

Splitting Modified Donor-Cell Schemes for Spectral Action Balance Equation

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Abstract—The spectral action balance equation is an equation that used to simulate short-crested wind-generated waves in shallow water areas such as coastal regions and inland waters. This equation consists of two spatial dimensions, wave direction, and wave frequency which can be solved by finite difference method. When this equation with dominating propagation velocity terms are discretized using central differences, stability problems occur when the grid spacing is chosen too coarse. In this paper, we introduce the splitting modified donor-cell scheme for avoiding stability problems and prove that it is consistent to the modified donor-cell scheme with same accuracy. The splitting modified donor-cell scheme was adopted to split the wave spectral action balance equation into four one-dimensional problems, which for each small problem obtains the independently tridiagonal linear systems. For each smaller system can be solved by direct or iterative methods at the same time which is very fast when performed by a multi-cores computer.

Keywords—donor-cell scheme, parallel algorithm, spectral action balance equation, splitting method.

I. INTRODUCTION

A third-generation model is a number of advanced spectral wind-wave models. It has been developed such as WAM model of WAMDI Group [9], in which all processes of wave generation, dissipation and nonlinear wave-wave interactions are accounted for explicitly. WAM model considers problems on oceanic scales, and make used of explicit propagation schemes in geographical and spectral spaces. Tolman [8] developed model base on spectral action balance equation, WAVEWATCH model incorporates all relevant wave-current interaction mechanism, including changes of absolute frequencies due to unsteadiness of depth and currents. The model explicitly accounts for growth and decay of wave energy and for nonlinear resonant wave-wave interactions. Booij [1] and Ris [7] et. al. summarized the research attainment in the wave energy, dissipation and nonlinear wave-wave interactions, and developed the third generation for coastal region in shallow water, SWAN(Simulating WAVE Nearshore) model, which can be applied in coastal zones, lake and estuaries. The model uses the spectral action balance equation to represents the process of wave shoaling, refraction, bottom friction, depth-induced wave breaking, whitcapping, wind input and nonlinear wave-wave interactions reasonably.

When the spectral action balance equation with dominating propagation velocity terms are discretized using central differences, stability problems occur when the grid spacing is

chosen too coarse. Maintaining stability for strongly propagation velocity problem would thus restrict the grid spacing to be very small, leading to excessively large system of equations, particularly in four dimensions. The possibility for avoiding stability problems is discretizing with the modified donor-cell scheme, although it also suffers from a lower order of approximation. The discretization yield a large 9-band linear system that can be solved by direct or iterative methods.

In recent years computers' evolution is going dramatically fast. Computers have been improved a lot and become much more powerful. One of the new types of computers is a multi-processing computer. So, we should develop algorithms that support and could be suitable for this evolution. In this paper we introduce the splitting donor-cell scheme for solving spectral action balance equation. This scheme splits the original four-dimensional spaces problem into a set of one-dimensional space problems. At each fractional step one has to solve m independent one-dimensional space systems of linear equations where m is the number of unknowns in one-dimensional space problems in appropriate direction. Therefore, we can solve each of these systems of linear equations at every step by m independent parallel processors. This method is preferable for multi-cores computers.

II. THE MODIFIED DONOR-CELL SCHEME FOR THE SPECTRAL ACTION BALANCE EQUATION

We consider the spectral action balance equation which described the wave characteristic:

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x}(c_x N) + \frac{\partial}{\partial y}(c_y N) + \frac{\partial}{\partial \sigma}(c_\sigma N) + \frac{\partial}{\partial \theta}(c_\theta N) = \frac{S}{\sigma}, \quad (1)$$

$$\forall (x, y, \sigma, \theta) \in \Omega \times \Gamma, \quad t \in [0, T],$$

$$\left. \frac{\partial N}{\partial n} \right|_{\partial \Omega \times \partial \Gamma} = 0, \quad t \in [0, T],$$

$$N|_{t=0} = N_0(x, y, \sigma, \theta), \quad \forall (x, y, \sigma, \theta) \in \Omega \times \Gamma,$$

where Ω and Γ are domain in geographical and spectral \mathbb{R}^2 , $\partial \Omega$ its boundary of Ω , $\partial \Gamma$ its boundary of Γ , $N_0(x, y, \sigma, \theta)$ is an initial values, and n is a normal direction of each variable. Which $N(x, y, \sigma, \theta, t)$ is the action density as a function of relative frequency σ , direction θ , horizontal coordinates x , and y , and time t . The coefficients c_x , c_y , c_σ and c_θ are propagation velocity in x -, y -, σ - and θ - direction, respectively. The first term of the left-hand side of equation (1) represents the local rate of change of action density in time, the second and the third term represent propagation of action density in geographical space, with propagation velocities c_x and c_y in x and y space respectively. The fourth term represents shifting of relative frequency due to variations in depths and currents with

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propagation velocity c_σ in σ space. The fifth term represents depth-induced and current-induced refraction with propagation velocity c_θ in θ space. And the right hand side the term S is the source term. More details are given in Booij et al. [1] and Ris et al. [7]

We choose a rectangular grid with constant mesh sizes Δx and Δy in x - and y - direction, respectively. The spectral space is divided into elementary bins with a constant directional resolution $\Delta\theta$ and a constant relative frequency resolution $\Delta\sigma/\sigma$. We denote the grid counters as $1 \leq i \leq N_x$, $1 \leq j \leq N_y$, $1 \leq l \leq N_\sigma$ and $1 \leq m \leq N_\theta$ in x -, y -, σ - and θ - spaces, respectively. Let N be given at the grid points, but assume that c_x , c_y , c_σ and c_θ are given at the interval midpoints as shown in the Figure 1. Firstly, we consider for the central

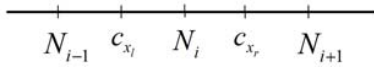


Fig. 1: Discretization using the donor-cell scheme in x -direction.

difference scheme of geographical space x at a grid point x_i

$$\left. \frac{\partial}{\partial x}(c_x N) \right|_i = \frac{(c_x N)_{xr} - (c_x N)_{xl}}{\Delta x}.$$

N_{xr} and N_{xl} are approximated by a linear interpolation, yields

$$N_{xr} = \frac{N_i + N_{i+1}}{2}, \quad N_{xl} = \frac{N_{i-1} + N_i}{2}.$$

Therefore,

$$\left. \frac{\partial}{\partial x}(c_x N) \right|_i = \frac{1}{\Delta x} \left[c_{xr} \left(\frac{N_i + N_{i+1}}{2} \right) - c_{xl} \left(\frac{N_{i-1} + N_i}{2} \right) \right]. \quad (2)$$

The donor-cell discretization of the propagation velocity c_x at a grid point x_i is given by

$$\left. \frac{\partial}{\partial x}(c_x N) \right|_i = \frac{1}{\Delta x} (c_{xr}^+ N_i + c_{xr}^- N_{i+1} - c_{xl}^+ N_{i-1} - c_{xl}^- N_i), \quad (3)$$

where

$$\begin{aligned} c_{xr}^+ &= \max\{c_{xr}, 0\}, & c_{xr}^- &= \min\{c_{xr}, 0\} \\ c_{xl}^+ &= \max\{c_{xl}, 0\}, & c_{xl}^- &= \min\{c_{xl}, 0\}. \end{aligned}$$

