# Pontrjagin Duality and Codes over Finite Commutative Rings 

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#### Abstract

We present linear codes over finite commutative rings which are not necessarily Frobenius. We treat the notion of syndrome decoding by using Pontrjagin duality. We also give a version of Delsarte's theorem over rings relating trace codes and subring subcodes.


Keywords-Codes, Finite Rings, Pontrjagin Duality, Trace Codes.

## I. Introduction

We recall some notions of algebra useful for the following discussion ( see [3] for more background). The reader may skip to the next sections.
Here, we consider an unitary ring $A$, which is not necessarily commutative. An $A$-module $I$ is said injective if for any injection $i: M \rightarrow N$ of $A$-modules and for every linear application $f: M \rightarrow I$, there exists a linear application $g: N \rightarrow I$ such that $f=g \circ i$. A submodule $N$ of an $A$-module $M$ is said essential if for every submodule $L \neq 0$, we have $N \cap L \neq 0$. We denote by $\operatorname{Soc}(M)$ the socle of an $A$-module $M$ which is the intersection of all essential submodules, and $J(A)$ is the Jacobson radical (it is the intersection of all non-trivial maximal ideals of $A$ ). For a left ideal $I$, resp. a right ideal $K$, of $A$ the annihilators are $l(I)=\{a \in A: a I=0\}$ and $r(I)=\{a \in A: I a=0\}$. An Artinian ring $A$ is said quasi-Frobenius if for any left ideal $I$ and right ideal $K$, we have

$$
\begin{equation*}
l(r(I))=I, r(l(K))=K . \tag{1}
\end{equation*}
$$

Furthermore, if $\operatorname{Soc}(A) \simeq A / J(A)$, then $A$ is said Frobenius. If the ring is commutative, then the two notions coincide, and the relation (1) becomes $l^{2}(I)=I$.

Wood [1] has shown the fundamental result that The MacWilliams relation holds for a code if and only if the ring is quasi-Frobenius, in the framework of linear functionalbased duality. This result singles out the class of codes over quasi-Frobenius rings. But we make emphasis here on finite commutative rings, not necessarily Frobenius, and on Pontrjagin duality.

In the sequel of this paper, we consider a finite commutative ring $A$ of cardinality $q$. Paragraph II introduces the concept of linear codes over a ring. Paragraph III recalls Pontrjagin

[^0]duality and basic facts ( such as a module over $A$ is isomorphic to its bidual) . Paragraph IV presents syndrome decoding. In Paragraph V the control matrix is introduced for a $A$-code with a free dual, and we also show how a code is decomposed in terms of its local codes (that is codes over local rings). In paragraph VI, we present an example. Paragraph VII gives a version of Delsarte's theorem for the ring extension $A \subseteq$ $B$, requiring the existence of a nondegenerate bilinear 'form' with values in the Pontrjagin dual of $A$ and that the subring $A$ is Frobenius. The last section is a conclusion for further investigations.

## II. Definitions

Definition 1: 1) A linear code $C$ over $A$ of length $n$ is a submodule of $A^{n}$. ( we also say that $C$ is a linear A-code.)
2) A linear code of length $n$ free over $A$ of rank $k$ is said an $[n, k]$-code over $A$.
Definition 2: 1) The Hamming distance between $x$ and $y$ in $A^{n}$ is $d(x, y)=\left|\left\{i \in\{1, \ldots, n\}: x_{i} \neq y_{i}\right\}\right|$.
2) The Hamming weight of $x \in A^{n}$ is $w(x)=d(x, 0)$.
3) The minimal distance of a code $C$ is $d(C)=$ $\min \{d(x, y): x \neq y \in C\}=\min \{w(c): c \in C \backslash\{0\}\}$.
Definition 3: Let $C$ be an $[n, k]$-code linear over $A$ with basis $\left(e_{1}, \ldots, e_{k}\right)$.

1) The matrix $G \in \mathscr{M}_{k, n}(A)$ with lines $e_{i}, 1 \leq i \leq k$ is said the generator matrix of $C$.
2) The message $m \in A^{k}$ is encoded by the codeword $c=$ $m G \in C$.

