Some Static Isotropic Perfect Fluid Spheres in General Relativity

Sachin Kumar, Y. K. Gupta, and J. R. Sharma

Abstract—In the present article**,** a new class of solutions of Einstein field equations is investigated for a spherically symmetric space-time when the source of gravitation is a perfect fluid. All the solutions have been derived by making some suitable arrangements in the field equations. The solutions so obtained have been seen to describe Schwarzschild interior solutions. Most of the solutions are subjected to the reality conditions. As far as the authors are aware the solutions are new.

Keywords—Einstein's equations, General Relativity, Perfect Fluid, Spherical symmetric.

I. INTRODUCTION

THE standard Friedman-Robertson-Walker (FRW) cosmological model prescribes a homogeneous and isotropic distribution for its matter in a description of the present state of the universe. At the present state of evolution the universe is spherically symmetric and the matter distribution in the universe is on the whole isotropic and homogeneous. The early attempt at the construction of such models have done by Tolman [18] and Bondi [4] who considered spherically symmetric model. Many workers like Senovilla [15, 16, 17], Ruiz and Senovilla [14], Dadhich et al. [5]**,** Patel et al. [8], Pradhan et al. [9, 10, 11, 12, 13], Barrow and Kunze [1, 2], Baysal et al. [3], Kandalkar and Gawande [6], Kilinc and Yavuz [7] have investigated cylindrically and spherically symmetric inhomogeneous cosmological models in various contexts like inclusion of electromagnetic and string source.

In this paper we have investigated the isotropic models through spherically symmetric perfect fluid distributions on the basis of reality conditions.

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II. ISOTROPIC FLUID DISTRIBUTIONS IN ISOTROPIC COORDINATE SYSTEM

The Spherically symmetric metric in the canonical coordinate system is expressed as:

$$ds^2 = -A\,dr^2 - B\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) + C\,dt^2 \tag{1}$$

Where A, B and C are functions of r and t only.

An appropriate transformation in the static case, i.e. $A = A(r)$, $B = B(r)$ and $C = C(r)$.

$$B = \bar{r}^2 A\left(\frac{d\,r}{d\,\bar{r}}\right)^2, \qquad r = r(\bar{r}) \tag{2}$$

Consequently (1) with reference (2), we have

$$ds^2 = -A(\bar{r})\left[d\,\bar{r}^2 + \bar{r}^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] + C(\bar{r})dt^2 \tag{3}$$

then the metric (1) is said to be in isotropic coordinate system.

III. FIELD EQUATIONS AND ISOTROPIC FLUID SPHERES

The Einstein's field equation for perfect fluid distributions can be written as below:

$$-8\pi T_{ij} = R_{ij} - \frac{1}{2}R\,g_{ij} = -8\pi\left[\left(p + \rho\right)v_i\,v_j - p\,g_{ij}\right] \tag{4}$$

where ρ being the density, p the pressure and v_i the flow vector.

The Einstein's field equations (4) for the metric (3) can be expressed as

$$8\pi T_1^1 = -\frac{1}{A}\left[\frac{1}{r}\left(\frac{C'}{C} + \frac{A'}{A}\right) + \frac{A'C'}{2AC} + \frac{A'^2}{4A^2}\right] = -8\pi p \tag{5}$$

$$8\pi T_2^2 = 8\pi T_3^3 = -\frac{1}{A}\left\{\begin{array}{c}\frac{1}{2}\left(\frac{A'}{A}\right)' + \frac{1}{2}\left(\frac{C'}{C}\right)' \\ + \frac{C'^2}{4C^2} + \frac{1}{2r}\left(\frac{C'}{C} + \frac{A'}{A}\right)\end{array}\right\} \quad (6)$$

$$= -8\pi p$$

$$8\pi T_4^4 = -\frac{1}{A}\left[\left(\frac{A'}{A}\right)' + \frac{A'^2}{4A^2} + \frac{2A'}{rA}\right] = 8\pi\rho \quad (7)$$

The consistency of (5) and (6) reveals

$$\frac{A''}{A} - \frac{3A'^2}{4A^2} + \frac{C''}{C} - \frac{C'^2}{2C^2} - \frac{1}{r}\left(\frac{A'}{A} + \frac{C'}{C}\right) - \frac{A'C'}{AC} = 0 \quad (8)$$

which can be reduced to a differential equation

$$LG'' = 2L''G \quad (9)$$

where $L = A^{-\frac{1}{2}}$, $G = \left(\frac{C}{A}\right)^{\frac{1}{2}}$

and the prime (') denotes the differentiation with respect to $x\left(= r^2\right)$.

The equation (9) involves two unknowns L and G. Therefore, we shall discuss some cases corresponding to the conditions adopted for L and G, respectively.

