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Comparison results of two-point fuzzy boundary value problems

Hsuan-Ku Liu

Abstract—This paper investigates the solutions of two-point fuzzy boundary value problems as the form $x'' = f(t, x(t))$, $x(0) = A$ and $x(l) = B$, where A and B are fuzzy numbers. There are four different solutions for the problems when the lateral type of H-derivative is employed to solve the problems. As $f(t, x)$ is a monotone function of x, these four solutions are reduced to two different solutions. As $f(t, x(t)) = \lambda x(t)$ or $f(t, x(t)) = -\lambda x(t)$, solutions and several comparison results are presented to indicate advantages of each solution.

Keywords—fuzzy derivative, lateral type of H-derivative, fuzzy differential equations, fuzzy boundary value problems, boundary value problems

I. Introduction

By using the H-derivative [9], several articles [5, 6, 9] have been denoted to the study of the solution of fuzzy differential equations. However, this approach has a disadvantage that it leads to solutions with increasing support [1]. To solve the disadvantage, Bede and Gal [1] introduced a generalized definition of fuzzy derivative for a fuzzy-number-valued function. He showed that the new generalization allows us to have $f'(x) = c \cdot g'(x)$ for all $x \in (a, b)$ when $g : [a, b] \to R$ is differentiable and $f(x) = c \cdot g(x)$, where c is a fuzzy number. Now, the generalized definition of fuzzy derivative has throw a new light on the subject of fuzzy differential equations.

Following the line of Bede and Gal [1], Chalco-Cano and Román-Flores [4] rewrote the generalized definition of fuzzy derivative to a lateral type of H-derivative. Under the lateral type of H-derivative, there usually exists two solutions of first order fuzzy initial value problems [3] and four solutions of second order initial value problems [7]. For the first order fuzzy initial value problems, the support of one solution become bigger and bigger while those of another solutions become smaller and smaller. For second-order initial value problems, a study on the support of solutions has been made by Liu [8].

So far, several published results are proposed to investigate the solution of two-point fuzzy boundary value problems (FBVPs). O'Regan et. al. [10] showed that a two-point FBVP is equivalent to a fuzzy integral equation. Bede [2] gave a counterexample to show that this statement does not hold and argued that two-point FBVPs have no solution in many solutions. To solve the inconvenient, Chen et al. [5] provided a proof of this statement under certain conditions. However, we did not find any published results which study on the support of solutions for two-point fuzzy boundary value problems. Hence, to investigate the support of solutions such that the solutions satisfy real world models becomes an interesting and important problem on the filed of fuzzy differential equations.

H.-K. Liu, Department of Mathematics and Information Education, National Taipei University of Education, Taiwan, Tel: 886-2-27321104 ext. 2321, Fax: 886-2-27373549 e-mail:HKLiu.nccu@gmail.com

In the current work, an investigation is made on the solution of two-point FBVPs by using the lateral type of H-derivative. To put it precisely, the two-point FBVP is given as the form

$$x''(t) = f(t, x(t)), \quad x(0) = A, \quad \text{and} \quad x(l) = B, \qquad (1)$$

where $t \in T = [0, l]$ and $[A]^\alpha = [\underline{A}_\alpha, \overline{A}_\alpha]$ and $[B]^\alpha = [\underline{B}_\alpha, \overline{B}_\alpha]$ are fuzzy numbers. There are four different solutions of the FBVP when the fuzzy derivative is consider by the lateral type of H-derivative [4]. As f is a monotone function of x, these four solutions are reduced to two different solutions. As the FBPVs are given as the form

$$x''(t) = \lambda x(t), \qquad x(0) = A \quad \text{and} \quad x(l) = B, \qquad (2)$$

$$x''(t) = -\lambda x(t), \qquad x(0) = A \quad \text{and} \quad x(l) = B, \qquad (3)$$

fuzzy solutions are developed, where the boundary conditions A and B are symmetric triangle fuzzy numbers. To make advantages of each solution clearly, comparison results on the properties of valid fuzzy level set and preserving shape of boundaries are provided to these two problems (2) and (3). That is, as boundary values are symmetric triangle fuzzy numbers, we show that all solutions are symmetric triangle fuzzy function of t but that some solutions are no longer a valid fuzzy level set. Several examples are presented to indicate advantages and the inconvenient of each solution clearly.

This paper is organized as follows. In Section 2, the basic results of the fuzzy numbers and fuzzy calculus are discussed. In Section 3, the four solutions obtained by the lateral type of H-derivative are reduced to two different solutions as $f(t, x)$ is a monotone function of x. In Section 4, solutions and comparison results for the given two-point FBVPs are provided. In Section 5, we close this paper with a concise conclusion.

II. Preliminary and generalized definition

There are various definitions for the concept of fuzzy numbers [1, 5]. In this section, an appropriate brief introduction to preliminary topics such as fuzzy numbers and fuzzy calculus will be introduced.

Definition 2.1: A fuzzy number is a function $u : R \to [0, 1]$ satisfying the following properties:

1) u is normal;
2) u is convex fuzzy set;
3) u is upper semi-continuous on R;

4) $\overline{\{x \in R^n | u(x) \geq 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

Definition 2.2: A fuzzy number in parametric form is presented by an ordered pair of functions $(\underline{u}_\alpha, \overline{u}_\alpha)$, $0 \leq \alpha \leq 1$, satisfying the following requirements:

1) $\underline{u}_\alpha$ is a bounded left-continuous non-decreasing function of α over $[0,1]$.
2) $\overline{u}_\alpha$ is a bounded left-continuous non-increasing function of α over $[0,1]$.
3) $\underline{u}_\alpha \leq \overline{u}_\alpha$, $0 \leq \alpha \leq 1$.

Definition 2.3: Let u be a fuzzy set in R^n. The α-level set of u, denoted $[u]^\alpha, 0 \leq \alpha \leq 1$, is

$$[u]^\alpha = \{x \in R^n | u(x) \geq \alpha\}.$$

If $\alpha = 0$, the support of u is defined as

$$[u]^0 = \overline{\{x \in R^n | u(x) \geq 0\}}.$$

Definition 2.4: If A is a symmetric triangular numbers with supports $[\underline{a}, \overline{a}]$, the α-level sets of $[A]^\alpha$ is

$$[A]^\alpha = \left[\underline{a} + (\frac{\overline{a}-\underline{a}}{2})\alpha, \overline{a} - (\frac{\overline{a}-\underline{a}}{2})\alpha\right].$$

If there is a $\alpha \in [0,1]$ such that $\underline{u}_\alpha > \overline{u}_\alpha$ in Definition 2.2, then the order pair $(\underline{u}_\alpha, \overline{u}_\alpha)$ is not a valid fuzzy level set.

It is well-known that the H-difference [9] for fuzzy sets was initially introduced by Hukuhara as follows.

Definition 2.5: Let u and v be two fuzzy sets. If there exists a fuzzy set w such that $u = v + w$, then w is called the H-difference of u and v and it is denoted by $u \underset{H}{-} v$.

The H-derivative for fuzzy mapping is based in the H-difference of sets.

Definition 2.6: Let $T = (a,b)$ and $F : (a,b) \to \mathcal{F}$ be a fuzzy mapping. We say that F is differentiable at $t_0 \in T$ if there exists an element $F'(t_0) \in \mathcal{F}$ such that the limits

$$\lim_{h \to o^+} \frac{F(t_0+h) - F(t_0)}{h} = \lim_{h \to o^-} \frac{F(t_0) - F(t_0-h)}{h}$$

exits and are equal to $F'(t_0)$.

