Position Vector of a Partially Null Curve Derived from a Vector Differential Equation

Süha Yılmaz, Emin Özyılmaz, Melih Turgut and Şuur Nizamoğlu

Abstract—In this paper, position vector of a partially null unit speed curve with respect to standard frame of Minkowski space-time is studied. First, it is proven that position vector of every partially null unit speed curve satisfies a vector differential equation of fourth order. In terms of solution of the differential equation, position vector of a partially null unit speed curve is expressed.

Keywords—Frenet Equations, Partially Null Curves, Minkowski Space-time, Vector Differential Equation.

I. INTRODUCTION

THE pseudo null and the partially null curves, lying fully in the Minkowski space–time are defined in [1] as space-like curves along which respectively the first binormal is null vector and the second binormal is null vector (but the first binormal is not). The Frenet equations of a pseudo null and a partially null curve, lying fully in Minkowski space-time are given in [8]. In [2], using those Frenet equations authors give some characterizations. Another work, in [7], authors define Frenet equations of such curves and study some of characterizations in Semi-Euclidean space.

Recently, a method has been developed by B. Y. Chen to classify curves with the solution of differential equations with constant coefficients. This method generally uses ordinary vector differential equations as well as Frenet equations. By this way, curves of a finite Chen type and some of classifications are given by the researchers in Euclidean space or another spaces, see [3], [4], [6]. Furthermore, classifications all space-like W-curves are given in [6].

In [9], author uses Frenet equations to obtain position vector of a space-like curve according to standard frame of Minkowski space E_1^3. He constructs a vector differential equation of fourth order and solves it within a special case, by this way, he expresses position vectors of all space-like W-curves in the space E_1^3.

Süha Yılmaz, Melih Turgut and Şuur Nizamoğlu are with Department of Mathematics, Buca Educational Faculty, Dokuz Eylül University, 35160, Buca-Izmir, Turkey. (Corresponding author, +902324208593, E-mail: melih.turgut@gmail.com)

Emin Özyılmaz is with Department of Mathematics, Faculty of Science, Ege University, Bornova-Izmir, Turkey. (E-mail: eminozyilmaz@hotmail.com)

In this work, in an analogous way as in the existing literature, vector differential equations are used. First, we prove that every partially null unit speed curve satisfies a vector differential equation of fourth order. Then, we investigate solution of it (in parametric form) by the method of variation of parameters. Position vector of an arbitrary partially null unit speed curve in Minkowksi space-time is obtained.

II. PRELIMINARIES

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E_1^4 are briefly presented (A more complete elementary treatment can be found in [5].)

Minkowski space-time E_1^4 is an Euclidean space E_1^4 provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2, \qquad (1)$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in E_1^4. Since g is an indefinite metric, recall that a vector $\vec{v} \in E_1^4$ can have one of the three causal characters; it can be space-like if $g(\vec{v}, \vec{v}) \rangle 0$ or $\vec{v} = 0$, time-like if $g(\vec{v}, \vec{v}) \langle 0$ and null (light-like) if $g(\vec{v}, \vec{v}) = 0$ and $\vec{v} \neq 0$. Similarly, an arbitrary curve $\vec{\alpha} = \vec{\alpha}(s)$ in E_1^4 can be locally be space-like, time-like or null (light-like), if all of its velocity vectors $\vec{\alpha}'(s)$ are respectively space-like, time-like or null. Also, recall the norm of a vector $\vec{v}$ is given by $\|\vec{v}\| = \sqrt{|g(\vec{v}, \vec{v})|}$. Therefore, $\vec{v}$ is a unit vector if $g(\vec{v}, \vec{v}) = \pm 1$. Next, vectors $\vec{v}, \vec{w}$ in E_1^4 are said to be orthogonal if $g(v, w) = 0$. The velocity of the curve $\vec{\alpha}$ is given by $\|\vec{\alpha}'\|$. Thus, a space-like or a time-like curve $\vec{\alpha}$ is said to be parametrized by arclength function s, if $g(\vec{\alpha}', \vec{\alpha}') = \pm 1$. The Lorentzian hypersphere of center $\vec{m} = (m_1, m_2, m_3, m_4)$ and radius $r \in R^+$ in the space E_1^4 defined by

$$S_1^3 = \left\{ \vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in E_1^4 : g(\vec{\alpha} - \vec{m}, \vec{\alpha} - \vec{m}) = r^2 \right\} \quad (2)$$

Denote by $\left\{ \vec{T}(s), \vec{N}(s), \vec{B}_1(s), \vec{B}_2(s) \right\}$ the moving Frenet frame along the curve $\vec{\alpha}$ in the space E_1^4. Then $\vec{T}, \vec{N}, \vec{B}_1, \vec{B}_2$ are, respectively, the tangent, the principal normal, the first

binormal and the second binormal vector fields. Recall that a space-like curve with time-like principal normal $\vec{N}$ and null first and second binormal is called a partially null curve in E_1^4 [1]. For a partially null unit speed curve $\vec{\alpha}$ in E_1^4 the following Frenet equations are given in [2], [8]