## III. Pontriagin Duality

A general background reference for Pontrjagin duality is [7]. Let $M$ be an $A$-module and $\mathbb{T}=\mathbb{R} / \mathbb{Z} \simeq\{z \in \mathbb{C}:|z|=1\}$ the one dimensional torus.

Definition 4: A charater of $M$ is a group homomorphism from $(M,+)$ to $\mathbb{T}$.
A character $\chi$ is trivial over a subset $N$ of $M$ if $\forall x \in$ $N, \chi(x)=1$. We define a addition on characters by : for $\chi$ and $\chi^{\prime}$ and all $x \in M$,

$$
\left(\chi+\chi^{\prime}\right)(x)=\chi(x) \chi^{\prime}(x) .
$$

The set of characters $\widehat{M}$. is an abelian group. We define for a character $\chi \in \widehat{M}$, a scalar $a \in A$ and an element $x \in M$

$$
(a \cdot \chi)(x)=\chi(a x) .
$$

Thus $\widehat{M}$ is an $A$-module.

Definition 5: 1) The orthogonal of a submodule $N$ of $M$ is the submodule of $\widehat{M}$ :

$$
N^{\perp}=\{\chi \in \widehat{M}: \chi=1 \text { over } N\}
$$

2) The dual of $\widehat{M}$, denoted $\widehat{\widehat{M}}$, is called the bidual of $M$.
3) The orthogonal of a submodule $H$ de $\widehat{M}$ is the submodule of $M$ :

$$
H^{\perp}=\{x \in M:(\forall \chi \in H), \chi(x)=1\} .
$$

4) The bi-orthogonal of $N$ is the orthogonal $N^{\perp \perp} \subset M$ of $N^{\perp}$ 。
Theorem 1 (Extension): If $N$ is a submodule of $M$, then the modules $\overline{M /} N$ and $N^{\perp}$ are isomorphic .

Theorem 2 (Separation): Suppose that $M$ is a module of finite cardinality. Let $N$ be a submodule and $x \in M$.

1) A necessary and sufficient condition for $x=0$ is that for all $\chi \in \widehat{M}, \chi(x)=1$.
2) $x \in N$ if and only if every character of $M$ trivial on $N$ is also trivial at $x$.
3) The module $M$ is rflexive (i.e., isomorphic to its bidual).
4) The modules $N$ and $N^{\perp \perp}$ are isomorphic.

## IV. Syndrome Decoding

Let $C$ be a linear $A$-code of length $n$, of minimal distance $d$ and $t=\left[\frac{d-1}{2}\right]$.

Definition 6: The syndrome of $x \in A^{n}$ is $s(x)=$ $(\chi(x))_{\chi \in C^{\perp}}$.

Proposition 1: Let $x \in A^{n}$. Then $x \in C$ iff $s(x)=1=$ $\left(1_{\chi}\right)_{\chi \in C^{\perp}}$ where $1_{\chi}=1 \in \mathbb{T}$.

## Proof

That the condition is necessary is a consequence of the orthogonal of $C$. Conversely, suppose that $s(x)=1$. then for all $\chi \in C^{\perp}, \chi(x)=1$. So, $x \in C^{\perp \perp}=C$.

Proposition 2: Two elements $x$ and $y$ of $A^{n}$ have the same class in $A^{n} /{ }_{C}$ iff they have the same syndrome.

## Proof

Note that every character $\chi: A^{n} \rightarrow \mathbb{T}$ is a group morphism to the multiplicative group $\mathbb{T}$. The group $\mathbb{T}^{\left|C^{\perp}\right|}$ is equiped with pointwise multiplication. We have:

$$
\begin{aligned}
\bar{x}=\bar{y} & \Longleftrightarrow x-y \in C \\
& \Longleftrightarrow s(x-y)=1 \\
& \Longleftrightarrow \chi(x-y)=1, \forall \chi \in C^{\perp} \\
& \Longleftrightarrow \chi(x) \chi(y)^{-1}=1, \forall \chi \in C^{\perp} \\
& \Longleftrightarrow \chi(x)=\chi(y), \forall \chi \in C^{\perp} \\
& \Longleftrightarrow s(x)=s(y)
\end{aligned}
$$

Remark 1: Let $y$ be the received word and $e$ the associated error vector. Then $c=y-e \in C$ and $s(y)=s(e)$.