However, Bede and Gel showed that if $f(t) = c \cdot g(t)$, where c is a fuzzy number and $g : [a,b] \to R$ is a real-valued function with $g'(t_0) < 0$, then f is not differential. To overcome the inconvenient, they introduced a generalized definition of derivative for fuzzy mapping. We consider the following definition which is called a lateral type of H-derivative in [4].

Definition 2.7: Let $T = (a,b)$ and $F : (a,b) \to \mathcal{F}$ be a fuzzy mapping. A F is differentiable at $t_0 \in T$ if:

1) there exists an element $F'(t_0) \in \mathcal{F}$ such that, for all $h > 0$ sufficiently near to 0, we have $F(t_0+h) - F(t_0)$, $F(t_0) - F(t_0-h)$ and the limits
$$\lim_{h \to o^+} \frac{F(t_0+h) - F(t_0)}{h} = \lim_{h \to o^-} \frac{F(t_0) - F(t_0-h)}{h}$$
exists and are equal to $F'(t_0)$.
2) there exists an element $F'(t_0) \in \mathcal{F}$ such that, for all $h < 0$ sufficiently near to 0, we have $F(t_0+h) - F(t_0)$, $F(t_0) - F(t_0-h)$ and the limits
$$\lim_{h \to o^+} \frac{F(t_0+h) - F(t_0)}{h} = \lim_{h \to o^-} \frac{F(t_0) - F(t_0-h)}{h}$$
exists and are equal to $F'(t_0)$.

When f is a fuzzy-value function, Chalco-Cano and Román-Flores [4] got the following results.

Theorem 2.8: Let $f : T \to F$ be a function and denote $[f(t)]^\alpha = [\underline{f}_\alpha(t), \overline{f}_\alpha(t)]$, for each $\alpha \in [0,1]$. Then

1) If f is differentiable in the first form, then $\underline{f}_\alpha(t)$ and $\overline{f}_\alpha(t)$ are differentiable functions and $[f']\alpha = [\underline{f}'_\alpha(t), \overline{f}'_\alpha(t)]$.
2) If f is differentiable in the first form, then $\underline{f}_\alpha(t)$ and $\overline{f}_\alpha(t)$ are differentiable functions and $[f']\alpha = [\overline{f}'_\alpha(t), \underline{f}'_\alpha(t)]$.

In the following sections, the generalized definition of fuzzy derivative and Theorem 2.8 will be used to discuss fuzzy boundary value problems.

III. Two-points fuzzy boundary value problems

Consider the two-point FBVP

$$x''(t) = f(t, x(t)), \quad x(0) = A, \quad \text{and} \quad x(l) = B, \tag{1}$$

where $t \in T = [0,l]$ and $[A]^\alpha = [\underline{A}_\alpha, \overline{A}_\alpha]$ and $[B]^\alpha = [\underline{B}_\alpha, \overline{B}_\alpha]$ are fuzzy numbers.

Let $y = x'$. The FBVP can be transformed into a system of first-order fuzzy differential equations (FDEs)

$$\begin{cases} x' = & y \\ y' = & f(t, x(t)) \end{cases} \tag{4}$$

with $x(0) = A$ and $x(l) = B$. Denote

$$[x(t)]^\alpha = [\underline{x}_\alpha(t), \overline{x}_\alpha(t)], \qquad [y(t)]^\alpha = [\underline{y}_\alpha(t), \overline{y}_\alpha(t)]. \tag{5}$$

Since $x(t)$ and $y(t)$ are differentiable in first and second form by Theorem 2.8, there are four s, which are labeled as (i,j)-solution, i,j=1,2, to the system of FDEs (4). Here, the (i,j)-solution means that x is differentiable in the i-th form and y is differentiable in the j-th form, i,j=1,2.

Theorem 3.1: Let $[x(t)]^\alpha = [\underline{x}_\alpha(t), \overline{x}_\alpha(t)]$ be a solution of (1).

(i) As f is a monotone increasing function of x, the lower and upper solutions, $\underline{x}_\alpha(t)$ and $\overline{x}_\alpha(t)$, solve the system

$$\begin{cases} \underline{x}''_\alpha(t) = f(t, \underline{x}_\alpha(t)), & \underline{x}_\alpha(0) = \underline{A}_\alpha \text{ and } \underline{x}_\alpha(l) = \underline{B}_\alpha; \\ \overline{x}''_\alpha(t) = f(t, \overline{x}_\alpha(t)), & \overline{x}_\alpha(0) = \overline{A}_\alpha \text{ and } \overline{x}_\alpha(l) = \overline{B}_\alpha, \end{cases}$$

for (1,1)-solution and (2,2)-solution; and the lower and upper solutions, $\underline{x}_\alpha(t)$ and $\overline{x}_\alpha(t)$, solve the system

$$\begin{cases} \underline{x}''_\alpha(t) = f(t, \overline{x}_\alpha(t)), & \underline{x}_\alpha(0) = \underline{A}_\alpha \text{ and } \underline{x}_\alpha(l) = \underline{B}_\alpha; \\ \overline{x}''_\alpha(t) = f(t, \underline{x}_\alpha(t)), & \overline{x}_\alpha(0) = \overline{A}_\alpha \text{ and } \overline{x}_\alpha(l) = \overline{B}_\alpha, \end{cases}$$

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for (1,2)-solution and (2,1)-solution.
(ii) As f is a monotone decreasing function of x, the lower and upper solutions, $\underline{x}_\alpha(t)$ and $\overline{x}_\alpha(t)$, solve the system

$$\begin{cases} \underline{x}''_\alpha(t) = f(t, \overline{x}_\alpha(t)), \quad \underline{x}_\alpha(0) = \underline{A}_\alpha \quad \text{and} \quad \underline{x}_\alpha(l) = \underline{B}_\alpha; \\ \overline{x}''_\alpha(t) = f(t, \underline{x}_\alpha(t)), \quad \overline{x}_\alpha(0) = \overline{A}_\alpha \quad \text{and} \quad \overline{x}_\alpha(l) = \overline{B}_\alpha, \end{cases}$$

for (1,1)-solution and (2,2)-solution; and the lower and upper solutions, $\underline{x}_\alpha(t)$ and $\overline{x}_\alpha(t)$, solve the system

$$\begin{cases} \underline{x}''_\alpha(t) = f(t, \underline{x}_\alpha(t)), \quad \underline{x}_\alpha(0) = \underline{A}_\alpha \quad \text{and} \quad \underline{x}_\alpha(l) = \underline{B}_\alpha; \\ \overline{x}''_\alpha(t) = f(t, \overline{x}_\alpha(t)), \quad \overline{x}_\alpha(0) = \overline{A}_\alpha \quad \text{and} \quad \overline{x}_\alpha(l) = \overline{B}_\alpha, \end{cases}$$

for (1,2)-solution and (2,1)-solution.

Proof: We only prove the solution of that f is a monotone increasing function of x. The similar idea can be applied to the discussion of the monotone decreasing function f.

