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Univalence of an integral operator defined by generalized operators

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Abstract—In this paper we define generalized differential operators from some well-known operators on the class A of analytic functions in the unit disk $U = \{z \in C : |z| < 1\}$. New classes containing these operators are investigated. Also univalence of integral operator is considered.

Keywords—Univalent functions, Integral operators, Differential operators.

I. Introduction

Let H be the class of analytic functions in $U = \{z \in C : |z| < 1\}$ and $H[a, n]$ be the subclass of H consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$
Let A be the subclass of H consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

Let $\hat{C}_\theta(\alpha)$ denote the class of functions $f \in A$ satisfying the following

$$\Re\left\{e^{i\theta}\left(\frac{zf''(z)}{f'(z)} + 1\right)\right\} > \alpha\cos\theta$$

for

$$\left(0 \le \alpha < 1\,,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}\,,\ z \in U\right).$$

If $\theta = 0$, the class $\hat{C}_\theta(\alpha) = \hat{C}(\alpha)$ is the well-known convex functions of order α. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in A$ then the Hadamard product or convolution of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U.$$

And for several functions $f_1(z), ..., f_m(z) \in A$

$$f_1(z) * .... * f_m(z) = z + \sum_{n=2}^{\infty} (a_{1n}.....a_{mn}) z^n, \quad z \in U$$

Our aim is to use the Hadamard product of K-th order to define generalized differential operators. For $f \in A$ of the form (1) first we define the following generalized differential operator

$$D^0 f(z) = f(z)$$

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$$D^1_{\alpha,\beta,\lambda,\delta} f(z) = [1 - (\lambda-\delta)(\beta-\alpha)]\, f(z) + (\lambda-\delta)(\beta-\alpha)\, zf'(z)$$
$$= z + \sum_{n=2}^{\infty} [(\lambda-\delta)(\beta-\alpha)(n-1) + 1]\, a_n z^n$$
$$\vdots$$
$$D^k_{\alpha,\beta,\lambda,\delta} f(z) = D^1_{\alpha,\beta,\lambda,\delta}\left(D^{k-1}_{\alpha,\beta,\lambda,\delta} f(z)\right)$$

$$D^k_{\alpha,\beta,\lambda,\delta} f(z) = z + \sum_{n=2}^{\infty} [(\lambda-\delta)(\beta-\alpha)(n-1) + 1]^k a_n z^n, \tag{2}$$

for $\alpha \ge 0, \beta \ge 0, \lambda \ge 0, \delta \ge 0, \lambda > \delta, \beta > \alpha$ and $k \in \{0, 1, 2, ...\}$.

Remark : (i) When $\alpha = 0$, $\delta = 0$, $\lambda = 1$, $\beta = 1$ we get Salagean differential operator (see [5]).

(ii) When $\alpha = 0$ we get M. Darus and R. Ibrahim differential operator (see [3]).

(iii) And when $\alpha = 0, \delta = 0, \beta = 1$ we get Al- Oboudi differential operator(see [1]).

Definition 1.1: A function $f \in A$ in the class $\hat{C}^k_\theta(\alpha)$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $0 \le \alpha < 1$, if it satisfies the following inequality:

$$\Re\left\{e^{i\theta}\left(\frac{z\left[D^k_{\alpha,\beta,\lambda,\delta} f(z)\right]''}{\left[D^k_{\alpha,\beta,\lambda,\delta} f(z)\right]'} + 1\right)\right\} > \alpha\cos\theta \tag{3}$$

Definition 1.2: For $m \in N \cup \{0\}$, $j \in \{1, 2, 3, ..., m\}$, $s_j \in C$ we introduce the integral operator

$$F_{s_1...s_m}(z) = \int_0^z [tf_1'(t)]^{s_1} .... [tf_m'(t)]^{s_m} dt, \quad (t > 0,\ s_j > 0) \tag{4}$$

when $t = 1$, the operator $F_{s_1...s_m}(z)$ reduced to an integral operator

$$F_s(z) = \int_0^z [f_1'(t)]^{s_1} .... [f_m'(t)]^{s_m} dt \quad (s_j > 0)$$

recently introduced and studied by D.Breaz, S.Owa, and N.Breaz in [2]. See also similar work given by [4] and [6].

In this paper, we consider the following integral operator which involving the generalized operator (2) and study its properties on the class $\hat{C}_\theta(\alpha)$.