Proposition 3: Let $e$ be the error vector of weight $\leq t$ and syndrome $s$. Then $e$ is the unique vector of weight $\leq t$ and with syndrome $s$.

Let $e^{\prime} \in A^{n}$ of weight $\leq t$ and $s(e)=s\left(e^{\prime}\right)$. Then, by the proposition 2, we have $e-e^{\prime} \in C$.

$$
\begin{aligned}
w\left(e-e^{\prime}\right) & =d\left(e-e^{\prime}\right) \\
& =d\left(e, e^{\prime}\right) \\
& \leq d(e, 0)+d\left(0, e^{\prime}\right) \\
& \leq w(e)+w\left(e^{\prime}\right) \\
& \leq 2 t<d
\end{aligned}
$$

So $e-e^{\prime}=0$ and $e=e^{\prime}$.
Based on these results, we describe a syndrome decoding algorithm which computes the error vector:
Input : received noisy word $y \in A^{n}$.
Output : codeword $c$ nearest to $y$.

1) Compute the syndrome $s(y)$ of $y$.
2) Determine the class $\bar{y}$ of $y$ modulo $C$.
3) Determine the vector $e \in \bar{y}$ of weight $\leq t$ with $s(y)=$ $s(e)$.
4) Return $c=y-e \in C$.

## V. Control Matrix

Let $C$ be an $[n, k]$-code linear over $A$ such that $C^{\perp}$ is free over $A$.

Proposition 4: All the bases of $C^{\perp}$ have the same cardinality and $C^{\perp}$ is an $[n, n-k]-$ code linear over $A$.

## Proof

Let $h$ be the cardinal of a basis of $C^{\perp}$. By theorem 1, $C^{\perp}$ and $\widehat{A^{n} C_{C}}$ are isomorphic, and

$$
\left|C^{\perp}\right|=\left|\widehat{A^{n} / C}\right|=\left|A^{n} / C\right|=\left[A^{n}: C\right]=\frac{\left|A^{n}\right|}{|C|}
$$

Therefore, $q^{n}=q^{k} q^{h}$ and $h=n-k$. Thus, $C^{\perp}$ is an $[n, n-$ $k]$-code.

A linear $A$-code is said local when the underlying ring is local. We will indicate how a general linear $A$-code $C$ is the product of local codes and will give a condition for $C^{\perp}$ to be free when $C$ is free. We are indebted to and inspired by the work of [5] for the next proposition. Let $M_{1}, \ldots, M_{l}$ be the set of maximal ideals of $A$. For each ideal $I$ denote by $i(I)=$ $\min \left\{j: I^{j}=I^{j+1}\right\}$ the nilpotency of $I$. Let $A_{i}=A / M_{i}^{i\left(M_{i}\right)}$ the local ring with maximal ideal $M_{i} / M_{i}^{i\left(M_{i}\right)}$. Then

$$
\begin{equation*}
A=\prod_{i=1}^{l} A_{i} \tag{2}
\end{equation*}
$$

By using the Chinese remainder theorem we get

$$
\begin{equation*}
C=\prod_{i=1}^{l} C_{i} \tag{3}
\end{equation*}
$$

Thus the decomposition of a ring as (2) induces a decomposition of a code (3) as a product of its 'local codes.' Also we have $d(C) \leq \min \left\{d\left(C_{i}\right): 1 \leq i \leq l\right\}$. Using the property that the character of a product is the product of characters, and from (3), we get a decomposition of the dual

$$
\begin{equation*}
C^{\perp}=\prod_{i=1}^{l} C_{i}^{\perp} \tag{4}
\end{equation*}
$$

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We recall that an $A$-module is said projective if it is a direct summand of a free $A$-module[9], and since a projective module over a local ring is free, we have
Proposition 5: Suppose $C$ is free. Then $C^{\perp}$ is free iff each of its local codes $C_{i}^{\perp}$ is projective.

Remark 2: Since a linear $A$-code $C$ is trivially Noetherian and Artinian, it admits a composition series ( or a JordanHölder series), and we may study the concept of its length ( as an $A$-module), even if the code is not free.

Definition 7: The matrix $H \in \mathscr{M}_{n-k, n}(A)$ formed by the lines of a basis vectors of the code $C^{\perp}$ is called the control matrix of $C$.