As f is a monotone increasing function of x, we have $f(t, \underline{x}_\alpha) < f(t, \overline{x}_\alpha)$ and

$$[f(t, x(t))]^\alpha = [f(t, \underline{x}_\alpha(t)), f(t, \overline{x}_\alpha(t))], \tag{6}$$

For (1,1)-solution, x and y are differentiable in the first form; that is

$$\begin{aligned} [x'(t)]^\alpha &= [\underline{x}'_\alpha(t), \overline{x}'_\alpha(t)], \\ [y'(t)]^\alpha &= [\underline{y}'_\alpha(t), \overline{y}'_\alpha(t)]. \end{aligned}$$

Substituting (5) and (6) into (4), we have

$$\begin{aligned} &[\underline{x}'_\alpha(t), \overline{x}'_\alpha(t)] = \left[\underline{y}_\alpha(t), \overline{y}_\alpha(t)\right], \\ &\left[\underline{y}'_\alpha(t), \overline{y}'_\alpha(t)\right] = \left[f(t, \underline{x}'_\alpha(t)), f(t, \overline{x}'_\alpha(t))\right], \end{aligned}$$

Hence to solve the fuzzy differential system (4) becomes to solve a real-valued system of the differential equation (DE) as below:

$$\underline{x}'_\alpha(t) = \underline{y}_\alpha(t), \tag{7}$$
$$\overline{x}'_\alpha(t) = \overline{y}_\alpha(t), \tag{8}$$
$$\underline{y}'_\alpha(t) = f(t, \underline{x}_\alpha(t)), \tag{9}$$
$$\overline{y}'_\alpha(t) = f(t, \overline{x}_\alpha(t)). \tag{10}$$

Differentiating (7) and (8) yields

$$\underline{x}''_\alpha(t) = \underline{y}'_\alpha \text{ and } \overline{x}''_\alpha(t) = \overline{y}'_\alpha(t). \tag{11}$$

Substituting (11) into (9) and (10) respectively, we get

$$\begin{cases} \underline{x}''_\alpha(t) = f(t, \underline{x}_\alpha(t)), \\ \overline{x}''_\alpha(t) = f(t, \overline{x}_\alpha(t)). \end{cases} \tag{12}$$

Since $[A]^\alpha = [\underline{A}_\alpha, \overline{A}_\alpha]$ and $[B]^\alpha = [\underline{B}_\alpha, \overline{B}_\alpha]$, boundary conditions of (12) are $\underline{x}_\alpha(0) = \underline{A}_\alpha$, $\underline{x}_\alpha(l) = \underline{B}_\alpha$, $\overline{x}_\alpha(0) = \overline{A}_\alpha$ and $\overline{x}_\alpha(l) = \overline{B}_\alpha$.

For (1,2)-solution, x is differentiable in the first form and y is differentiable in the second form; that is

$$\begin{aligned} [x'(t)]^\alpha &= [\underline{x}'_\alpha(t), \overline{x}'_\alpha(t)], \\ [y'(t)]^\alpha &= [\overline{y}'_\alpha(t), \underline{y}'_\alpha(t)]. \end{aligned}$$

This becomes to solve the differential system

$$\begin{aligned} &[\underline{x}'_\alpha(t), \overline{x}'_\alpha(t)] = \left[\underline{y}_\alpha(t), \overline{y}_\alpha(t)\right], \\ &\left[\underline{y}'_\alpha(t), \overline{y}'_\alpha(t)\right] = \left[f(t, \overline{x}'_\alpha(t)), f(t, \underline{x}'_\alpha(t))\right]. \end{aligned}$$

Using the same procedure in (1,1)-solution produces the following systems of BVPs

$$\begin{aligned} \underline{x}''_\alpha(t) &= f(t, \underline{x}_\alpha(t)), \quad \underline{x}_\alpha(0) = \underline{A}_\alpha, \quad \underline{x}_\alpha(l) = \underline{B}_\alpha \\ \overline{x}''_\alpha(t) &= f(t, \overline{x}_\alpha(t)), \quad \overline{x}_\alpha(0) = \overline{A}_\alpha, \quad \overline{x}_\alpha(l) = \overline{B}_\alpha. \end{aligned} \tag{13}$$

For (2,1)-solution, x is differentiable in the second form and y is differentiable in the first form; that is

$$\begin{aligned} [x'(t)]^\alpha &= [\overline{x}'_\alpha(t), \underline{x}'_\alpha(t)], \\ [y'(t)]^\alpha &= \left[\underline{y}'_\alpha(t), \overline{y}'_\alpha(t)\right]. \end{aligned}$$

This becomes to solve the differential system

$$\begin{aligned} &[\overline{x}'_\alpha(t), \underline{x}'_\alpha(t)] = \left[\underline{y}_\alpha(t), \overline{y}_\alpha(t)\right], \\ &\left[\underline{y}'_\alpha(t), \overline{y}'_\alpha(t)\right] = \left[f(t, \overline{x}'_\alpha(t)), f(t, \underline{x}'_\alpha(t))\right]. \end{aligned}$$

Using the same procedure in (1,2)-solution produces the following systems of BVPs

$$\begin{aligned} \underline{x}''_\alpha(t) &= f(t, \underline{x}_\alpha(t)), \quad \underline{x}_\alpha(0) = \underline{A}_\alpha, \quad \underline{x}_\alpha(l) = \underline{B}_\alpha \\ \overline{x}''_\alpha(t) &= f(t, \overline{x}_\alpha(t)), \quad \overline{x}_\alpha(0) = \overline{A}_\alpha, \quad \overline{x}_\alpha(l) = \overline{B}_\alpha. \end{aligned}$$

For (2,2)-solution, x and y are differentiable in the second form; that is

$$\begin{aligned} x'(t)^\alpha &= [\overline{x}'_\alpha(t), \underline{x}'_\alpha(t)], \\ y'(t)^\alpha &= [\overline{y}'_\alpha(t), \underline{y}'_\alpha(t)]. \end{aligned}$$

This becomes to solve the differential system

$$\begin{aligned} &[\overline{x}'_\alpha(t), \underline{x}'_\alpha(t)] = \left[\underline{y}_\alpha(t), \overline{y}_\alpha(t)\right], \\ &\left[\overline{y}'_\alpha(t), \underline{y}'_\alpha(t)\right] = \left[f(t, \overline{x}'_\alpha(t)), f(t, \underline{x}'_\alpha(t))\right]. \end{aligned}$$

Using the same procedure in (1,1)-solution produces the following systems of BVPs

$$\begin{aligned} \underline{x}''_\alpha(t) &= f(t, \underline{x}_\alpha(t)), \quad \underline{x}_\alpha(0) = \underline{A}_\alpha, \quad \underline{x}_\alpha(l) = \underline{B}_\alpha \\ \overline{x}''_\alpha(t) &= f(t, \overline{x}_\alpha(t)), \quad \overline{x}_\alpha(0) = \overline{A}_\alpha, \quad \overline{x}_\alpha(l) = \overline{B}_\alpha. \end{aligned}$$

■

IV. Second order fuzzy linear differential equations with constant coefficients

A. The case of positive constant coefficients

Consider the FBVP

$$\begin{cases} x''(t) = \lambda x(t), \\ x(0) = A, \quad \text{and} \quad x(l) = B, \end{cases} \tag{2}$$

where $\lambda > 0$. Boundary conditions $[A]^\alpha = [\underline{a} + (\frac{\overline{a}-\underline{a}}{2})\alpha, \overline{a} - (\frac{\overline{a}-\underline{a}}{2})\alpha]$ and $[B]^\alpha = [\underline{b} + (\frac{\overline{b}-\underline{b}}{2})\alpha, \overline{b} - (\frac{\overline{b}-\underline{b}}{2})\alpha]$ are symmetric triangular numbers.