Because the propagation velocities in the equation (1) become dominant at high velocities, it is necessary to use a mixture of the central differences described above and the donor-cell discretization at grid point $(x_i, y_j, \sigma_l, \theta_m)$. We set

$$\begin{aligned} \left. \frac{\partial}{\partial x}(c_x N) \right|_{i,j,l,m} &= \frac{(1-\gamma)}{\Delta x} \left[(c_{xr}^+ + c_{xr}^-) \frac{N_i + N_{i+1}}{2} \right. \\ &\quad \left. - (c_{xl}^+ + c_{xl}^-) \frac{N_{i-1} + N_i}{2} \right]_{j,l,m} \\ &\quad + \frac{\gamma}{\Delta x} [c_{xr}^+ N_i + c_{xr}^- N_{i+1} \\ &\quad - c_{xl}^+ N_{i-1} - c_{xl}^- N_i]_{j,l,m}, \end{aligned}$$

$$\begin{aligned} \left. \frac{\partial}{\partial y}(c_y N) \right|_{i,j,l,m} &= \frac{(1-\gamma)}{\Delta y} \left[(c_{yr}^+ + c_{yr}^-) \frac{N_j + N_{j+1}}{2} \right. \\ &\quad \left. - (c_{yl}^+ + c_{yl}^-) \frac{N_{j-1} + N_j}{2} \right]_{i,l,m} \\ &\quad + \frac{\gamma}{\Delta y} [c_{yr}^+ N_j + c_{yr}^- N_{j+1} \\ &\quad - c_{yl}^+ N_{j-1} - c_{yl}^- N_j]_{i,l,m}, \end{aligned}$$

$$\begin{aligned} \left. \frac{\partial}{\partial \sigma}(c_\sigma N) \right|_{i,j,l,m} &= \frac{(1-\gamma)}{\Delta \sigma} \left[(c_{\sigma r}^+ + c_{\sigma r}^-) \frac{N_l + N_{l+1}}{2} \right. \\ &\quad \left. - (c_{\sigma l}^+ + c_{\sigma l}^-) \frac{N_{l-1} + N_l}{2} \right]_{i,j,m} \\ &\quad + \frac{\gamma}{\Delta \sigma} [c_{\sigma r}^+ N_l + c_{\sigma r}^- N_{l+1} \\ &\quad - c_{\sigma l}^+ N_{l-1} - c_{\sigma l}^- N_l]_{i,j,m}, \end{aligned}$$

$$\begin{aligned} \left. \frac{\partial}{\partial \theta}(c_\theta N) \right|_{i,j,l,m} &= \frac{(1-\gamma)}{\Delta \theta} \left[(c_{\theta r}^+ + c_{\theta r}^-) \frac{N_m + N_{m+1}}{2} \right. \\ &\quad \left. - (c_{\theta l}^+ + c_{\theta l}^-) \frac{N_{m-1} + N_m}{2} \right]_{i,j,l} \\ &\quad + \frac{\gamma}{\Delta \theta} [c_{\theta r}^+ N_m + c_{\theta r}^- N_{m+1} \\ &\quad - c_{\theta l}^+ N_{m-1} - c_{\theta l}^- N_m]_{i,j,l}. \end{aligned}$$

Therefore, the modified donor-cell scheme for the equation (1) is the following

$$\begin{aligned} &\frac{N^n - N^{n-1}}{\Delta t} \\ &+ \frac{(1-\gamma)}{\Delta x} \left[(c_{xr}^+ + c_{xr}^-) \frac{N_i + N_{i+1}}{2} - (c_{xl}^+ + c_{xl}^-) \frac{N_{i-1} + N_i}{2} \right]_{j,l,m}^n \\ &+ \frac{\gamma}{\Delta x} [c_{xr}^+ N_i + c_{xr}^- N_{i+1} - c_{xl}^+ N_{i-1} - c_{xl}^- N_i]_{j,l,m}^n \\ &+ \frac{(1-\gamma)}{\Delta y} \left[(c_{yr}^+ + c_{yr}^-) \frac{N_j + N_{j+1}}{2} - (c_{yl}^+ + c_{yl}^-) \frac{N_{j-1} + N_j}{2} \right]_{i,l,m}^n \\ &+ \frac{\gamma}{\Delta y} [c_{yr}^+ N_j + c_{yr}^- N_{j+1} - c_{yl}^+ N_{j-1} - c_{yl}^- N_j]_{i,l,m}^n \\ &+ \frac{(1-\gamma)}{\Delta \sigma} \left[(c_{\sigma r}^+ + c_{\sigma r}^-) \frac{N_l + N_{l+1}}{2} - (c_{\sigma l}^+ + c_{\sigma l}^-) \frac{N_{l-1} + N_l}{2} \right]_{i,j,m}^n \\ &+ \frac{\gamma}{\Delta \sigma} [c_{\sigma r}^+ N_l + c_{\sigma r}^- N_{l+1} - c_{\sigma l}^+ N_{l-1} - c_{\sigma l}^- N_l]_{i,j,m}^n \\ &+ \frac{(1-\gamma)}{\Delta \theta} \left[(c_{\theta r}^+ + c_{\theta r}^-) \frac{N_m + N_{m+1}}{2} - (c_{\theta l}^+ + c_{\theta l}^-) \frac{N_{m-1} + N_m}{2} \right]_{i,j,l}^n \\ &+ \frac{\gamma}{\Delta \theta} [c_{\theta r}^+ N_m + c_{\theta r}^- N_{m+1} - c_{\theta l}^+ N_{m-1} - c_{\theta l}^- N_m]_{i,j,l}^n \\ &= \frac{S}{\sigma} \Big|_{i,j,l,m}. \end{aligned} \quad (4)$$

The parameter γ in the above formulas lies between 0 and 1. For $\gamma = 0$ we recover the central difference discretization, and for $\gamma = 1$, a pure donor-cell scheme results. According to [5], γ should be chosen such that

$$\gamma \geq \Delta t \max \left(\left| \frac{(c_x)_{ijlm}}{\Delta x} \right|, \left| \frac{(c_y)_{ijlm}}{\Delta y} \right|, \left| \frac{(c_\sigma)_{ijlm}}{\Delta \sigma} \right|, \left| \frac{(c_\theta)_{ijlm}}{\Delta \theta} \right| \right).$$

When rearrange them, then we have the following equation

$$\begin{aligned}
 & a_{i,j,l,m} N_{i,j,l,m}^n \\
 & + \frac{\Delta t}{2\Delta x} [(1-\gamma)c_{xr}^+ + (1+\gamma)c_{xr}^-]_{i,j,l,m} N_{i+1,j,l,m}^n \\
 & - \frac{\Delta t}{2\Delta x} [(1+\gamma)c_{xl}^+ + (1-\gamma)c_{xl}^-]_{i,j,l,m} N_{i-1,j,l,m}^n \\
 & + \frac{\Delta t}{2\Delta y} [(1-\gamma)c_{yr}^+ + (1+\gamma)c_{yr}^-]_{i,j,l,m} N_{i,j+1,l,m}^n \\
 & - \frac{\Delta t}{2\Delta y} [(1+\gamma)c_{yl}^+ + (1-\gamma)c_{yl}^-]_{i,j,l,m} N_{j-1,j,l,m}^n \\
 & + \frac{\Delta t}{2\Delta \sigma} [(1-\gamma)c_{\sigma r}^+ + (1+\gamma)c_{\sigma r}^-]_{i,j,l,m} N_{i,j,l+1,m}^n \\
 & - \frac{\Delta t}{2\Delta \sigma} [(1+\gamma)c_{\sigma l}^+ + (1-\gamma)c_{\sigma l}^-]_{i,j,l,m} N_{i,j,l-1,m}^n \\
 & + \frac{\Delta t}{2\Delta \theta} [(1-\gamma)c_{\theta r}^+ + (1+\gamma)c_{\theta r}^-]_{i,j,l,m} N_{i,j,l,m+1}^n \\
 & - \frac{\Delta t}{2\Delta \theta} [(1+\gamma)c_{\theta l}^+ + (1-\gamma)c_{\theta l}^-]_{i,j,l,m} N_{i,j,l,m-1}^n \\
 & = \frac{S^{n-1}}{\sigma_l} + 4N_{i,j,l,m}^{n-1} \quad (5)
 \end{aligned}$$