Proposition 6: Suppose that $A=\mathbb{T}_{m} \mathbb{Z}^{\text {and that }} H$ is the control matrix of $C$. Then, for all $x, y \in A^{n}$ :

1) $s(x)=1$ iff $H x^{t}=1$.
2) $s(x)=s(y)$ iff $H x^{t}=H y^{t}$.

## Proof

1) That the condition is necessary is trivial .

Conversely, Suppose that $H x^{t}=1$. Let

$$
H=\left(\begin{array}{ccc}
\chi_{11} & \cdots & \chi_{1 n} \\
\vdots & \ddots & \vdots \\
\chi_{n-k, 1} & \cdots & \chi_{n-k, n}
\end{array}\right)
$$

Then for all $i=1 \ldots n-k, e_{i}=\left(\chi_{i 1}, \ldots, \chi_{i n}\right) \in C^{\perp}$. Let $\chi \in C^{\perp}$, then there exists $a_{1}, \ldots, a_{n-k}$ in $A$ such that $\chi=\sum_{i=1}^{n-k} a_{i} e_{i}$. We have

$$
\begin{aligned}
\chi(x) & =\prod_{i=1}^{n-k}\left(a_{i} e_{i}\right)(x) \\
& =\prod_{i=1}^{n-k} e_{i}\left(a_{i} x\right) \\
& =\prod_{i=1}^{n-k}\left(e_{i}(x)\right)^{a_{i}}=1
\end{aligned}
$$

Therefore, $s(x)=1$.
2) uses 1).

Proposition 7: Let $H$ be the control matrix of a code $C$. Then the minimal distance of $C$ is the minimal number of dependent columns of $H$.

## Proof

Let $d$ be the minimal distance of $C, v_{1}, \ldots, v_{n}$ column vectors of $H, c=\left(c_{1}, \ldots, c_{n}\right) \in C$ of weight $w(c)=d$ and $I=\left\{i \in\{1, \ldots, n\}: c_{i} \neq 0\right\}$. Then $|I|=d$ and $\sum_{i \in I} c_{i} v_{i}=0$. So, $\left(v_{i}\right)_{i \in I}$ is a dependent family.
Conversely, let $\left(v_{i}\right)_{i \in J}$ be a dependent family. Then there exists $\left(\alpha_{i}\right)_{i \in J} \subset A$ such that $\sum_{i \in J} \alpha_{i} v_{i}=0$. Let $c=$ $\left(\alpha_{1}, \ldots, \alpha_{n}\right) \in A^{n}$ such that for all $i \notin J, \alpha_{i}=0$. Then $H c=1$ and $c \in C$. So, $|J| \geq d$.

## VI. An Example

We give a simple example illustrating the concepts studied above.

Let $A=\mathbb{Z}_{4 \mathbb{Z}}$ and $C$ the linear code over $A$ of length 5 and generator matrix

$$
G=\left(\begin{array}{lllll}
1 & 0 & 1 & 3 & 0 \\
0 & 1 & 1 & 0 & 3
\end{array}\right)
$$

Let $v_{1}=(1,0,1,3,0)$ and $v_{2}=(0,1,1,0,3)$ be the lines of $G$. We have $\hat{A}=\left\{1, \psi, \psi^{2}, \psi^{3}\right\}$, such that $\psi(1)=\omega$ and $1(1)=1 \in \mathbb{T}$ where $\omega$ is a primitive root of order four of unity. The Pontrjagin dual of $A^{5}$ is $\widehat{A}^{5}$. We will determine a control matrix $H$ of $C$. Let $\chi=\left(\chi_{1}, \chi_{2}, \chi_{3}, \chi_{4}, \chi_{5}\right) \in \widehat{A^{5}}$ such that $\chi_{1}(1)=\omega_{1}, \quad \chi_{2}(1)=\omega_{2}, \chi_{3}(1)=\omega_{3}, \chi_{4}(1)=\omega_{4}$ and $\chi_{5}(1)=\omega_{5}$. We have

$$
\begin{aligned}
\chi \in C^{\perp} & \Longleftrightarrow\left\{\begin{array} { l } 
{ \chi ( v _ { 1 } ) = 1 } \\
{ \chi ( v _ { 2 } ) = 1 } \\
{ } \\
{ }
\end{array} \Longleftrightarrow \left\{\begin{array}{l}
\omega_{1} \omega_{3} \omega_{4}^{3}=1 \\
\omega_{2} \omega_{3} \omega_{5}^{3}=1 \\
\omega_{4}=\omega_{1} \omega_{3} \\
\omega_{5}=\omega_{2} \omega_{3}
\end{array}\right.\right.
\end{aligned}
$$

