end{aligned}$$

Now, by $u_{X \times X}$ and u_X we mean the uniform Topology on $X \times X$ and X respectively.

Theorem 3.6. Let X be a medial BCI-algebra. Then the operation $*$: $X \times X \rightarrow X$ is continuous.

Proof. Let $f : X \times X \rightarrow X$ be defined by

$$\begin{aligned} f(x, y) &= x * y, \quad \forall x, y \in X, G \in u_X \text{ and} \\ (x, y) &\in f^{-1}(G). \end{aligned}$$

Then there is $U \in K_X$ such that $U[x * y] \subseteq G$. Hence

$U_\mu \subseteq U$, for some $\mu \in FI(X)$. Now we define fuzzy subset η of $X \times X$ by

$$\eta(x, y) = \mu(x * y).$$

we show that $\eta \in FI(X \times X)$.

$\eta(0,0) = \mu(0 * 0) = \mu(0) \geq \mu(x * y) = \eta(x, y)$, for all $x, y \in X$. On the other hand

$$\begin{aligned} \min\{\eta((x, y) * (z, u)), \eta(z, u)\} &= \\ \min\{\mu((x * z) * (y * u)), \mu(z * u)\} &= \\ \min\{\mu((x * y) * (z, u)), \mu(z, u)\} &= \\ \leq \mu(x * y) &= \\ = \eta(x, y), \quad \forall x, y, z, u \in X. \end{aligned}$$

Therefore $\eta \in FI(X \times X)$. Now consider U_η in $K_{X \times X}^*$.

We show that $U_\eta[(x, y)] \subseteq f^{-1}(G)$. Let

$$(z, u) \in U_\eta[(x, y)],$$

then

$$\min\{\eta((x, y) \otimes (z, u)), \eta((z, u) \otimes (x, y))\} > 0.$$

So,

$$\min\{\eta(x * z, y * u), \eta(z * x, u * y)\} > 0.$$

In other words,

$$\min\{\mu((x * z) * (y * u)), \mu((z * x) * (u * y))\} > 0.$$

Hence

$$\mu((x * y) * (z * u)) > 0$$

and

$$\mu((z * u), (x * y)) > 0.$$

.It follows that,

$$(x * y, z * u) \in U_\mu \subseteq U \text{ and so } z * u \in U[x * y] \subseteq G.$$

It means that $(z * u) = f(z, u) \in G$ or $(z, u) \in f^{-1}(G)$.

Consequently, $f^{-1}(G) \in u_{X \times X}$.

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