where $i = 1, \dots, N_x$; $j = 1, \dots, N_y$; $l = 1, \dots, N_\sigma$; $m = 1, \dots, N_\theta$ and

$$\begin{aligned}
 a_{i,j,l,m} = 1 & + \frac{\Delta t}{2\Delta x} [c_{xr}^+ + c_{xr}^- - c_{xl}^+ - c_{xl}^- \\
 & + \gamma(c_{xr}^+ - c_{xr}^- + c_{xl}^+ - c_{xl}^-)] \\
 & + \frac{\Delta t}{2\Delta y} [c_{yr}^+ + c_{yr}^- - c_{yl}^+ - c_{yl}^- \\
 & + \gamma(c_{yr}^+ - c_{yr}^- + c_{yl}^+ - c_{yl}^-)] \\
 & + \frac{\Delta t}{2\Delta \sigma} [c_{\sigma r}^+ + c_{\sigma r}^- - c_{\sigma l}^+ - c_{\sigma l}^- \\
 & + \gamma(c_{\sigma r}^+ - c_{\sigma r}^- + c_{\sigma l}^+ - c_{\sigma l}^-)] \\
 & + \frac{\Delta t}{2\Delta \theta} [c_{\theta r}^+ + c_{\theta r}^- - c_{\theta l}^+ - c_{\theta l}^- \\
 & + \gamma(c_{\theta r}^+ - c_{\theta r}^- + c_{\theta l}^+ - c_{\theta l}^-)]_{i,j,l,m}.
 \end{aligned}$$

We can see that the structure of the coefficient matrix of the linear system (5) is in the form of 9-band matrix. This linear system can be solved by any direct or iterative methods under the diagonal dominant condition, that is the sum of off diagonal entry must be less than the main diagonal of coefficient matrix.

Now, we are analyzing the criteria of $\Delta t, \Delta x, \Delta y, \Delta \sigma$ and $\Delta \theta$ for exists and unques solution of this linear system. Let

us consider the diagonal dominant condition

$$\begin{aligned}
 |a_{i,j,l,m}| & > \frac{\Delta t}{2\Delta x} [(1-\gamma)c_{xr}^+ + (1+\gamma)c_{xr}^- \\
 & + (1+\gamma)c_{xl}^+ + (1-\gamma)c_{xl}^-]_{i,j,l,m} \\
 & + \frac{\Delta t}{2\Delta y} [(1-\gamma)c_{yr}^+ + (1+\gamma)c_{yr}^- \\
 & + (1+\gamma)c_{yl}^+ + (1-\gamma)c_{yl}^-]_{i,j,l,m} \\
 & + \frac{\Delta t}{2\Delta \sigma} [(1-\gamma)c_{\sigma r}^+ + (1+\gamma)c_{\sigma r}^- \\
 & + (1+\gamma)c_{\sigma l}^+ + (1-\gamma)c_{\sigma l}^-]_{i,j,l,m} \\
 & + \frac{\Delta t}{2\Delta \theta} [(1-\gamma)c_{\theta r}^+ + (1+\gamma)c_{\theta r}^- \\
 & + (1+\gamma)c_{\theta l}^+ + (1-\gamma)c_{\theta l}^-]_{i,j,l,m} \\
 & \equiv \text{cond} \quad (6)
 \end{aligned}$$

where $i = 1, \dots, N_x$; $j = 1, \dots, N_y$; $l = 1, \dots, N_\sigma$; $m = 1, \dots, N_\theta$.

Next, we try to simplify this stability criteria, by let

$$\begin{aligned}
 M_x & \equiv \max_{\forall i,j,l,m} |c_x|_{i,j,l,m}, \quad M_y \equiv \max_{\forall i,j,l,m} |c_y|_{i,j,l,m}, \\
 M_\sigma & \equiv \max_{\forall i,j,l,m} |c_\sigma|_{i,j,l,m}, \quad M_\theta \equiv \max_{\forall i,j,l,m} |c_\theta|_{i,j,l,m}. \quad (7)
 \end{aligned}$$

Since

$$\begin{aligned}
 |a_{i,j,l,m}| & = \left| 1 + \frac{\Delta t}{2\Delta x} [c_{xr}^+ + c_{xr}^- - c_{xl}^+ - c_{xl}^- \right. \\
 & \quad \left. + \gamma(c_{xr}^+ - c_{xr}^- + c_{xl}^+ - c_{xl}^-)] \right. \\
 & \quad \left. + \frac{\Delta t}{2\Delta y} [c_{yr}^+ + c_{yr}^- - c_{yl}^+ - c_{yl}^- \right. \\
 & \quad \left. + \gamma(c_{yr}^+ - c_{yr}^- + c_{yl}^+ - c_{yl}^-)] \right. \\
 & \quad \left. + \frac{\Delta t}{2\Delta \sigma} [c_{\sigma r}^+ + c_{\sigma r}^- - c_{\sigma l}^+ - c_{\sigma l}^- \right. \\
 & \quad \left. + \gamma(c_{\sigma r}^+ - c_{\sigma r}^- + c_{\sigma l}^+ - c_{\sigma l}^-)] \right. \\
 & \quad \left. + \frac{\Delta t}{2\Delta \theta} [c_{\theta r}^+ + c_{\theta r}^- - c_{\theta l}^+ - c_{\theta l}^- \right. \\
 & \quad \left. + \gamma(c_{\theta r}^+ - c_{\theta r}^- + c_{\theta l}^+ - c_{\theta l}^-)] \right|_{i,j,l,m} \\
 & \geq \left| 1 + \frac{\Delta t}{2\Delta x} (c_{xr}^- + c_{xl}^+) + \frac{\Delta t}{2\Delta y} (c_{yr}^- + c_{yl}^+) \right. \\
 & \quad \left. + \frac{\Delta t}{2\Delta \sigma} (c_{\sigma r}^- + c_{\sigma l}^+) + \frac{\Delta t}{2\Delta \theta} (c_{\theta r}^- + c_{\theta l}^+) \right|_{i,j,l,m} \\
 & \geq 1 - \frac{\Delta t M_x}{\Delta x} - \frac{\Delta t M_y}{\Delta y} - \frac{\Delta t M_\sigma}{\Delta \sigma} - \frac{\Delta t M_\theta}{\Delta \theta} \quad (8)
 \end{aligned}$$

and substituting the notation (7) into the equation (6), yields

$$\text{cond} \leq 2\Delta t \left[\frac{M_x}{\Delta x} + \frac{M_y}{\Delta y} + \frac{M_\sigma}{\Delta \sigma} + \frac{M_\theta}{\Delta \theta} \right]. \quad (9)$$