Definition 1.3: for $m \in \{0,1,2,...\}$, $s_j > 0$, $f \in A$ we define an integral operator $F^k_{s_1...s_m}(z)$ as the following:

$$F^k_{s_1...s_m}(z) = \int_0^z \left[t\left(D^k_{\alpha,\beta,\lambda,\delta}f_1(t)\right)'\right]^{s_1} .... \left[t\left(D^k_{\alpha,\beta,\lambda,\delta}f_m(t)\right)'\right]^{s_m} dt, \tag{5}$$

for

$$(t > 0,\ t \in U).$$

Lemma 1.4: If $f \in A$ satisfies the inequality

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \quad for\ \ all\ \ z \in U$$

then the function f is univalent in U.

II. Main results

We begin with the following theorem:

Theorem 2.1: Let $f_j \in A,\ s_j \in C,\ j \in \{1,2,...,m\}$. If

$$\left|\frac{z\left[D^k_{\alpha,\beta,\lambda,\delta}f_j(z)\right]''}{\left[D^k_{\alpha,\beta,\lambda,\delta}f_j(z)\right]'} + 1\right| \le 1,$$

$|s_1| + |s_2| + ... + |s_m| \le 1,\ z \in U$, then $F^k_{s_1...s_m}(z)$ given by (5) is univalent.

Proof: From (5) we obtain

$$\left[F^k_{s_1...s_m}(z)\right]' = \left[z\left(D^k_{\alpha,\beta,\lambda,\delta}f_1(z)\right)'\right]^{s_1} .... \left[z\left(D^k_{\alpha,\beta,\lambda,\delta}f_m(z)\right)'\right]^{s_m}$$

for $z > 0,\ z \in U$ which implies that

$$\ln\left[F^k_{s_1...s_m}(z)\right]' = s_1 \ln\left[z\left(D^k_{\alpha,\beta,\lambda,\delta}f_1(z)\right)'\right] + ... + s_m \ln\left[z\left(D^k_{\alpha,\beta,\lambda,\delta}f_m(z)\right)'\right]$$

and taking the derivative for the above equality, we have

$$\frac{\left[F^k_{s_1...s_m}(z)\right]''}{\left[F^k_{s_1...s_m}(z)\right]'} = s_1\left[\frac{\left(D^k_{\alpha,\beta,\lambda,\delta}f_1(z)\right)''}{\left(D^k_{\alpha,\beta,\lambda,\delta}f_1(z)\right)'} + \frac{1}{z}\right] + ... + s_m\left[\frac{\left(D^k_{\alpha,\beta,\lambda,\delta}f_m(z)\right)''}{\left(D^k_{\alpha,\beta,\lambda,\delta}f_m(z)\right)'} + \frac{1}{z}\right]. \tag{6}$$

By multiplying the relation (6) with z we obtain

$$\frac{z\left[F^k_{s_1...s_m}(z)\right]''}{\left[F^k_{s_1...s_m}(z)\right]'} = s_1\left[\frac{z\left(D^k_{\alpha,\beta,\lambda,\delta}f_1(z)\right)''}{\left(D^k_{\alpha,\beta,\lambda,\delta}f_1(z)\right)'} + 1\right] + ... + s_m\left[\frac{z\left(D^k_{\alpha,\beta,\lambda,\delta}f_m(z)\right)''}{\left(D^k_{\alpha,\beta,\lambda,\delta}f_m(z)\right)'} + 1\right] \tag{7}$$

On multiplying the modulus of equation (7) by $\left(1-|z|^2\right)$, we obtain

$$\begin{aligned}
&\left(1-|z|^2\right)\left|\frac{z\left[F^k_{s_1...s_m}(z)\right]''}{\left[F^k_{s_1...s_m}(z)\right]'}\right| \\
&\le \left(1-|z|^2\right)\left[|s_1|\left|\frac{z\left(D^k_{\alpha,\beta,\lambda,\delta}f_1(z)\right)''}{\left(D^k_{\alpha,\beta,\lambda,\delta}f_1(z)\right)'} + 1\right| + ... + |s_m|\left|\frac{z\left(D^k_{\alpha,\beta,\lambda,\delta}f_m(z)\right)''}{\left(D^k_{\alpha,\beta,\lambda,\delta}f_m(z)\right)'} + 1\right|\right] \\
&\le \left(1-|z|^2\right)\left[|s_1| + |s_2| + ... |s_m|\right] \\
&\le |s_1| + |s_2| + ... + |s_m| \le 1
\end{aligned}$$

From Lemma (4), we have that $F^k_{s_1...s_m}(z)$ is univalent.