Solution (I) Let us assume

$$\frac{2L''}{L} = k^2 \quad \text{and hence} \quad \frac{G''}{G} = k^2 \quad (10)$$

which integrates to

$$A = \frac{1}{L^2} = \frac{1}{B_1^2 \cosh^2\psi}, \quad C = \frac{G^2}{L^2} = \frac{B_2^2 \cosh^2\phi}{B_1^2 \cosh^2\psi} \quad (11)$$

where $\phi = kr^2 + \alpha$, $\psi = \frac{k}{\sqrt{2}}r^2 + \beta$; B_1, B_2, α and β being the arbitrary constants.

The density and pressure of the fluid can then be computed as

$$8\pi p = B_1^2 \cosh^2\psi\left\{\begin{array}{c}-4\sqrt{2}k\tanh\psi + 6k^2r^2\tanh^2\psi \\ + 4k\tanh\phi \\ -4\sqrt{2}k^2r^2\tanh\phi\tanh\psi\end{array}\right\},$$

$$8\pi\rho = B_1^2 \cosh^2\psi\left\{\begin{array}{c}6\sqrt{2}k\tanh\psi + 4k^2r^2 \\ -6k^2r^2\tanh^2\psi\end{array}\right\} \quad (12)$$

To ensure the positivity of p, ρ and $\rho - p$ at the centre, let us restrict the constants k, α, β, B_1 and B_2 follows:

$$\left(p\right)_{r=0} = 4B_1^2 k\cosh^2\beta\left[-\sqrt{2}\tanh\beta + \tanh\alpha\right] \succ 0,$$

$$\left(\rho\right)_{r=0} = 3\sqrt{2}B_1^2 k\sinh 2\beta \succ 0,$$

$$\left(\rho - p\right)_{r=0} = 2B_1^2 k\cosh^2\beta\left[5\sqrt{2}\tanh\beta - 2\tanh\alpha\right] \succ 0 \quad (13)$$

which suggest the following restrictions

if $k \succ 0$, then α and β should be such that

$$\frac{1}{\sqrt{2}} \succ \frac{\tanh\beta}{\tanh\alpha} \succ \frac{\sqrt{2}}{5}, \quad \text{also } \beta \succ 0 \text{ and hence } \alpha \succ 0. \quad (14)$$

and $k \prec 0$, (13) should be reversed which is unacceptable. So, the inequality (14) implies the positivity of k, α and β in addition to (13).

The pressure free surface can be had in this case too, but the radius of the isolated sphere cannot be calculated explicitly, which is very clear if we equate the pressure equal to zero. However, the extremum of the pressure and density exists at the centre of the sphere as we have

$$p_r = 0 = \rho_r \text{ and } \rho_{rr}, \ p_{rr} \succ 0 \text{ at } r = 0.$$

Adding all the conclusions we can say that the density and pressure are positive non-zero at the centre and expected to increase with the increase of r.

Solution (II) Let us take

$$\frac{L''}{2L} = -\frac{1}{8x^2} \quad \text{and so,} \quad \frac{G''}{G} = -\frac{1}{2x^2} \quad (15)$$

Both the above equations are of Cauchy-Euler's type and the substitution

$$z = \log x,$$

would make, both of them to be linear and integration of which provides us

$$A = \frac{1}{L^2} = \frac{1}{r^2 \left(C_1 + 2C_2 \log r\right)^2},$$

$$C = \frac{G^2}{L^2} = \frac{C_3^{\ 2} \cos^2 \left(\log r + \beta\right)}{\left(C_1 + 2C_2 \log r\right)^2} \qquad (16)$$

where C_1, C_2, C_3 and β being the arbitrary constants.

Also, the pressure and density are

$$8\pi p = 4C_2 \left(C_1 + 2C_2 \log r\right) \tan \left(\log r + \beta\right)$$
$$+ 12C_2^{\ 2} - \left(C_1 + 2C_2 \log r\right)^2, \qquad (17)$$

$$8\pi \rho = 3\left(C_1 + 2C_2 \log r\right)^2 - 12C_2^{\ 2}$$

The present solution is singular at the origin $r = 0$ and so are the density and pressure. The above singularity has its origin in the coordinate system itself. Let us transform the coordinate system through the transformation.