From relations (8) and (9), we can choose

$$\begin{aligned}
 1 - \Delta t \left[\frac{M_x}{\Delta x} + \frac{M_y}{\Delta y} + \frac{M_\sigma}{\Delta \sigma} + \frac{M_\theta}{\Delta \theta} \right] \\
 > 2\Delta t \left[\frac{M_x}{\Delta x} + \frac{M_y}{\Delta y} + \frac{M_\sigma}{\Delta \sigma} + \frac{M_\theta}{\Delta \theta} \right]. \quad (10)
 \end{aligned}$$

Thus the condition of Δt that satisfy the diagonal dominant of the linear system is following

$$\Delta t < \frac{1}{12 \max\left\{\frac{M_x}{\Delta x}, \frac{M_y}{\Delta y}, \frac{M_\sigma}{\Delta \sigma}, \frac{M_\theta}{\Delta \theta}\right\}}. \quad (11)$$

III. THE SPLITTING MODIFIED DONOR-CELL SCHEME

In the previous section, the numerical solution of the spectral action balance equation was described with a very huge coefficient matrix that needs to be solved by any direct and iterative methods that take a lots of computer's memory and operation count.

In this section, we will design a new numerical method that reduce the size of the original problem by splitting the original problem into four smaller problems. For each smaller problem can be solved easier than the original problem and take less computer's resource such as memory and operation counts. This method is called "splitting method".

Let us consider the spectral action balance equation on a domain $\Omega \times \Gamma$ with boundary $\partial\Omega \times \partial\Gamma$:

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x}(c_x N) + \frac{\partial}{\partial y}(c_y N) + \frac{\partial}{\partial \sigma}(c_\sigma N) + \frac{\partial}{\partial \theta}(c_\theta N) = \frac{S}{\sigma}. \quad (12)$$

Let Λ_x , Λ_y , Λ_σ and Λ_θ be approximation operator of $\frac{\partial}{\partial x}c_x(\cdot)$, $\frac{\partial}{\partial y}c_y(\cdot)$, $\frac{\partial}{\partial \sigma}c_\sigma(\cdot)$ and $\frac{\partial}{\partial \theta}c_\theta(\cdot)$, respectively. For each point $(x_i, y_j, \sigma_l, \theta_m, t_k)$, the approximate operators are represented as following

$$\begin{aligned} \left. \frac{\partial}{\partial x}(c_x N) \right|_{ijlm}^k &\approx \Lambda_x N_{ijlm}^k, \\ \left. \frac{\partial}{\partial y}(c_y N) \right|_{ijlm}^k &\approx \Lambda_y N_{ijlm}^k, \\ \left. \frac{\partial}{\partial \sigma}(c_\sigma N) \right|_{ijlm}^k &\approx \Lambda_\sigma N_{ijlm}^k, \\ \left. \frac{\partial}{\partial \theta}(c_\theta N) \right|_{ijlm}^k &\approx \Lambda_\theta N_{ijlm}^k. \end{aligned}$$

Therefore, the equation (12) can be approximated at each point $(x_i, y_j, \sigma_l, \theta_m, t)$ by the following

$$\left. \frac{\partial N}{\partial t} \right|_{ijlm} + \Lambda_x N_{ijlm} + \Lambda_y N_{ijlm} + \Lambda_\sigma N_{ijlm} + \Lambda_\theta N_{ijlm} = \frac{S_{ijlm}}{\sigma_l}.$$

For convenient writing, the indices i, j, l, m are neglected, yields

$$\frac{\partial N}{\partial t} + \Lambda_x N + \Lambda_y N + \Lambda_\sigma N + \Lambda_\theta N = \frac{S}{\sigma}.$$

Let us consider the backward time of spectral action balance equation at point $(x_i, y_j, \sigma_l, \theta_m)$

$$\frac{N^{k+1} - N^k}{\tau} + \Lambda_x N^{k+1} + \Lambda_y N^{k+1} + \Lambda_\sigma N^{k+1} + \Lambda_\theta N^{k+1} = \frac{S^{k+1}}{\sigma}. \quad (13)$$

and we introduce the splitting scheme

$$\frac{N^{k+\frac{1}{4}} - N^k}{\tau} + \Lambda_x N^{k+\frac{1}{4}} = 0 \quad (14)$$

$$\frac{N^{k+\frac{2}{4}} - N^{k+\frac{1}{4}}}{\tau} + \Lambda_y N^{k+\frac{2}{4}} = 0 \quad (15)$$

$$\frac{N^{k+\frac{3}{4}} - N^{k+\frac{2}{4}}}{\tau} + \Lambda_\sigma N^{k+\frac{3}{4}} = 0 \quad (16)$$

$$\frac{N^{k+1} - N^{k+\frac{3}{4}}}{\tau} + \Lambda_\theta N^{k+1} = \frac{S^{k+1}}{\sigma}. \quad (17)$$

Now, we will prove that equations (14)-(17) are consistent with the equation (13) by rearranging equations (14)-(17), yields

$$-N^k + (I + \tau\Lambda_x)N^{k+\frac{1}{4}} = 0 \quad (18)$$

$$-N^{k+\frac{1}{4}} + (I + \tau\Lambda_y)N^{k+\frac{2}{4}} = 0 \quad (19)$$

$$-N^{k+\frac{2}{4}} + (I + \tau\Lambda_\sigma)N^{k+\frac{3}{4}} = 0 \quad (20)$$

$$-N^{k+\frac{3}{4}} + (I + \tau\Lambda_\theta)N^{k+1} = \frac{\tau S^{k+1}}{\sigma} \quad (21)$$

where I is an identity approximation operator. To eliminate $N^{k+\frac{1}{4}}$ in equations (18) and (19), we multiply the equation (19) by $(I + \tau\Lambda_x)$ and adding the result to the equation (18), then we obtain

$$-N^k + (I + \tau\Lambda_x)(I + \tau\Lambda_y)N^{k+\frac{2}{4}} = 0. \quad (22)$$

To eliminate $N^{k+\frac{2}{4}}$ in equations (20) and (22), we multiply the equation (20) by $(I + \tau\Lambda_x)(I + \tau\Lambda_y)$ and adding the result to the equation (22) then

$$-N^k + (I + \tau\Lambda_x)(I + \tau\Lambda_y)(I + \tau\Lambda_\sigma)N^{k+\frac{3}{4}} = 0. \quad (23)$$

Similarly to eliminate $N^{k+\frac{3}{4}}$ in equations (21) and (23), we multiply the equation (21) by $(I + \tau\Lambda_x)(I + \tau\Lambda_y)(I + \tau\Lambda_\sigma)$ and adding the result to the equation (23) then

$$\begin{aligned} -N^k + (I + \tau\Lambda_x)(I + \tau\Lambda_y)(I + \tau\Lambda_\sigma)(I + \tau\Lambda_\theta)N^{k+1} \\ = (I + \tau\Lambda_x)(I + \tau\Lambda_y)(I + \tau\Lambda_\sigma)\left(\frac{\tau S^{k+1}}{\sigma}\right). \end{aligned} \quad (24)$$