Theorem 4.1: 1) For the (1,1)-solution and (2,2)-solution, the lower and upper solutions are

$$\begin{aligned} \underline{x}(t) &= \underline{c}_1 e^{\sqrt{\lambda} t} + \underline{c}_2 e^{-\sqrt{\lambda} t}, \\ \overline{x}(t) &= \overline{c}_1 e^{\sqrt{\lambda} t} + \overline{c}_2 e^{-\sqrt{\lambda} t}, \end{aligned}$$

where

$$\begin{aligned}\underline{c}_1 &= \frac{\left(\underline{b}+(\frac{\overline{b}-\underline{b}}{2})\alpha\right)-\left(\underline{a}+(\frac{\overline{a}-\underline{a}}{2})\alpha\right)e^{-\sqrt{\lambda}l}}{e^{\sqrt{\lambda}l}-e^{-\sqrt{\lambda}l}},\\ \underline{c}_2 &= \frac{\left(\underline{a}+(\frac{\overline{a}-\underline{a}}{2})\alpha\right)e^{\sqrt{\lambda}l}-\left(\underline{b}+(\frac{\overline{b}-\underline{b}}{2})\alpha\right)}{e^{\sqrt{\lambda}l}-e^{-\sqrt{\lambda}l}},\\ \overline{c}_1 &= \frac{\left(\overline{b}-(\frac{\overline{b}-\underline{b}}{2})\alpha\right)-\left(\overline{a}-(\frac{\overline{a}-\underline{a}}{2})\alpha\right)e^{-\sqrt{\lambda}l}}{e^{\sqrt{\lambda}l}-e^{-\sqrt{\lambda}l}},\\ \overline{c}_2 &= \frac{\left(\overline{a}-(\frac{\overline{a}-\underline{a}}{2})\alpha\right)e^{\sqrt{\lambda}l}-\left(\overline{b}-(\frac{\overline{b}-\underline{b}}{2})\alpha\right)}{e^{\sqrt{\lambda}l}-e^{-\sqrt{\lambda}l}}.\end{aligned}$$

2) For the (1,2)-solution and (2,1)-solution, the lower and upper solutions are

$$\underline{x}_\alpha(t) = a_1^\alpha e^{\sqrt{\lambda}t}+a_2^\alpha e^{-\sqrt{\lambda}t}-a_3^\alpha \sin(\sqrt{\lambda}t)-a_4^\alpha \cos(\sqrt{\lambda}t), \quad (14)$$

$$\overline{x}_\alpha(t) = a_1^\alpha e^{\sqrt{\lambda}t}+a_2^\alpha e^{-\sqrt{\lambda}t}+a_3^\alpha \sin(\sqrt{\lambda}t)+a_4^\alpha \cos(\sqrt{\lambda}t), \quad (15)$$

where a_1, a_2, a_3 and a_4 are

$$\begin{aligned}a_1^\alpha &= \frac{(\overline{b}+\underline{b})-e^{-\sqrt{\lambda}l}(\overline{a}+\underline{a})}{2(e^{\sqrt{\lambda}l}-e^{-\sqrt{\lambda}l})},\\ a_2^\alpha &= \frac{e^{\sqrt{\lambda}l}(\overline{a}+\underline{a})-(\overline{b}+\underline{b})}{2(e^{\sqrt{\lambda}l}-e^{-\sqrt{\lambda}l})},\\ a_3^\alpha &= \left(\frac{1-\alpha}{2}\right)\left(\frac{(\overline{b}-\underline{b})-(\overline{a}-\underline{a})\cos(\sqrt{\lambda}l)}{\sin(\sqrt{\lambda}l)}\right),\\ a_4^\alpha &= \left(\frac{1-\alpha}{2}\right)(\overline{a}-\underline{a}).\end{aligned}$$

Proof: Using Theorem 3.1, the lower solution and the upper solution of (2), satisfy the following differential equations

$$\begin{cases}\underline{x}''_\alpha(t) = \lambda\underline{x}_\alpha(t), \underline{x}_\alpha(0) = \underline{a}+(\frac{\overline{a}-\underline{a}}{2})\alpha, \underline{x}_\alpha(l) = \underline{b}+(\frac{\overline{b}-\underline{b}}{2})\alpha,\\ \overline{x}''_\alpha(t) = \lambda\overline{x}_\alpha(t), \overline{x}_\alpha(0) = \overline{a}-(\frac{\overline{a}-\underline{a}}{2})\alpha,\ \overline{b}-(\frac{\overline{b}-\underline{b}}{2})\alpha,\end{cases}$$

respectively.

Hence the solution can be obtained as

$$\begin{aligned}\underline{x}(t) &= \underline{c}_1 e^{\sqrt{\lambda}t}+\underline{c}_2 e^{-\sqrt{\lambda}t},\\ \overline{x}(t) &= \overline{c}_1 e^{\sqrt{\lambda}t}+\overline{c}_2 e^{-\sqrt{\lambda}t}.\end{aligned}$$

On the other hand, the FBVP (2) is transformed into a linear system of real-valued DEs

$$\begin{cases}\overline{x}''_\alpha(t) = \lambda\underline{x}_\alpha(t),\\ \underline{x}''_\alpha(t) = \lambda\overline{x}_\alpha(t),\end{cases} \quad (16)$$

with

$$\underline{x}_\alpha(0) = \underline{a}+(\frac{\overline{a}-\underline{a}}{2})\alpha \text{ and } \underline{x}_\alpha(l) = \underline{b}+(\frac{\overline{b}-\underline{b}}{2})\alpha, \quad (17)$$

$$\overline{x}_\alpha(0) = \overline{a}-(\frac{\overline{a}-\underline{a}}{2})\alpha \text{ and } \overline{x}_\alpha(l) = \overline{b}-(\frac{\overline{b}-\underline{b}}{2})\alpha. \quad (18)$$

The solutions of (16) are given as

$$\underline{x}_\alpha(t) = a_1^\alpha e^{\sqrt{\lambda}t}+a_2^\alpha e^{-\sqrt{\lambda}t}-a_3^\alpha \sin(\sqrt{\lambda}t)-a_4^\alpha \cos(\sqrt{\lambda}t), \quad (19)$$

$$\overline{x}_\alpha(t) = a_1^\alpha e^{\sqrt{\lambda}t}+a_2^\alpha e^{-\sqrt{\lambda}t}+a_3^\alpha \sin(\sqrt{\lambda}t)+a_4^\alpha \cos(\sqrt{\lambda}t). \quad (20)$$

Substituting boundary conditions (17) and (18) into (19) and (20) respectively, coefficients a_1, a_2, a_3 and a_4 are solved as

$$\begin{aligned}a_1^\alpha &= \frac{(\overline{b}+\underline{b})-e^{-\sqrt{\lambda}l}(\overline{a}+\underline{a})}{2(e^{\sqrt{\lambda}l}-e^{-\sqrt{\lambda}l})},\\ a_2^\alpha &= \frac{e^{\sqrt{\lambda}l}(\overline{a}+\underline{a})-(\overline{b}+\underline{b})}{2(e^{\sqrt{\lambda}l}-e^{-\sqrt{\lambda}l})},\\ a_3^\alpha &= \left(\frac{1-\alpha}{2}\right)\left(\frac{(\overline{b}-\underline{b})-(\overline{a}-\underline{a})\cos(\sqrt{\lambda}l)}{\sin(\sqrt{\lambda}l)}\right),\\ a_4^\alpha &= \left(\frac{1-\alpha}{2}\right)(\overline{a}-\underline{a}).\end{aligned}$$

■

Proposition 4.2: 1) For (1,1)-solution and (2,2)-solution, the solution $[x(t)]^\alpha = [\underline{x}_\alpha(t), \overline{x}_\alpha(t)]$ of (2) is a valid fuzzy level set if $(\overline{b}-\underline{b}) \geq (\overline{a}-\underline{a})e^{-\sqrt{\lambda}l}$.