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}_1' \\ \vec{B}_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & -\tau & 0 & \sigma \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B}_1 \\ \vec{B}_2 \end{bmatrix}, \quad (3)$$

where $\vec{T}, \vec{N}, \vec{B}_1$ and $\vec{B}_2$ are mutually orthogonal vectors satisfying equations

$$g(\vec{T}, \vec{T}) = g(\vec{N}, \vec{N}) = g(\vec{B}_1, \vec{B}_2) = 1$$
$$g(\vec{B}_1, \vec{B}_1) = g(\vec{B}_2, \vec{B}_2) = 0.$$

And here, $\kappa(s), \tau(s)$ and $\sigma(s)$ are first, second and third curvature of the curve $\vec{\alpha}$, respectively.

In the same space, the authors, in [2], expressed a characterization of partially null curves with the following theorem.

Theorem 1. A partially null unit speed curve $\vec{\varphi} = \vec{\varphi}(s)$ in E_1^4, with curvatures $\kappa \neq 0, \tau \neq 0$ for each $s \in I \subset R$ has $\sigma = 0$ for each s.

III. VECTOR DIFFERENTIAL EQUATION OF FOURTH ORDER CHARACTERIZES PARTIALLY NULL CURVES

Theorem 2. Position vector of a partially null unit speed curve with $\kappa \neq 0, \tau \neq 0$ in E_1^4 satisfies a vector differential equation of fourth order.

Proof. Let $\vec{\varphi} = \vec{\varphi}(s)$ be a partially null unit speed curve in $\kappa \neq 0, \tau \neq 0$ in E_1^4. Then, the system of ordinary differential equations (3) hold. By $(3)_1$, we may write

$$\vec{N} = \frac{1}{\kappa} \frac{d\vec{T}}{ds}. \quad (4)$$

Using $(3)_2$, we easily form

$$\vec{B}_1 = \frac{\kappa}{\tau} \vec{T} + \frac{1}{\tau} \frac{d}{ds}\left(\frac{1}{\kappa} \frac{d\vec{T}}{ds} \right). \quad (5)$$

The differential equation $(3)_3$ gives us

$$\frac{d}{ds}\left(\frac{\kappa}{\tau} \vec{T} \right) + \frac{d}{ds}\left[\frac{1}{\tau} \frac{d}{ds}\left(\frac{1}{\kappa} \frac{d\vec{T}}{ds} \right) \right] = \vec{0}. \quad (6)$$

Denoting $\frac{d\vec{\varphi}}{ds} = \vec{T}$, we express

$$\frac{d}{ds}\left(\frac{\kappa}{\tau} \frac{d\vec{\varphi}}{ds} \right) + \frac{d}{ds}\left[\frac{1}{\tau} \frac{d}{ds}\left(\frac{1}{\kappa} \frac{d^2\vec{\varphi}}{ds^2} \right) \right] = \vec{0}, \quad (7)$$

which completes the proof.

IV. SOLUTION OF THE VECTOR DIFFERENTIAL EQUATION (6)

In this section, we shall obtain position vector of a partially null curve in terms of solution of the vector differential equation (6).

One can easily see that the first binormal vector is a constant vector. Since, we denote it by $\vec{B}_1 = \vec{C} = (c_1, c_2, c_3, c_4)$. Rewriting (5), we have

$$\frac{\kappa}{\tau} \vec{T} + \frac{1}{\tau} \frac{d}{ds}\left(\frac{1}{\kappa} \frac{d\vec{T}}{ds} \right) = \vec{C} \quad (7)$$

or in parametric form

$$\left\{ \frac{\kappa}{\tau} t_i + \frac{1}{\tau} \frac{d}{ds}\left(\frac{1}{\kappa} \frac{dt_i}{ds} \right) = c_i \right. \quad (8)$$

where $\vec{T} = (t_1, t_2, t_3, t_4)$ is the tangent vector of the partially null curve $\vec{\varphi} = \vec{\varphi}(s)$, according to standard frame of E_1^4, for $1 \leq i \leq 4$. Using an exchange variable $\phi = \int_0^s \kappa \, ds$ in (8), we immediately have

$$\left\{ \frac{d^2 t_i}{d\phi^2} + \phi = \frac{\tau}{\kappa} c_i \right. \quad (9)$$

which is a system of non-homogenous ordinary differential equations. The general solution of the system can be expressed by the method of variation of parameters. We may write parts of our solution as follow:

$$\left\{ \begin{array}{l} t_{ic} = \delta_i \cos\phi + \delta_{i+4} \sin\phi \\ t_p = u(\phi)\cos\phi + v(\phi)\sin\phi, \end{array} \right. \quad (10)$$

for the real numbers δ_i and real valued functions $u(\phi), v(\phi)$. Thus, we express

$$\left\{ \begin{array}{l} -\dfrac{du}{d\phi}\sin\phi + \dfrac{dv}{d\phi}\cos\phi = \dfrac{\tau}{\kappa} c_i \\ \dfrac{du}{d\phi}\cos\phi + \dfrac{dv}{d\phi}\sin\phi = 0 \end{array} \right. . \quad (11)$$