Since

$$(I + \tau\Lambda_x)(I + \tau\Lambda_y)(I + \tau\Lambda_\sigma) = I + \tau(\Lambda_\sigma + \Lambda_x + \Lambda_y) + O(\tau^2) \quad (25)$$

and

$$\begin{aligned} (I + \tau\Lambda_x)(I + \tau\Lambda_y)(I + \tau\Lambda_\sigma)(I + \tau\Lambda_\theta) \\ = I + \tau(\Lambda_x + \Lambda_y + \Lambda_\sigma + \Lambda_\theta) + O(\tau^2), \end{aligned} \quad (26)$$

substituting equations (25) and (26) into the equation (24),

$$\begin{aligned} -N^k + [I + \tau(\Lambda_x + \Lambda_y + \Lambda_\sigma + \Lambda_\theta) + O(\tau^2)]N^{k+1} \\ = [I + \tau(\Lambda_x + \Lambda_y + \Lambda_\sigma) + O(\tau^2)]\left[\frac{\tau S^{k+1}}{\sigma}\right]. \end{aligned} \quad (27)$$

Then

$$\frac{N^{k+1} - N^k}{\tau} + \Lambda_x N^{k+1} + \Lambda_y N^{k+1} + \Lambda_\sigma N^{k+1} + \Lambda_\theta N^{k+1} = \frac{S^{k+1}}{\sigma}. \quad (28)$$

Therefore, the scheme (28) and the equivalent schemes (14)-(17) approximate the spectral action balance equation with the same accuracy $O(\tau)$ as the scheme (13).

For the stability criteria for each system, we must choose the type of approximate operator Λ_x , Λ_y , Λ_σ , and Λ_θ . Here, we choose these approximate operators as modified donor-cell scheme approximation and apply this approximation with the

equations (14)-(17), yields

$$\begin{aligned} & \frac{N^{n+\frac{1}{4}} - N^n}{\tau_x} \Big|_{i,j,l,m} \\ & + \frac{(1-\gamma)}{\Delta x} \left[c_{xr} \frac{N_i + N_{i+1}}{2} - c_{xl} \frac{N_{i-1} + N_i}{2} \right]_{j,l,m}^{n+\frac{1}{4}} \\ & + \frac{\gamma}{\Delta x} (c_{xr}^+ N_i + c_{xr}^- N_{i+1} - c_{xl}^+ N_{i-1} - c_{xl}^- N_i) \Big|_{j,l,m}^{n+\frac{1}{4}} \\ & = 0 \end{aligned} \quad (29)$$

$$\begin{aligned} & \frac{N^{n+\frac{2}{4}} - N^{n+\frac{1}{4}}}{\tau_y} \Big|_{i,j,l,m} \\ & + \frac{(1-\gamma)}{\Delta y} \left[c_{yr} \frac{N_j + N_{j+1}}{2} - c_{yl} \frac{N_{j-1} + N_j}{2} \right]_{i,l,m}^{n+\frac{2}{4}} \\ & + \frac{\gamma}{\Delta y} (c_{yr}^+ N_j + c_{yr}^- N_{j+1} - c_{yl}^+ N_{j-1} - c_{yl}^- N_j) \Big|_{i,l,m}^{n+\frac{2}{4}} \\ & = 0 \end{aligned} \quad (30)$$

$$\begin{aligned} & \frac{N^{n+\frac{3}{4}} - N^{n+\frac{2}{4}}}{\tau_\sigma} \Big|_{i,j,l,m} \\ & + \frac{(1-\gamma)}{\Delta \sigma} \left[c_{\sigma r} \frac{N_l + N_{l+1}}{2} - c_{\sigma l} \frac{N_{l-1} + N_l}{2} \right]_{i,j,m}^{n+\frac{3}{4}} \\ & + \frac{\gamma}{\Delta \sigma} (c_{\sigma r}^+ N_l + c_{\sigma r}^- N_{l+1} - c_{\sigma l}^+ N_{l-1} - c_{\sigma l}^- N_l) \Big|_{i,j,m}^{n+\frac{3}{4}} \\ & = 0 \end{aligned} \quad (31)$$

$$\begin{aligned} & \frac{N^{n+1} - N^{n+\frac{3}{4}}}{\tau_\theta} \Big|_{i,j,l,m} \\ & + \frac{(1-\gamma)}{\Delta \theta} \left[c_{\theta r} \frac{N_m + N_{m+1}}{2} - c_{\theta l} \frac{N_{m-1} + N_m}{2} \right]_{i,j,l}^{n+1} \\ & + \frac{\gamma}{\Delta \theta} (c_{\theta r}^+ N_m + c_{\theta r}^- N_{m+1} - c_{\theta l}^+ N_{m-1} - c_{\theta l}^- N_m) \Big|_{i,j,l}^{n+1} \\ & = \frac{S^{n-1}}{\sigma_l} \Big|_{i,j,l,m} \end{aligned} \quad (32)$$

Rearranging equations (29)-(32), we have four linear systems as follows

$$\begin{aligned} a_{i,j,l,m} N_{i,j,l,m}^{n+\frac{1}{4}} + \tau_x \frac{(1-\gamma)c_{xr}^+ + (1+\gamma)c_{xr}^-}{2\Delta x} N_{i+1,j,l,m}^n \\ - \tau_x \frac{(1+\gamma)c_{xl}^+ + (1-\gamma)c_{xl}^-}{2\Delta x} N_{i-1,j,l,m}^n \\ = N_{i,j,l,m}^n \end{aligned} \quad (33)$$

$$\begin{aligned} b_{i,j,l,m} N_{i,j,l,m}^{n+\frac{2}{4}} + \tau_y \frac{(1-\gamma)c_{yr}^+ + (1+\gamma)c_{yr}^-}{2\Delta y} N_{i,j+1,l,m}^n \\ - \tau_y \frac{(1+\gamma)c_{yl}^+ + (1-\gamma)c_{yl}^-}{2\Delta y} N_{i,j-1,l,m}^n \\ = N_{i,j,l,m}^{n+\frac{1}{4}} \end{aligned} \quad (34)$$

$$\begin{aligned} c_{i,j,l,m} N_{i,j,l,m}^{n+\frac{3}{4}} + \tau_\sigma \frac{(1-\gamma)c_{\sigma r}^+ + (1+\gamma)c_{\sigma r}^-}{2\Delta \sigma} N_{i,j,l+1,m}^n \\ - \tau_\sigma \frac{(1+\gamma)c_{\sigma l}^+ + (1-\gamma)c_{\sigma l}^-}{2\Delta \sigma} N_{i,j,l-1,m}^n \\ = N_{i,j,l,m}^{n+\frac{2}{4}} \end{aligned} \quad (35)$$

$$\begin{aligned} d_{i,j,l,m} N_{i,j,l,m}^{n+1} + \tau_\theta \frac{(1-\gamma)c_{\theta r}^+ + (1+\gamma)c_{\theta r}^-}{2\Delta \theta} N_{i,j,l,m+1}^n \\ - \tau_\theta \frac{(1+\gamma)c_{\theta l}^+ + (1-\gamma)c_{\theta l}^-}{2\Delta \theta} N_{i,j,l,m-1}^n \\ = \frac{\tau_\theta S}{\sigma_l} \Big|_{i,j,l,m} + N_{i,j,l,m}^{n+\frac{3}{4}} \end{aligned} \quad (36)$$

where $i = 1, \dots, N_x$; $j = 1, \dots, N_y$; $l = 1, \dots, N_\sigma$; $m = 1, \dots, N_\theta$,