2) For (1,2)-solution and (2,1)-solution, the solution $[x(t)]^\alpha = [\underline{x}_\alpha(t), \overline{x}_\alpha(t)]$ of (2) is no longer a valid fuzzy level set if

$$t > \frac{1}{\sqrt{\lambda}}\cot^{-1}(-\left(\frac{(\overline{b}-\underline{b})-(\overline{a}-\underline{a})\cos(\sqrt{\lambda}l)}{\overline{a}-\underline{a}\sin(\sqrt{\lambda}l)}\right)).$$

Proof: Given $t \in [0, l]$,

$$\begin{aligned}&\overline{x}_\alpha(t)-\underline{x}_\alpha(t)\\ &= (\overline{c}_1-\underline{c}_1)e^{\sqrt{\lambda}t}+(\overline{c}_2-\underline{c}_2)e^{-\sqrt{\lambda}t}\\ &= e^{-\sqrt{\lambda}t}[(\overline{c}_1-\underline{c}_1)e^{2\sqrt{\lambda}t}+(\overline{c}_2-\underline{c}_2)].\end{aligned}$$

Let $f(t) = (\overline{c}_1-\underline{c}_1)e^{2\sqrt{\lambda}t}+(\overline{c}_2-\underline{c}_2)$ then $f(0) = \overline{a}-\underline{a} > 0$ and

$$\begin{aligned}f'(t) &= (2\sqrt{\lambda}\overline{c}_1-\underline{c}_1)e^{2\sqrt{\lambda}t})\\ &= 2\sqrt{\lambda}\left(\frac{\overline{b}-\underline{b})-(\overline{a}-\underline{a})e^{-\sqrt{\lambda}l}}{e^{-\sqrt{\lambda}l}-e^{-\sqrt{\lambda}l}}\right)e^{2\sqrt{\lambda}t} > 0,\end{aligned}$$

if $(\overline{b}-\underline{b}) \geq (\overline{a}-\underline{a})e^{-\sqrt{\lambda}l}$.

For (1,2)-solution and (2,1)-solution, the difference between $\overline{x}_\alpha$ and $\underline{x}_\alpha$ is $\overline{x}_\alpha(t)-\underline{x}_\alpha(t) = 2a_3^\alpha\sin(\sqrt{\lambda}t)+2a_4^\alpha\cos(\sqrt{\lambda}t)$. The solution of this inequality $\overline{x}_\alpha(t)-\underline{x}_\alpha(t) \geq 0$ can be obtained by solving

$$a_3^\alpha\sin(\sqrt{\lambda}t) \geq -a_4^\alpha\cos(\sqrt{\lambda}t).$$

As $\sqrt{\lambda}t \in (0,\pi)$, we have $\sin(\sqrt{\lambda}t) > 0$. Since $-a_4^\alpha = -(\frac{1-\alpha}{2})(\overline{a}-\underline{a}) < 0$, the inequality is transformed to

$$-\frac{a_3^\alpha}{a_4^\alpha} \leq \frac{\cos(\sqrt{\lambda}t)}{\sin(\sqrt{\lambda}t)} = \cot(\sqrt{\lambda}t);$$

that is

$$0 \leq t \leq \frac{1}{\sqrt{\lambda}}\cot^{-1}(-\frac{a_3^\alpha}{a_4^\alpha}),$$

where

$$\frac{a_3^\alpha}{a_4^\alpha} = \left(\frac{(\overline{b}-\underline{b})-(\overline{a}-\underline{a})\cos(\sqrt{\lambda}l)}{\overline{a}-\underline{a}\sin(\sqrt{\lambda}l)}\right).$$

Therefore, as $t > \frac{1}{\sqrt{\lambda}}\cot^{-1}(-\frac{a_3^\alpha}{a_4^\alpha})$ we have $\overline{x}_\alpha-\underline{x}_\alpha(t) < 0$. This implies that the solution $[x(t)]^\alpha = [\underline{x}_\alpha(t), \overline{x}_\alpha(t)]$ is not a valid fuzzy level set if $t > \frac{1}{\sqrt{\lambda}}\cot^{-1}(-\frac{a_3^\alpha}{a_4^\alpha})$. ■

Proposition 4.3: For any $t \in [0, l]$, the solution $[x(t)]^\alpha = [\underline{x}_\alpha(t), \overline{x}_\alpha(t)]$ of (2) is a symmetric triangle fuzzy number.

Proof: The solutions $[x(t)]^\alpha$ of (19) and (20) are triangle fuzzy numbers since solutions are a straight line of α. It

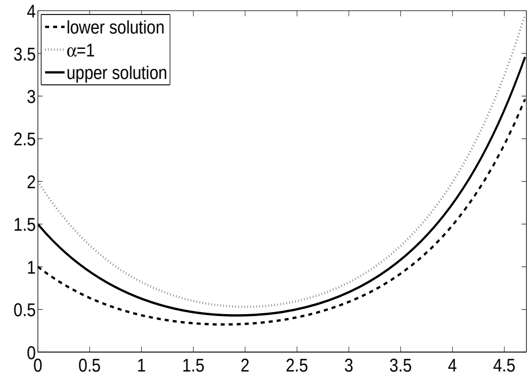

Fig. 1. The upper line, mid-line and lower line are the graphs of $\overline{x}_0(t)$, $\overline{x}_1(t) = \underline{x}_1(t)$ and $\underline{x}_0(t)$, respectively.

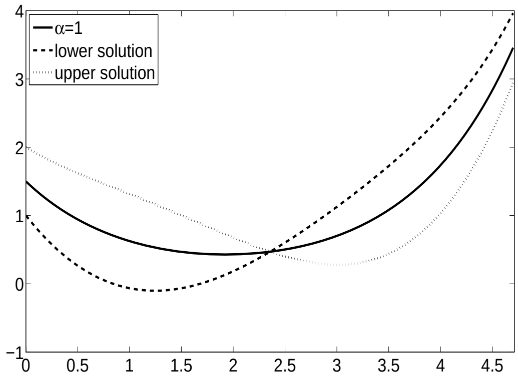

Fig. 2. The upper line, mid-line and lower line are the graphs of $\overline{x}_0(t)$, $\overline{x}_1(t) = \underline{x}_1(t)$ and $\underline{x}_0(t)$, respectively.

suffices to verify the symmetry of the solutions. For (1,1)-solution and (2,2)-solution, we have

$$\begin{aligned}&\underline{x}_1(t) - \underline{x}_\alpha(t)\\ &= \frac{(\frac{\overline{b}-\underline{b}}{2})(1-\alpha) - (\frac{\overline{a}-\underline{a}}{2})(1-\alpha)e^{-\sqrt{\lambda}l}}{e^{\sqrt{\lambda}l} - e^{-\sqrt{\lambda}l}} e^{\sqrt{\lambda}t}\\ &\quad + \frac{-(\frac{\overline{b}-\underline{b}}{2})(1-\alpha) + (\frac{\overline{a}-\underline{a}}{2})(1-\alpha)e^{\sqrt{\lambda}l}}{e^{\sqrt{\lambda}l} - e^{-\sqrt{\lambda}l}} e^{-\sqrt{\lambda}t}\\ &= \overline{x}_\alpha(t) - \overline{x}_1(t).\end{aligned}$$

For (1,2)-solution and (2,1)-solution, we have $\overline{x}_1(t) = a_1^\alpha e^{\sqrt{\lambda}t} + a_2^\alpha e^{-\sqrt{\lambda}t} = \underline{x}_1(t)$ and

$$\begin{aligned}\underline{x}_1(t) - \underline{x}_\alpha(t) &= a_3^\alpha \sin(\sqrt{\lambda}t) + a_4^\alpha \cos(\sqrt{\lambda}t)\\ &= \overline{x}_\alpha(t) - \overline{x}_1(t).\end{aligned}$$