$$R = \log r \quad \text{or} \quad r = e^{R} \qquad (18)$$

As a consequence, the metric (3) takes the form

$$ds^2 = \frac{1}{\left(C_1 + 2C_2 R\right)^2} \left\{ \begin{array}{c} -dR^2 - \left(d\theta^2 + \sin^2 \theta \, d\phi^2\right) \\ + C_3^{\ 2} \cos^2 \left(R + \beta\right) dt^2 \end{array} \right\} \qquad (19)$$

and the expressions for pressure and density are given as

$$8\pi p = 4C_2 \left(C_1 + 2C_2 R\right) \tan \left(R + \beta\right)$$
$$+ 12C_2^{\ 2} - \left(C_1 + 2C_2 R\right)^2,$$

$$8\pi \rho = 3\left(C_1 + 2C_2 R\right)^2 - 12C_2^{\ 2} \qquad (20)$$

Now, for p, ρ and $\rho - p$ at the centre $R = 0$, we have

$$\left(p\right)_{R=0} = 4C_1 C_2 \tan \beta + 12C_2^{\ 2} - C_1^{\ 2} \succ 0,$$

$$\left(\rho\right)_{R=0} = 3C_1^{\ 2} - 12C_2^{\ 2} \succ 0,$$

$$\left(\rho - p\right)_{R=0} = 2C_1^{\ 2} - 24C_2^{\ 2} - 4C_1 C_2 \tan \beta \succ 0 \qquad (21)$$

The above inequality imply that

$$C_1 C_2 \prec 0, \quad \tan \beta \prec 0 \, (\beta \prec 0, \text{ let})$$

$$2C_1^{\ 2} - 24C_2^{\ 2} \succ 4C_1 C_2 \tan \beta \succ C_1^{\ 2} - 12C_2^{\ 2},$$

which further imply $p_R \prec 0$ and $\rho_R \prec 0$ at $R = 0$.

It indicates that the pressure and density are positive and maximum at the centre and will keep on decreasing with the increase of R.

Solution (III)   In this case, let us takes

$$\frac{2L''}{L} = -\frac{2}{\left(ax + b\right)^4},$$

and so, we have, the equation for G

$$\frac{G''}{G} = -\frac{2}{\left(ax + b\right)^4}, \qquad (22)$$

which yield the solutions

$$A = \frac{1}{L^2} = \frac{1}{D_1^{\ 2} \left(ar^2 + b\right)^2 \cos^2 \psi},$$

$$C = \frac{G^2}{L^2} = \frac{D_2^{\ 2} \cos^2 \phi}{D_1^{\ 2} \cos^2 \psi} \qquad (23)$$

where

$$\phi = \frac{\sqrt{2}}{a\left(ar^2 + b\right)} + \alpha, \ \psi = \frac{1}{a\left(ar^2 + b\right)} + \beta; \ D_1, D_2, \alpha,$$

β, a and b being the arbitrary constants. On inserting (23) into the expressions for pressure and density, we get

$$8\pi p = D_1^{\ 2} \cos^2 \psi \left\{ \begin{array}{c} \dfrac{4\sqrt{2}\left(b - ar^2\right)}{\left(ar^2 + b\right)} \tan \phi \\[2mm] + \dfrac{12r^2}{\left(ar^2 + b\right)^2} \tan^2 \psi - 4ab \\[2mm] + \dfrac{16ar^2}{\left(ar^2 + b\right)} \tan \psi - 8 \tan \psi \\[2mm] - \dfrac{8\sqrt{2}\, r^2}{\left(ar^2 + b\right)^2} \tan \phi \tan \psi \end{array} \right\}$$

$$8\pi\,\rho = D_1^{\,2}\cos^2\psi\left\{\begin{array}{l}-\dfrac{12\left(b-ar^2\right)}{\left(ar^2+b\right)}\tan\psi\\[2mm]-\dfrac{12r^2}{\left(ar^2+b\right)^2}\tan^2\psi\\[2mm]-\dfrac{8\,r^2}{\left(ar^2+b\right)^2}+12ab\end{array}\right\} \tag{24}$$

The reality conditions, i.e. $p \succ 0$, $\rho \succ 0$ and $\rho - p \succ 0$ and $p_{rr} \prec 0$ at $r = 0$ imply

$$ab \succ -\tan\left(\frac{1}{ab}+\beta\right),$$

$$\sqrt{2}\tan\left(\frac{\sqrt{2}}{ab}+\alpha\right) \succ 2\tan\left(\frac{1}{ab}+\beta\right)+ab,$$

$$5\tan\left(\frac{1}{ab}+\beta\right)+4ab \succ \sqrt{2}\tan\left(\frac{\sqrt{2}}{ab}+\alpha\right),$$

$$-2\sqrt{2}\,ab\tan\left(\frac{\sqrt{2}}{ab}+\alpha\right)-2\tan^2\left(\frac{\sqrt{2}}{ab}+\alpha\right)$$
$$+2ab\tan\left(\frac{1}{ab}+\beta\right)-\tan^2\left(\frac{1}{ab}+\beta\right) \prec 0. \tag{25}$$

All the above inequalities hold good if we take

$$\frac{\tan^2\left(\dfrac{\sqrt{2}}{ab}+\alpha\right)+\tan^2\left(\dfrac{1}{ab}+\beta\right)}{\tan\left(\dfrac{1}{ab}+\beta\right)-\sqrt{2}\tan\left(\dfrac{\sqrt{2}}{ab}+\alpha\right)} \succ a\,b$$

$$\succ -\tan\left(\frac{1}{ab}+\beta\right). \tag{26}$$

Further, it is seen that the pressure and density gradients at the centre are vanishing and hence, we have the pressure and density, are maximum and will keep on decreasing towards the surface of the fluid sphere.