$$\begin{aligned} a_{i,j,l,m} &= 1 + \frac{\tau_x}{2\Delta x} [c_{xr}^+ - c_{xr}^- - c_{xl}^+ - c_{xl}^- \\ &\quad + \gamma(c_{xr}^+ - c_{xr}^- + c_{xl}^+ - c_{xl}^-)], \\ b_{i,j,l,m} &= 1 + \frac{\tau_y}{2\Delta y} [c_{yr}^+ - c_{yr}^- - c_{yl}^+ - c_{yl}^- \\ &\quad + \gamma(c_{yr}^+ - c_{yr}^- + c_{yl}^+ - c_{yl}^-)], \\ c_{i,j,l,m} &= 1 + \frac{\tau_\sigma}{2\Delta \sigma} [c_{\sigma r}^+ - c_{\sigma r}^- - c_{\sigma l}^+ - c_{\sigma l}^- \\ &\quad + \gamma(c_{\sigma r}^+ - c_{\sigma r}^- + c_{\sigma l}^+ - c_{\sigma l}^-)], \\ d_{i,j,l,m} &= 1 + \frac{\tau_\theta}{2\Delta \theta} [c_{\theta r}^+ - c_{\theta r}^- - c_{\theta l}^+ - c_{\theta l}^- \\ &\quad + \gamma(c_{\theta r}^+ - c_{\theta r}^- + c_{\theta l}^+ - c_{\theta l}^-)]. \end{aligned}$$

We can see that the structure of the coefficient matrix of these linear systems (33)-(36) are tridiagonal matrices. These linear systems can be solved by any direct or iterative methods under the diagonal dominant condition, that is the sum of off diagonal entry must less than the main diagonal of coefficient matrix.

Now, we are analyzing the criteria of $\Delta t, \Delta x, \Delta y, \Delta \sigma$ and $\Delta \theta$ for exists and uniques solution of this linear system. Let us consider the diagonal dominant condition

$$\begin{aligned} |a_{i,j,l,m}| &> \frac{\tau_x}{2\Delta x} (|(1-\gamma)c_{xr}^+ + (1+\gamma)c_{xr}^-| \\ &\quad + |(1+\gamma)c_{xl}^+ + (1-\gamma)c_{xl}^-|)_{i,j,l,m} \\ &\equiv \text{cond}_a \end{aligned} \quad (37)$$

$$\begin{aligned} |b_{i,j,l,m}| &> \frac{\tau_y}{2\Delta y} (|(1-\gamma)c_{yr}^+ + (1+\gamma)c_{yr}^-| \\ &\quad + |(1+\gamma)c_{yl}^+ + (1-\gamma)c_{yl}^-|)_{i,j,l,m} \\ &\equiv \text{cond}_b \end{aligned} \quad (38)$$

$$\begin{aligned} |c_{i,j,l,m}| &> \frac{\tau_\sigma}{2\Delta \sigma} (|(1-\gamma)c_{\sigma r}^+ + (1+\gamma)c_{\sigma r}^-| \\ &\quad + |(1+\gamma)c_{\sigma l}^+ + (1-\gamma)c_{\sigma l}^-|)_{i,j,l,m} \\ &\equiv \text{cond}_c \end{aligned} \quad (39)$$

$$\begin{aligned} |d_{i,j,l,m}| &> \frac{\tau_\theta}{2\Delta \theta} (|(1-\gamma)c_{\theta r}^+ + (1+\gamma)c_{\theta r}^-| \\ &\quad + |(1+\gamma)c_{\theta l}^+ + (1-\gamma)c_{\theta l}^-|)_{i,j,l,m} \\ &\equiv \text{cond}_d \end{aligned} \quad (40)$$

where $i = 1, \dots, N_x$; $j = 1, \dots, N_y$; $l = 1, \dots, N_\sigma$; $m = 1, \dots, N_\theta$.

Next, we try to simplify these stability criteria, by letting

$$\begin{aligned} M_x &\equiv \max_{i,j,l,m} |c_x|_{i,j,l,m}, & M_y &\equiv \max_{i,j,l,m} |c_y|_{i,j,l,m}, \\ M_\sigma &\equiv \max_{i,j,l,m} |c_\sigma|_{i,j,l,m}, & M_\theta &\equiv \max_{i,j,l,m} |c_\theta|_{i,j,l,m}. \end{aligned} \quad (41)$$

Since

$$\begin{aligned} |a_{i,j,l,m}| &= \left| 1 + \frac{\tau_x}{2\Delta x} [c_{xr}^+ - c_{xr}^- - c_{xl}^+ - c_{xl}^- + \gamma(c_{xr}^+ - c_{xr}^- + c_{xl}^+ - c_{xl}^-)] \right|_{i,j,l,m} \\ &\geq \left| 1 + \frac{\tau_x}{2\Delta x} (c_{xr}^- - c_{xl}^+) \right|_{i,j,l,m} \\ &\geq 1 - \frac{\tau_x}{2\Delta x} |c_{xr}|_{i,j,l,m} - \frac{\tau_x}{2\Delta x} |c_{xl}|_{i,j,l,m} \\ &\geq 1 - \frac{\tau_x}{\Delta x} M_x, \end{aligned} \quad (42)$$

$$\begin{aligned} |b_{i,j,l,m}| &= \left| 1 + \frac{\tau_y}{2\Delta y} [c_{yr}^+ - c_{yr}^- - c_{yl}^+ - c_{yl}^- + \gamma(c_{yr}^+ - c_{yr}^- + c_{yl}^+ - c_{yl}^-)] \right|_{i,j,l,m} \\ &\geq \left| 1 + \frac{\tau_y}{2\Delta y} (c_{yr}^- - c_{yl}^+) \right|_{i,j,l,m} \\ &\geq 1 - \frac{\tau_y}{2\Delta y} |c_{yr}|_{i,j,l,m} - \frac{\tau_y}{2\Delta y} |c_{yl}|_{i,j,l,m} \\ &\geq 1 - \frac{\tau_y}{\Delta y} M_y, \end{aligned} \quad (43)$$

$$\begin{aligned} |c_{i,j,l,m}| &= \left| 1 + \frac{\tau_\sigma}{2\Delta\sigma} [c_{\sigma r}^+ - c_{\sigma r}^- - c_{\sigma l}^+ - c_{\sigma l}^- + \gamma(c_{\sigma r}^+ - c_{\sigma r}^- + c_{\sigma l}^+ - c_{\sigma l}^-)] \right|_{i,j,l,m} \\ &\geq \left| 1 + \frac{\tau_\sigma}{2\Delta\sigma} (c_{\sigma r}^- - c_{\sigma l}^+) \right|_{i,j,l,m} \\ &\geq 1 - \frac{\tau_\sigma}{2\Delta\sigma} |c_{\sigma r}|_{i,j,l,m} - \frac{\tau_\sigma}{2\Delta\sigma} |c_{\sigma l}|_{i,j,l,m} \\ &\geq 1 - \frac{\tau_\sigma}{\Delta\sigma} M_\sigma, \end{aligned} \quad (44)$$

$$\begin{aligned} |d_{i,j,l,m}| &= \left| 1 + \frac{\tau_\theta}{2\Delta\theta} [c_{\theta r}^+ - c_{\theta r}^- - c_{\theta l}^+ - c_{\theta l}^- + \gamma(c_{\theta r}^+ - c_{\theta r}^- + c_{\theta l}^+ - c_{\theta l}^-)] \right|_{i,j,l,m} \\ &\geq \left| 1 + \frac{\tau_\theta}{2\Delta\theta} (c_{\theta r}^- - c_{\theta l}^+) \right|_{i,j,l,m} \\ &\geq 1 - \frac{\tau_\theta}{2\Delta\theta} |c_{\theta r}|_{i,j,l,m} - \frac{\tau_\theta}{2\Delta\theta} |c_{\theta l}|_{i,j,l,m} \\ &\geq 1 - \frac{\tau_\theta}{\Delta\theta} M_\theta, \end{aligned} \quad (45)$$

and substituting the notations (41) into the relations (37)-(40), yields

$$\begin{aligned} cond_a &\leq 2\tau_x \frac{M_x}{\Delta x}, & cond_b &\leq 2\tau_y \frac{M_y}{\Delta y}, \\ cond_c &\leq 2\tau_\sigma \frac{M_\sigma}{\Delta\sigma}, & cond_d &\leq 2\tau_\theta \frac{M_\theta}{\Delta\theta}. \end{aligned} \quad (46)$$