Hence solutions (19) and (20) are symmetric fuzzy function of t. ■

Example 4.4: Consider the FBVP

$$\begin{cases} x''(t) = x(t), t \in (0, \frac{3}{2}\pi),\\ x(0) = [1 + \frac{1}{2}\alpha, 2 - \frac{1}{2}\alpha], \quad \text{and} \quad x(\frac{3\pi}{2}) = [3 + \frac{1}{2}\alpha, 4 - \frac{1}{2}\alpha].\end{cases}$$

For (1,1)-solution and (2,2)-solution, the fuzzy solution is obtained as

$$\begin{aligned}\underline{x}_\alpha(t) &= \underline{c}_1 e^t + \underline{c}_2 e^{-t},\\ \overline{x}_\alpha(t) &= \overline{c}_1 e^t + \overline{c}_2 e^{-t},\end{aligned}$$

where

$$\begin{aligned}\underline{c}_1 &= \frac{(3+\frac{1}{2}\alpha) - (1+\frac{1}{2}\alpha)e^{-\frac{3\pi}{2}}}{e^{\frac{3\pi}{2}} - e^{-\frac{3\pi}{2}}}, \qquad \underline{c}_2 = \frac{(1+\frac{1}{2}\alpha)e^{\frac{3\pi}{2}} - (3+\frac{1}{2}\alpha)}{e^{\frac{3\pi}{2}} - e^{-\frac{3\pi}{2}}},\\ \overline{c}_1 &= \frac{(4-\frac{1}{2}\alpha) - (2-\frac{1}{2}\alpha)e^{-\frac{3\pi}{2}}}{e^{\frac{3\pi}{2}} - e^{-\frac{3\pi}{2}}}, \qquad \underline{c}_2 = \frac{(2-\frac{1}{2}\alpha)e^{\frac{3\pi}{2}} - (4-\frac{1}{2}\alpha)}{e^{\frac{3\pi}{2}} - e^{-\frac{3\pi}{2}}}.\end{aligned}$$

For (1,2)-solution and (2,1)-solution, the fuzzy solution is obtained as

$$\begin{aligned}\underline{x}_\alpha = &\left(\frac{7-3e^{-\pi/2}}{e^{\pi/2}-e^{-\pi/2}}\right)e^t + \left(\frac{3e^{3\pi/2}-7}{e^{3\pi/2}-e^{-3\pi/2}}\right)e^{-t}\\ &- \left(\frac{1-\alpha}{2}\right)\sin(t) - \left(\frac{1-\alpha}{2}\right)\cos(t)\\ \overline{x}_\alpha = &\left(\frac{7-3e^{-3\pi/2}}{e^{3\pi/2}-e^{-3\pi/2}}\right)e^t + \left(\frac{3e^{3\pi/2}-7}{e^{3\pi/2}-e^{-3\pi/2}}\right)e^{-t}\\ &+ \left(\frac{1-\alpha}{2}\right)\sin(t) + \left(\frac{1-\alpha}{2}\right)\cos(t)\end{aligned}$$

For (1,1)-solution and (2,2)-solution, the graph of the fuzzy function $x(t)$ is displayed in Figure 1.

As we can see, the solution is a valid fuzzy level set and a symmetric fuzzy function for all $t \in [0, \frac{3\pi}{2}]$.

For (1,2)-solution and (2,1)-solution, the graph of the fuzzy function $x(t)$ is displayed in Figure 2.

As we can see, the solution is a not valid fuzzy level set.

B. The case of negative constant coefficients

Consider the FBVP

$$\begin{cases} x''(t) = -\lambda x(t),\\ x(0) = A, \quad \text{and} \quad x(l) = B,\end{cases} \tag{3}$$

where $\lambda > 0$ and $[A]^\alpha = [\underline{a} + (\frac{\overline{a}-\underline{a}}{2})\alpha, \overline{a} - (\frac{\overline{a}-\underline{a}}{2})\alpha]$ and $[B]^\alpha = [\underline{b} + (\frac{\overline{b}-\underline{b}}{2})\alpha, \overline{b} - (\frac{\overline{b}-\underline{b}}{2})\alpha]$ are symmetric fuzzy numbers.

Theorem 4.5:

1) For the (1,2)-solution and (2,1)-solution, the upper and lower solutions of (3) are

$$\begin{aligned}\underline{x}_\alpha(t) &= \left(\underline{a} + (\tfrac{\overline{a}-\underline{a}}{2})\alpha\right)\cos(\sqrt{\lambda}t) + \underline{c}\sin(\sqrt{\lambda}t),\\ \overline{x}_\alpha(t) &= \left(\overline{a} - (\tfrac{\overline{a}-\underline{a}}{2})\alpha\right)\cos(\sqrt{\lambda}t) + \overline{c}\sin(\sqrt{\lambda}t),\end{aligned}$$

where

$$\underline{c} = \frac{\left[\left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right) - \left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\cos(\sqrt{\lambda}l)\right]}{\sin(\sqrt{\lambda}l)},$$

$$\overline{c} = \frac{\left[\left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right) - \left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\cos(\sqrt{\lambda}l)\right]}{\sin(\sqrt{\lambda}l)}.$$

and $l \neq \frac{n\pi}{\sqrt{\lambda}}$, $n \in \mathbb{N}$;

2) For the (1,1)-solution and (2,2)-solution, the upper and lower solutions of (3) are

$$\underline{x}_\alpha(t) = -a_1^\alpha e^{\sqrt{\lambda}t} - a_2^\alpha e^{-\sqrt{\lambda}t} + a_3^\alpha \sin(\sqrt{\lambda}t) + a_4^\alpha \cos(\sqrt{\lambda}t), \tag{21}$$

$$\overline{x}_\alpha(t) = a_1^\alpha e^{\sqrt{\lambda}t} + a_2^\alpha e^{-\sqrt{\lambda}t} + a_3^\alpha \sin(\sqrt{\lambda}t) + a_4^\alpha \cos(\sqrt{\lambda}t), \tag{22}$$

where a_1, a_2, a_3 and a_4 are

$$\begin{aligned} a_1^\alpha &= \left(\tfrac{1-\alpha}{2}\right)\left(\tfrac{(\overline{b}-\underline{b})-e^{-\sqrt{\lambda}l}(\overline{a}-\underline{a})}{e^{\sqrt{\lambda}l}-e^{-\sqrt{\lambda}l}}\right)\\ a_2^\alpha &= \left(\tfrac{1-\alpha}{2}\right)\left((\overline{a}-\underline{a})-\tfrac{(\overline{b}-\underline{b})-e^{-\sqrt{\lambda}l}(\overline{a}-\underline{a})}{e^{\sqrt{\lambda}l}-e^{-\sqrt{\lambda}l}}\right)\\ a_3^\alpha &= \tfrac{(\overline{b}+\underline{b})-(\overline{a}+\underline{a})\cos(\sqrt{\lambda}l)}{2\sin(\sqrt{\lambda}l)},\\ a_4^\alpha &= \tfrac{\overline{a}+\underline{a}}{2}. \end{aligned}$$

Proof: Using Theorem 3.1, the lower solution $\underline{x}(t)$ and the upper solution $\overline{x}(t)$ satisfy

$$\begin{cases} \underline{x}''_\alpha(t) = -\lambda \underline{x}_\alpha(t), \\ \underline{x}_\alpha(0) = \underline{a} + (\frac{\overline{a}-\underline{a}}{2})\alpha,\ \underline{x}_\alpha(l) = \underline{b} + (\frac{\overline{b}-\underline{b}}{2})\alpha, \end{cases} \tag{23}$$

and

$$\begin{cases} \overline{x}''_\alpha(t) = -\lambda \overline{x}_\alpha(t), \\ \overline{x}_\alpha(0) = \overline{a} - (\frac{\overline{a}-\underline{a}}{2})\alpha,\ \overline{x}_\alpha(l) = \overline{b} - (\frac{\overline{b}-\underline{b}}{2})\alpha, \end{cases} \tag{24}$$

respectively.