Solution (IV) Let us assume

$$\frac{2L''}{L} = -\frac{2b^2}{\left(x^2+a^2\right)^2},$$

and $\dfrac{G''}{G} = -\dfrac{2b^2}{\left(x^2+a^2\right)^2}.$ \hfill (27)

which integrates to

$$A = \frac{1}{L^2} = \frac{1}{E_1^{\,2}\left(r^4+a^2\right)\cos^2\psi},\quad C = \frac{G^2}{L^2} = \frac{E_2^{\,2}\cos^2\phi}{E_1^{\,2}\cos^2\psi} \tag{28}$$

where $\phi = \dfrac{\sqrt{a^2+2b^2}}{a}\tan^{-1}\left(\dfrac{r^2}{a}\right)+\alpha,$

$\psi = \dfrac{\sqrt{a^2+b^2}}{a}\tan^{-1}\left(\dfrac{r^2}{a}\right)+\beta;$ E_1, E_2, α, β, a and

b are arbitrary constants.

The pressure and density of the fluid can be computed as:

$$8\pi p = 4E_1^{\,2}\cos^2\psi\left\{\begin{array}{l}-\sqrt{a^2+2b^2}\,\tan\phi\\[1mm]+2\sqrt{a^2+b^2}\,\tan\psi - r^2\\[1mm]+\dfrac{r^4}{r^4+a^2}\left(\begin{array}{l}r^2+2\sqrt{a^2+2b^2}\,\tan\phi\\-3\sqrt{a^2+b^2}\,\tan\psi\end{array}\right)\\[2mm]\times\left(r^2-\sqrt{a^2+b^2}\,\tan\psi\right)\end{array}\right\}$$

$$8\pi\,\rho = 4E_1^{\,2}\cos^2\psi\left\{\begin{array}{l}\dfrac{3r^2\left(a^2+b^2\right)}{r^4+a^2}\tan^2\psi-\dfrac{5r^6}{r^4+a^2}\\[2mm]+3\sqrt{a^2+b^2}\,\tan\psi+\dfrac{2r^2\left(a^2+b^2\right)}{r^4+a^2}\\[2mm]-\dfrac{6r^4\sqrt{a^2+b^2}}{r^4+a^2}\tan\psi+5r^2\end{array}\right\} \tag{29}$$

Positivity of p, ρ and $\rho - p$ at the centre $\left(r = 0\right)$ requires:

$$\left(p \right)_{r=0} = 4E_1^{\,2}\cos^2\alpha \left[\begin{array}{c} -\sqrt{a^2+2b^2}\,\tan\beta \\ +2\sqrt{a^2+b^2}\,\tan\alpha \end{array} \right] \succ 0 ,$$

$$\left(\rho \right)_{r=0} = -4E_1^{\,2}\cos^2\alpha \left[3\sqrt{a^2+b^2}\,\tan\alpha \right] \succ 0 ,$$

$$\left(\rho - p \right)_{r=0} = 4E_1^{\,2}\cos^2\alpha \left[\begin{array}{c} \sqrt{a^2+2b^2}\,\tan\beta \\ -5\sqrt{a^2+b^2}\,\tan\alpha \end{array} \right] \succ 0$$

(30)

which further demands: $\tan\alpha \prec 0, \ \tan\beta \prec 0$ and

$$2 \prec \sqrt{\frac{a^2+2b^2}{a^2+b^2}}\,\frac{\tan\beta}{\tan\alpha} \prec 5 \qquad (31)$$

From (29), we have

$$p_r = 0 = \rho_r \ \text{and} \ \rho_{rr}, \ p_{rr} \prec 0 \ \text{at} \ r = 0 .$$

Therefore, we can say that the density and pressure are maximum at the centre and will go on decreasing as one approaches the surface of the fluid sphere.

IV. CONCLUSION

In the preceding section, we conclude that all the solutions have been derived by making some suitable arrangements in the field equations. As a result, some new types of static perfect fluid spheres together with the well known, e.g. Schwarzschild interior solutions are presented. The solutions are subjected to the reality conditions. It has been found that the pressure and density gradients at the centre are vanishing, which is a suitable criterion for a physically acceptable model. Further, it is established that the pressure and density keep on decreasing with the increase of radius, except for the case in which $\rho_{rr}, \ p_{rr} \succ 0$.

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