From relations (42) - (46), we can choose

$$\begin{aligned} 1 - \tau_x \frac{M_x}{\Delta x} &> 2\tau_x \frac{M_x}{\Delta x}, & 1 - \tau_y \frac{M_y}{\Delta y} &> 2\tau_y \frac{M_y}{\Delta y}, \\ 1 - \tau_\sigma \frac{M_\sigma}{\Delta\sigma} &> 2\tau_\sigma \frac{M_\sigma}{\Delta\sigma}, & 1 - \tau_\theta \frac{M_\theta}{\Delta\theta} &> 2\tau_\theta \frac{M_\theta}{\Delta\theta}. \end{aligned}$$

Thus the condition of $\tau_x, \tau_y, \tau_\sigma$ and τ_θ that satisfy the diagonal dominant of the linear system are following

$$\tau_x < \frac{\Delta x}{3M_x}, \quad \tau_y < \frac{\Delta y}{3M_y}, \quad \tau_\sigma < \frac{\Delta\sigma}{3M_\sigma}, \quad \tau_\theta < \frac{\Delta\theta}{3M_\theta}.$$

Thus the condition of τ that satisfy the diagonal dominant of all linear systems in the splitting scheme is following

$$\tau < \frac{1}{3} \min \left\{ \frac{\Delta x}{M_x}, \frac{\Delta y}{M_y}, \frac{\Delta\sigma}{M_\sigma}, \frac{\Delta\theta}{M_\theta} \right\}. \quad (47)$$

IV. NUMERICAL EXPERIMENTS

In this section, we collect some results calculated using the modified donor-cell scheme and compare to the central difference scheme in [2]. We wish to emphasize the diversity of the possible applications.

We begin with spectral action balance equation:

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x}(c_x N) + \frac{\partial}{\partial y}(c_y N) + \frac{\partial}{\partial \sigma}(c_\sigma N) + \frac{\partial}{\partial \theta}(c_\theta N) = \frac{S}{\sigma}, \quad \forall (x, y, \sigma, \theta) \in \Omega \times \Gamma$$

where $t \in [0, T]$, and the initial and boundary conditions are defined as follows:

$$\begin{aligned} N|_{t=0} &= N_0(x, y, \sigma, \theta), \quad \forall (x, y, \sigma, \theta) \in \Omega \times \Gamma; \\ \frac{\partial N}{\partial n} &= 0, \quad \forall (x, y, \sigma, \theta) \in \partial\Omega \times \partial\Gamma, \quad t \in [0, T]. \end{aligned}$$

The specific parameters used in our calculations are as follows:

$$\begin{aligned} x_l \leq x \leq x_r, & \quad y_l \leq y \leq y_r, & \sigma_l \leq \sigma \leq \sigma_r, & \quad \theta_l \leq \theta \leq \theta_r, \\ x_l = -1, & \quad x_r = 1, & y_l = -1, & \quad y_r = 1, \\ \sigma_l = 0.04, & \quad \sigma_r = 1, & \theta_l = 0, & \quad \theta_r = 2\pi, \\ N_x = 20, & \quad N_y = 20, & N_\sigma = 20, & \quad N_\theta = 20, \\ \Delta x = \frac{x_r - x_l}{N_x - 1}, & \quad \Delta y = \frac{y_r - y_l}{N_y - 1}, & \Delta\sigma = \frac{\sigma_r - \sigma_l}{N_\sigma - 1}, & \quad \Delta\theta = \frac{\theta_r - \theta_l}{N_\theta - 1}, \end{aligned}$$

source terms:

$$\begin{aligned} S(0, 9 : 11, N_y - 2, N_\sigma/2, N_\theta/2) &= 100 \\ S(0, 14 : 17, N_y - 2, N_\sigma/2, N_\theta/2) &= 100 \\ S(t, x, y, \sigma, \theta) &= 0, \\ \forall x, y, \sigma, \theta \in \Omega \times \Gamma, \quad t > 0. \end{aligned}$$

and propagation velocity terms:

$$\begin{aligned} c_x(i, j, N_\sigma/2, N_\theta/2) &= \cos(\pi(i+j)/N_x) \\ c_y(i, j, N_\sigma/2, N_\theta/2) &= \sin(\pi(i+j)/N_y) \\ c_\sigma(:, :, :, :) &= 0.01 \\ c_\theta(:, :, :, :) &= 0.01 \end{aligned}$$

In this experiment, we simulate a spectral action balance equation in a square domain. The physical configuration consists of a square container filled with wave energy. The central difference and the modified donor-cell scheme with $\gamma = 1$ are presented. Firstly, we set the initial values of

N as zero for very nodes in the domain $\Omega \times \Gamma$. At the initial time, we filled the wave energy into the top-left of the domain. At first time step, the energy peaked at those grid point and after that its moves along the direction field of the propagation velocities. The numerical results by these two methods are shown in Figures 2 and 3. From these two figures, we can see that the result from a central difference scheme is unphysical oscillations. The reason for this lies in the fact that, for grid spaces are too large, certain properties of the continuous equations are no longer correctly captured by discrete equation. But the result from modified donor-cell with $\gamma = 1$ is very stable with the same grid spaces as a central difference scheme and moves along the direction field of the propagation velocities.

V. CONCLUSION AND DISCUSSION

The spectral action balance equation is an equation that used to simulate short-crested wind-generated waves in shallow water areas such as coastal regions and inland waters. This equation consists of two spatial dimensions, wave direction, and wave frequency which can be solved by finite difference method. When this equation with dominating propagation velocity terms are discretized using central differences, stability problems occur when the grid spacing is chosen too coarse. For avoiding stability problems, we applied the modified donor-cell scheme and the splitting modified donor-cell scheme for numerical solution of the spectral action balance equation with time splitting, although it also suffers from a lower order of approximation. The splitting schemes were adopted to split the wave spectral action balance equation into four one-dimensional problems, which for each small problem obtains the independently tridiagonal linear systems. Therefore, we can solve these systems by direct or iterative methods at the same time which is very fast when performed by a multi-cores computer.