Hence the solutions are

$$\begin{aligned} \underline{x}_\alpha(t) &= \left(\underline{a} + (\tfrac{\overline{a}-\underline{a}}{2})\alpha\right)\cos(\sqrt{\lambda}t) + \underline{c}\sin(\sqrt{\lambda}t),\\ \overline{x}_\alpha(t) &= \left(\overline{a} - (\tfrac{\overline{a}-\underline{a}}{2})\alpha\right)\cos(\sqrt{\lambda}t) + \overline{c}\sin(\sqrt{\lambda}t). \end{aligned}$$

On the other hand, the FBVP (3) is transformed into a linear system of real-valued DEs

$$\begin{cases} \overline{x}''_\alpha(t) = -\lambda \underline{x}_\alpha(t), \\ \underline{x}''_\alpha(t) = -\lambda \overline{x}_\alpha(t), \end{cases} \tag{25}$$

with

$$\underline{x}_\alpha(0) = \underline{a} + (\frac{\overline{a}-\underline{a}}{2})\alpha \text{ and } \underline{x}_\alpha(l) = \underline{b} + (\frac{\overline{b}-\underline{b}}{2})\alpha, \tag{26}$$

$$\overline{x}_\alpha(0) = \overline{a} - (\frac{\overline{a}-\underline{a}}{2})\alpha \text{ and } \overline{x}_\alpha(l) = \overline{b} - (\frac{\overline{b}-\underline{b}}{2})\alpha, \tag{27}$$

The solutions of (25) have been known as

$$\underline{x}_\alpha(t) = -a_1^\alpha e^{\sqrt{\lambda}t} - a_2^\alpha e^{-\sqrt{\lambda}t} + a_3^\alpha \sin(\sqrt{\lambda}t) + a_4^\alpha \cos(\sqrt{\lambda}t), \tag{28}$$

$$\overline{x}_\alpha(t) = a_1^\alpha e^{\sqrt{\lambda}t} + a_2^\alpha e^{-\sqrt{\lambda}t} + a_3^\alpha \sin(\sqrt{\lambda}t) + a_4^\alpha \cos(\sqrt{\lambda}t). \tag{29}$$

Substituting (26) and (27) into (28) and (29) respectively, coefficients a_1, a_2, a_3 and a_4 are obtained as

$$\begin{aligned} a_1^\alpha &= \left(\tfrac{1-\alpha}{2}\right)\left(\tfrac{(\overline{b}-\underline{b})-e^{-\sqrt{\lambda}l}(\overline{a}-\underline{a})}{e^{\sqrt{\lambda}l}-e^{-\sqrt{\lambda}l}}\right)\\ a_2^\alpha &= \left(\tfrac{1-\alpha}{2}\right)\left((\overline{a}-\underline{a})-\tfrac{(\overline{b}-\underline{b})-e^{-\sqrt{\lambda}l}(\overline{a}-\underline{a})}{e^{\sqrt{\lambda}l}-e^{-\sqrt{\lambda}l}}\right)\\ a_3^\alpha &= \tfrac{(\overline{b}+\underline{b})-(\overline{a}+\underline{a})\cos(\sqrt{\lambda}l)}{2\sin(\sqrt{\lambda}l)},\\ a_4^\alpha &= \tfrac{\overline{a}+\underline{a}}{2}. \end{aligned}$$

■

Proposition 4.6: For (1,2)-solution and (2,1)-solution, the solution $[x(t)]^\alpha = [\underline{x}_\alpha(t), \overline{x}_\alpha(t)]$ of (3) is no longer a valid fuzzy level set as

$$t > \frac{1}{\sqrt{\lambda}}\cot^{-1}\left(\frac{-c}{(1-\alpha)(\overline{a}-\underline{a})}\right),$$

where

$$c = \frac{\left[(\overline{b}-\underline{b})(1-\alpha) - (\overline{a}-\underline{a})(1-\alpha)\cos(\sqrt{\lambda}l)\right]}{\sin(\sqrt{\lambda}l)}.$$

For (1,1)-solution and (2,2)-solution, the solution $[x(t)]^\alpha = [\underline{x}_\alpha(t), \overline{x}_\alpha(t)]$ of (25) is a valid fuzzy level set for all $t \in [0, l]$ if $a_1^\alpha > 0$.

Proof: We first consider the (1,2)-solution and (2,1)-solution. To show that $[x(t)]^\alpha$ is a valid fuzzy level set, it suffices to show that the difference

$$z_\alpha(t) = \overline{x}_\alpha(t) - \underline{x}_\alpha(t) \geq 0$$

for all $t \in [0, l]$, $l \leq l^*$ and $\alpha \in [0, 1]$. The difference $z_\alpha(t)$ is obtained as

$$z_\alpha(t) = (\overline{a}-\underline{a})(1-\alpha)\cos(\sqrt{\lambda}t) - c\sin(\sqrt{\lambda}t),$$

where

$$c = \frac{\left[(\overline{b}-\underline{b})(1-\alpha) - (\overline{a}-\underline{a})(1-\alpha)\cos(\sqrt{\lambda}l)\right]}{\sin(\sqrt{\lambda}l)}$$

and $l \neq \frac{n\pi}{\sqrt{\lambda}}$, for all integer n. To find the support of $z_\alpha \geq 0$, it is equal to solve the inequality

$$(\overline{a}-\underline{a})(1-\alpha)\cos(\sqrt{\lambda}t) \geq -c\sin(\sqrt{\lambda}t).$$

As $t \in [0, \pi]$, we have

$$\cot(\sqrt{\lambda}t) = \frac{\cos(\sqrt{\lambda}t)}{\sin(\sqrt{\lambda}t)} \geq \frac{-c}{(1-\alpha)(\overline{a}-\underline{a})};$$

that is

$$0 \leq t \leq \frac{1}{\sqrt{\lambda}}\cot^{-1}\left(\frac{-c}{(1-\alpha)(\overline{a}-\underline{a})}\right).$$

Secondly, we consider the (1,1)-solution and (2,2)-solution. the difference of $\overline{x}_\alpha$ and $\underline{x}_\alpha$ is

$$\begin{aligned} \overline{x}_\alpha(t) - \underline{x}_\alpha(t) &= 2(a_1^\alpha e^{\sqrt{\lambda}t} + a_2^\alpha e^{-\sqrt{\lambda}t})\\ &= 2e^{-\sqrt{\lambda}t}(a_1^\alpha e^{2\sqrt{\lambda}t} + a_2^\alpha). \end{aligned}$$

Let $f(t) = a_1^\alpha e^{2\sqrt{\lambda}t} + a_2^\alpha$). Then $f(0) = \overline{a} - \underline{a} > 0$ and $f'(t) = 2\sqrt{\lambda}a_1^\alpha e^{2\sqrt{\lambda}t} > 0$ as $a_1^\alpha > 0$. Hence $f(t) \geq 0$ as $a_1^\alpha > 0$ since $f(t)$ is an increasing function with $f(0) > 0$. Therefore, $\overline{x}_\alpha(t) - \underline{x}_\alpha(t) = 2e^{-\sqrt{\lambda}t}f(t) > 0$ as $a_1^\alpha > 0$. ■

Remark 4.7: If $l > -\frac{1}{\sqrt{\lambda}}\log\frac{\overline{b}-\underline{b}}{\overline{a}-\underline{a}}$ then $a_1^\alpha > 0$.