Then $\chi(1)=\left(\omega_{1}, \omega_{2}, \omega_{3}, \omega_{2}, \omega_{1} \omega_{3}^{2}\right)$. There exists $\alpha_{1}, \alpha_{2}, \alpha_{3}$ and $\alpha_{3}$ in $A$ such that for all $i=1 . .3, \omega_{i}=\omega^{\alpha_{i}}$. Then $\chi=$ $\alpha_{1} \varphi_{1}+\alpha_{2} \varphi_{2}+\alpha_{3} \varphi_{3}$ where $\varphi_{1}(1)=(\omega, 1,1, \omega, 1), \varphi_{2}(1)=$ $(1, \omega, 1,1, \omega)$ et $\varphi_{3}(1)=(1,1, \omega, \omega, \omega)$. Therefore, $C^{\perp}$ is free and the following matrix

$$
H=\left(\begin{array}{ccccc}
\omega & 1 & 1 & \omega & 1 \\
1 & \omega & 1 & 1 & \omega \\
1 & 1 & \omega & \omega & \omega
\end{array}\right)
$$

is the control matrix of $C$. By proposition 7, the minimal distance of $C$ is $d=3$ and this permits to detect 2 errors and to correct one error, $t=\left[\frac{d-1}{2}\right]=1$. Let $c=11233 \in C$ the transmitted message and $y=11213$ the received noisy word.

$$
H y^{t}=\left(\begin{array}{c}
\omega^{2} \\
1 \\
\omega^{2}
\end{array}\right) \neq 1
$$

Thus the message is erroneous. The class of $y$ modulo $C$ is $\bar{y}=\left\{x=\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}\right): s(x)=s(y)\right\}$.

$$
\begin{aligned}
& s(x)=s(y) \Longleftrightarrow H x^{t}=H y^{t} \\
& \Longleftrightarrow\left\{\begin{array}{r}
x_{1}+x_{4}=2 \\
x_{2}+x_{5}=0 \\
x_{3}+x_{4}+x_{5}=2
\end{array}\right. \\
& x_{4}=2+3 x_{1} \\
& x_{5}=3 x_{2} \\
& x_{3}=x_{1}+x_{2}
\end{aligned}
$$

By theorem 2, $C$ is defined by the equations

$$
\left\{\begin{array}{l}
x_{4}=3 x_{1} \\
x_{5}=3 x_{2} \\
x_{3}=x_{1}+x_{2}
\end{array}\right.
$$

and $\bar{y}=\bar{e}$, with $e=00020$. Since $d=3$ and $w(e)=1 \leq$ $t, e$ is the convenient error. Thus the transmitted codeword is $y-e=11233=c$.

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## VII. Trace Codes

## A. Generalities

Let $k=\mathbb{F}_{q} \subset K=\mathbb{F}_{q^{m}}$ be a finite Galois field extension. For a $K$-linear code $C$, we denote by $C^{\perp}$ the dual code with respect to the usual inner product, $\operatorname{Res}(C)=C_{\mid k}=C \cap k^{n}$ the code restricted to $k$, and $\operatorname{Tr}(C)=\{\operatorname{Tr}(c): c \in C\}$ the trace code.
Theorem 3 (Delsarte): For a $K$-linear code $C$, we have

$$
\operatorname{Res}(C)^{\perp}=\operatorname{Tr}\left(C^{\perp}\right)
$$

Our aim is to give a version of this theorem over finite commutative rings, following the proof given in [4].
In the following $A, B$ designate finite commutative rings with identities and $n$ is a nonnegative integer. We recall the
Definition 1: We say that $C \subseteq B^{n}$ is a linear $B$-code if $C$ is a submodule of $B^{n}$.
Suppose that $A \subseteq B$ is a ring extension such that $B$ is an $A$-algebra with dimension $\operatorname{dim}_{A} B=m$ and basis $\left(w_{i}\right)_{1 \leq i \leq m}$. Then we define the trace function