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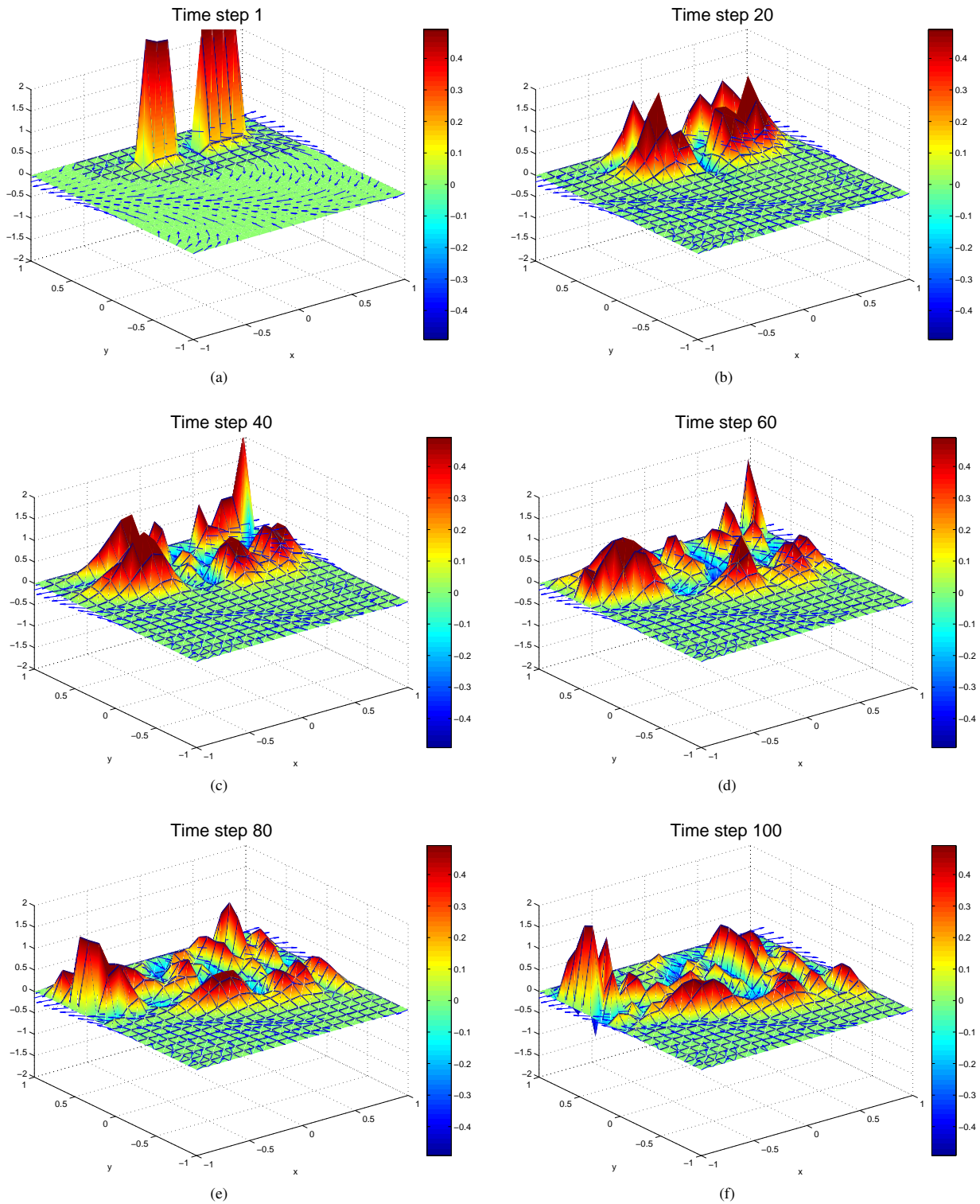


Fig. 2: Numerical results for every 20 time step by using a central difference scheme.

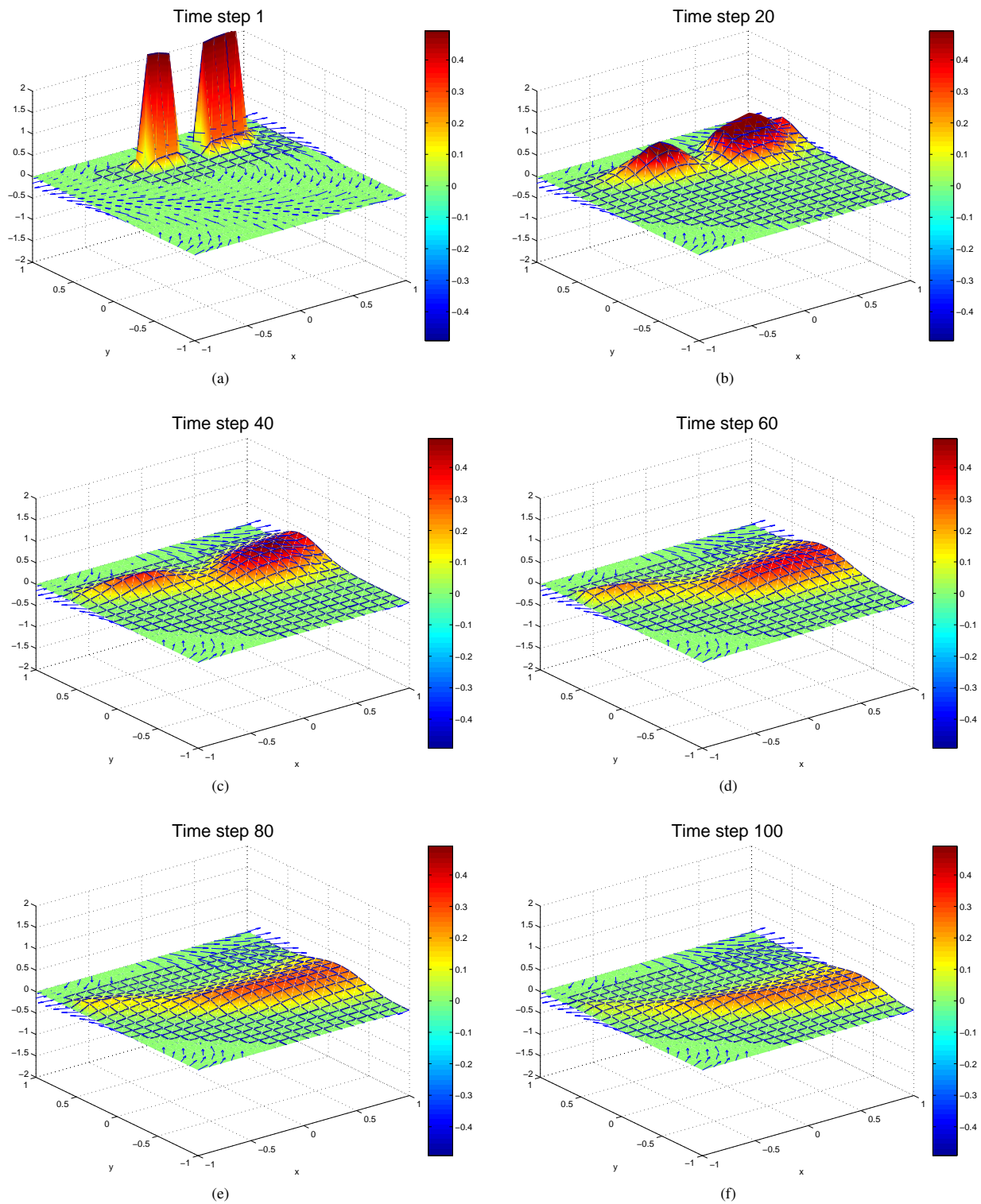


Fig. 3: Numerical results for every 20 time step by using the modified donor-cell scheme with $\gamma = 1$.