Proposition 4.8: For (1,2)-solution and (2,1)-solution, the solution $[x(t)]^\alpha = [\underline{x}_\alpha(t), \overline{x}_\alpha(t)]$ of (25) is a symmetric triangle fuzzy number as $t \in [0, l]$. For (1,2)-solution and (2,1)-solution, the solution $[x(t)]^\alpha = [\underline{x}_\alpha(t), \overline{x}_\alpha(t)]$ of (25) is a symmetric fuzzy number as $t \in [0, l]$.

Proof: The solutions $[x(t)]^\alpha$ of (25) are triangle fuzzy numbers since solutions are a straight line of α. It suffices to verify the symmetry of the solutions. For (1,2)-solution and (2,1)-solution, given $t \in [0, l]$ fixed, we have

$$\begin{aligned} &\underline{x}_1(t) - \underline{x}_\alpha(t)\\ &= (\tfrac{\overline{a}-\underline{a}}{2})(1-\alpha)\cos(\sqrt{\lambda}t)\\ &\quad + \tfrac{(\frac{\overline{b}-\underline{b}}{2})(1-\alpha)-(\frac{\overline{a}-\underline{a}}{2})(1-\alpha)\cos(\sqrt{\lambda}l)}{\sin(\sqrt{\lambda}l)}\sin(\sqrt{\lambda}t)\\ &= \overline{x}_\alpha(t) - \overline{x}_1(t). \end{aligned}$$

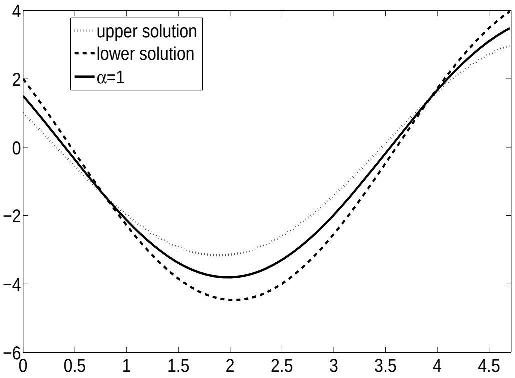

Fig. 3. The upper line, mid-line and lower line are the graphs of $\overline{x}_0(t)$, $\overline{x}_1(t) = \underline{x}_1$ and $\underline{x}_0$, respectively.

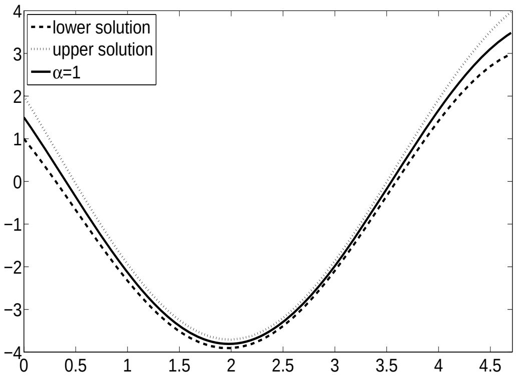

Fig. 4. The upper line, mid-line and lower line are the graphs of $\overline{x}_0(t)$, $\overline{x}_1(t) = \underline{x}_1$ and $\underline{x}_0$, respectively.

For (1,1)-solution and (2,2)-solution, given $t \in [0, l]$ fixed, we have

$$\underline{x}_1(t) - \underline{x}_\alpha(t) = a_1^\alpha e^{\sqrt{\lambda}t} + a_2^\alpha e^{-\sqrt{\lambda}t} = \overline{x}_\alpha(t) - \overline{x}_1(t)$$

since $\overline{x}_1(t) = \underline{x}_1(t) = a_3^\alpha \sin(\sqrt{\lambda}t) + a_4^\alpha \cos(\sqrt{\lambda}t)$. Hence the solution $[x(t)]^\alpha$ of (25) is a symmetric fuzzy number. ■

Example 4.9: Consider the FBVP

$$\begin{cases} x''(t) = -x(t), & t \in [0, \frac{3\pi}{2}], \\ x(0) = [1, 2], & \text{and} \quad x(\frac{3\pi}{2}) = [3, 4]. \end{cases}$$

The lower and upper solutions are obtained as

$$\begin{aligned} \underline{x}_\alpha(t) &= (1 + \tfrac{1}{2}\alpha)\cos(t) - (3 + \tfrac{1}{2}\alpha)\sin(t), \quad & t \in [0, \tfrac{3\pi}{2}], \\ \overline{x}_\alpha(t) &= (2 - \tfrac{1}{2}\alpha)\cos(t) - (4 - \tfrac{1}{2}\alpha)\sin(t), \quad & t \in [0, \tfrac{3\pi}{2}]. \end{aligned}$$

for (1,2)-solution and (2,1)-solution; and the lower and upper solutions are obtained as

$$\begin{aligned} \overline{x}_\alpha(t) = \ & \left(\tfrac{1-\alpha}{2}\right)\left(\frac{1-e^{-\frac{3}{2}\pi}}{e^{\frac{3}{2}\pi}-e^{-\frac{3}{2}\pi}}\right)e^t + \left(\tfrac{1-\alpha}{2}\right)\left(\frac{e^{\frac{3}{2}\pi}-1}{e^{\frac{3}{2}\pi}-e^{-\frac{3}{2}\pi}}\right)e^{-t} \\ & -\tfrac{7}{2}\sin(t) + \tfrac{3}{2}\cos(t), \\ \underline{x}_\alpha(t) = \ & \left(\tfrac{1-\alpha}{2}\right)\left(\frac{e^{-\frac{3}{2}\pi}-1}{e^{\frac{3}{2}\pi}-e^{-\frac{3}{2}\pi}}\right)e^t + \left(\tfrac{1-\alpha}{2}\right)\left(\frac{1-e^{\frac{3}{2}\pi}}{e^{\frac{3}{2}\pi}-e^{-\frac{3}{2}\pi}}\right)e^{-t} \\ & -\tfrac{7}{2}\sin(t) + \tfrac{3}{2}\cos(t). \end{aligned}$$

for (1,1)-solution and (2,2)-solution.

As $\alpha = 0$ and $\alpha = 1$, the graphs of (1,2)-solution and (2,1)-solution is displayed in Figure 3.

Figure 3 indicates that the solution is not a valid fuzzy level set when $l > \cot^{-1}(-1)$.

As $\alpha = 0$ and $\alpha = 1$, the graphs of (1,1)-solution and (2,2)-solution are displayed in Figure 4. As we can see, the solution $[\underline{x}_\alpha(t), \overline{x}_\alpha(t)]$ of (3) is a symmetric triangle valid fuzzy level set as $t \in [0, \frac{3}{2}\pi]$.

V. Conclusions

There are four different solutions for two-point FBVPs when the fuzzy derivative is regarded as a lateral type of H-derivative. This paper reduces these four solutions to two different solutions as $f(t, x)$ in (1) is a monotone function of x. Moreover, the fuzzy solutions of (2) and (3) are provided respectively, when boundary conditions are given as symmetric triangle fuzzy numbers. As boundary values are symmetric triangle fuzzy numbers, we show that solutions are symmetric triangle fuzzy function of t, but that some solutions are not a valid fuzzy level set. The presented examples indicate advantages and the inconvenient of each solution clearly.

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