By the Cramer method, one can find

$$\left\{ \begin{array}{l} \dfrac{du}{d\phi} = -(\sin\phi)\dfrac{\tau}{\kappa} c_i \\ \dfrac{dv}{d\phi} = \dfrac{\tau}{\kappa} c_i (\cos\phi). \end{array} \right. \quad (12)$$

Solving the system (12) and rewriting the exchange variable, we have

$$\begin{cases} u = -c_i \int\limits_0^s \left[\tau \sin \int\limits_0^s \kappa \, ds \right] ds \\ v = c_i \int\limits_0^s \left[\tau \cos \int\limits_0^s \kappa \, ds \right] ds \end{cases}. \qquad (13)$$

We have the solution

$$t_i = \left\{ \begin{array}{l} \cos \int\limits_0^s \kappa \, ds \left[\delta_i - c_i \int\limits_0^s \left(\tau \sin \int\limits_0^s \kappa \, ds \right) ds \right] \\ + \sin \int\limits_0^s \kappa \, ds \left[\delta_{i+4} - c_i \int\limits_0^s \left(\tau \cos \int\limits_0^s \kappa \, ds \right) ds \right] \end{array} \right\}. \qquad (14)$$

Since, in parametric form, position vector of a partially null curve may be expressed as

$$\varphi_i = \int\limits_0^s \left\{ \begin{array}{l} \cos \int\limits_0^s \kappa \, ds \left[\delta_i - c_i \int\limits_0^s \left(\tau \sin \int\limits_0^s \kappa \, ds \right) ds \right] \\ + \sin \int\limits_0^s \kappa \, ds \left[\delta_{i+4} - c_i \int\limits_0^s \left(\tau \cos \int\limits_0^s \kappa \, ds \right) ds \right] \end{array} \right\} ds + L_i, \qquad (15)$$

where $\vec{\varphi} = \vec{\varphi}(s) = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ is the position vector of $\vec{\varphi}$ according to standard frame of E_1^4 for the real numbers L_i.

Theorem 3. Position vector of a partially null unit speed curve with respect to standard frame of E_1^4 can be composed by above components.

We finally give a characterization by means of constant vector (5).

We have expressed that $\vec{B}_1 = \vec{C} = (c_1, c_2, c_3, c_4)$. Moreover, we know that the first binormal vector of a partially null curve is a null curve and the constant vector $\vec{C}$ is composed with respect to standard frame. Since, by the equality $g(\vec{C}, \vec{C}) = 0$, we may write

$$c_1^2 = c_2^2 + c_3^2 + c_4^2. \qquad (16)$$

Considering (8), it is safe to report that there is a relation between components of the tangent vector of the partially null curve as follows:

$$\sum_{j=2}^4 \left\{ \kappa t_j^2 + \frac{d}{ds}\left(\frac{1}{\kappa} \frac{dt_j}{ds} \right) \left[2t_j + \frac{d}{ds}\left(\frac{1}{\kappa} \frac{dt_j}{ds} \right) \right] \right\}$$
$$- \left\{ \kappa t_1^2 + \frac{d}{ds}\left(\frac{1}{\kappa} \frac{dt_1}{ds} \right) \left[2t_1 + \frac{d}{ds}\left(\frac{1}{\kappa} \frac{dt_1}{ds} \right) \right] \right\} = 0. \qquad (17)$$

REFERENCES

[1] W.B. Bonnor, "Null curves in a Minkowski space-time", *Tensor.* Vol. 20, pp. 229-242, 1969.
[2] C. Camci, K. Ilarslan and E. Sucurovic, "On pseudohyperbolical curves in Minkowski space-time". *Turk J.Math.* vol. 27, pp. 315-328, 2003.
[3] B.Y. Chen, "A report on submanifolds of finite type", *Soochow J. Math.* vol. 22 pp. 1–128, 1996.
[4] B.Y. Chen, F. Dillen and L. Verstraelen, "Finite type space curves", *Soochow J. Math.* vol. 12 pp. 1–10, 1986.
[5] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.
[6] M. Petrovic-Torgasev and E. Sucurovic, "W-Curves in Minkowski Space-time". *Novi Sad J. Math.* vol. 32, no. 2 pp. 55-65, 2002.
[7] M. Petrovic-Torgasev, K.Ilarslan and E. Nesovic, "On partially null and pseudo null curves in the semi-euclidean space R_2^4". *J. of Geometry.* vol. 84, pp. 106-116, 2005.
[8] J. Walrave, "Curves and surfaces in Minkowski space" Ph.D. dissertation, K. U. Leuven, Fac. of Science, Leuven 1995.
[9] S. Yılmaz, "Determination of Space-like curves by Vector Differential Equations in Minkowski space E_1^3". *J. Avd. Res.Pure Math.* vol. 1 no. 1 pp. 10-14, 2009.