Taking $k = 0$ in Theorem 2.1 ,we have

Corollary 2.2: If $f_j \in A,\ s_j \in C,\ j = \{1,2,...,m\}$. If

$$\left|\frac{zf''(z)}{f'(z)} + 1\right| \le 1,\ |s_1| + |s_2| + ... + |s_m| \le 1,\ z \in U$$

then $F_{s_1...s_m}(z)$ given by (4) is univalent.

Theorem 2.3: Let $s_1, s_2, ..., s_m$ be real number with the properties $s_j > 0$ for $j \in \{1,2,....,m\}$ and

$$0 \le \sum_{j=1}^m s_j\alpha_j + 1 < 1$$

then the integral operator $F^k_{s_1...s_m}(z) \in \hat{C}^k_\theta(\gamma)$ where $\gamma = \sum_{j=1}^m s_j\alpha_j + 1,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Proof: Using (7), we obtain

$$\frac{z\left[F^k_{s_1...s_m}(z)\right]''}{\left[F^k_{s_1...s_m}(z)\right]'} = \sum_{j=1}^m s_j\left[\frac{z\left(D^k_{\alpha,\beta,\lambda,\delta}f_j(z)\right)''}{\left(D^k_{\alpha,\beta,\lambda,\delta}f_j(z)\right)'} + 1\right] \tag{8}$$

the relation (8) is the equivalent to

$$\frac{z\left[F^k_{s_1...s_m}(z)\right]''}{\left[F^k_{s_1...s_m}(z)\right]'} + 1 = \sum_{j=1}^m s_j\left[\frac{z\left(D^k_{\alpha,\beta,\lambda,\delta}f_j(z)\right)''}{\left(D^k_{\alpha,\beta,\lambda,\delta}f_j(z)\right)'} + 1\right] + 1 \tag{9}$$

by multiplying the relation (9) by $e^{i\theta}$ we get

$$\Re\left\{e^{i\theta}\left(\frac{z\left[F^k_{s_1...s_m}(z)\right]''}{\left[F^k_{s_1...s_m}(z)\right]'} + 1\right)\right\} \tag{10}$$

$$= \sum_{j=1}^m s_j \Re\left\{e^{i\theta}\left(\frac{z\left(D^k_{\alpha,\beta,\lambda,\delta}f_j(z)\right)''}{\left(D^k_{\alpha,\beta,\lambda,\delta}f_j(z)\right)'} + 1\right)\right\} + \Re e^{i\theta}$$

since each $f_j \in \hat{C}_\theta(\alpha_j)$ for $j \in \{1,2,3,...,m\}$ by using (3) in (10) we have

$$\Re\left\{e^{i\theta}\left(\frac{z\left[F^k_{s_1\ldots s_m}(z)\right]''}{\left[F^k_{s_1\ldots s_m}(z)\right]'}+1\right)\right\}>\cos\theta\left(\sum_{j=1}^{m}s_j\alpha_j+1\right),$$
$-\frac{\pi}{2}<\theta<\frac{\pi}{2}$.

Since by hypothesis $0\le\sum_{j=1}^{m}s_j\alpha_j+1<1$, we obtain $F^k_{s_1\ldots s_m}(z)\in\hat{C}^k_\theta(\gamma)$ where $\gamma=\sum_{j=1}^{m}s_j\alpha_j+1,\ -\frac{\pi}{2}<\theta<\frac{\pi}{2}$.

By taking $k=0$ in Theorem 2.3 we have

Corollary 2.4: Let $s_1,s_2,...,s_m$ be real number with the properties $s_j>0$ for $j\in\{1,2,3,...,m\}$ and

$$0\le\sum_{j=1}^{m}s_j\alpha_j+1<1$$

If $f_j(z)\in\hat{C}_\theta(\alpha_j)$ for $j\in\{1,2,....,m\}$, then $F_{s_1\ldots s_m}(z)$ given by (4) belongs to $\hat{C}_\theta(\gamma)$ where $\gamma=\sum_{j=1}^{m}s_j\alpha_j+1,\ -\frac{\pi}{2}<\theta<\frac{\pi}{2}$.

III. Conclusion

Univalence condition in the area of studies is very important. The class of functions introduced needed to be varified its univalency. The criteria is indeed proven. The operator given can also be extended further and can generate more new results.

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