$$
\begin{gathered}
\operatorname{Tr}_{B / A}=\operatorname{Tr}: B \rightarrow A \\
b \mapsto \operatorname{Tr}_{B / A}(b)=\operatorname{Tr}_{B / A}\left(b w_{i}\right)_{1 \leq i \leq m}=\operatorname{Tr}\left(\left(\alpha_{i j}\right)_{1 \leq j \leq m}\right)
\end{gathered}
$$

where $b w_{i}=\sum_{i=1}^{m} \alpha_{i j} w_{j}$, extended to

$$
\operatorname{Tr}: B^{n} \rightarrow A^{n},\left(b_{i}\right)_{i} \mapsto\left(\operatorname{Tr}\left(b_{i}\right)\right)_{i} .
$$

It is easy to see that for every $a \in A$, we have $\operatorname{Tr}_{B / A}(a)=$ $m a$. Recall that the characteristic of a ring $A$ denoted $\operatorname{char} A$ is the least positive integer $p$ such that for all $x \in A, p x=0$.

Lemma 1: If char $A$ does not divide $m$, then the trace function is nonzero.
For an extension $A \subseteq B$ and a linear $B$-code $C$, we define the trace code and the restricted code, respectively:

$$
\begin{aligned}
& \operatorname{Tr}(C)=\{\operatorname{Tr}(c): c \in C\} \\
& \operatorname{Res}(C)=C_{\mid A}=C \cap A^{n}
\end{aligned}
$$

We recall the following fundamental and well-known
Lemma 2: [3] If $A$ is a ring, then $\hat{A}$ is in injective as an $A$-module.

## B. A form of Delsarte's theorem

We make the following hypothesis : there exists a nondegenerate bilinear form $\beta_{A}=\beta: A \times A \rightarrow \hat{A}$ extended to

$$
\beta_{A}: A^{n} \times A^{n} \rightarrow \hat{A},(x, y) \mapsto \sum_{i=1}^{n} \beta_{A}\left(x_{i}, y_{i}\right)
$$

For $C$ a linear $A$-code, we define its $\beta$-dual as

$$
l_{\beta_{A}}(C)=\left\{a \in A^{n}: \beta_{A}(a, b)=0, \forall b \in C\right\} \subseteq A^{n}
$$

We need the following result which is called the double annihilator property, which is well documented in [1], [2]
Lemma 3: let $C \subseteq C^{\prime}$ be a linear codes over a Frobenius ring $A$. Then

$$
l_{\beta_{A}}^{2}(C)=C \text { and } l_{\beta_{A}}\left(C^{\prime}\right) \subseteq l_{\beta_{A}}(C)
$$

Suppose $A \subseteq B$ is a ring extension such that $A$ is equipped with the form $\beta_{A}=\beta$. Then, by lemma 2, there exists $\beta^{\prime}$ : $B^{n} \times B^{n} \rightarrow \hat{A}$ such that $\beta_{\mid A^{2}}^{\prime}=\beta$ (extension). Furthermore, we will use the existence of an isomorphism $\rho: \hat{A} \rightarrow A$ and that $\beta: A^{n} \times A^{n} \rightarrow \hat{A} \simeq A$ is given by the matrix $M=\left(m_{i j}\right)_{n \times n}$ with values in $A$ :

$$
\beta(x, y)=\sum_{i, j=1}^{n} m_{i j} x_{i} y_{j}
$$

Note that the group isomorphism $\rho: \hat{A} \rightarrow A$ is obtained by a standard use of the main theorem of the decomposition of a finite abelian group in terms of cyclic groups. In the following, we will identify $\rho \beta$ with $\beta$ and make use of the matrix representation $M$ of $\beta$.

Lemma 4: Let $f: B \rightarrow A$ be a linear map, extended to $B^{n}$ by $f\left(b_{1}, \ldots, b_{n}\right)=\left(f\left(b_{1}\right), \ldots, f\left(b_{n}\right)\right)$. Then for every $a, b \in$ $B^{n}$, we have $\beta^{\prime}(f(b), a)=f\left(\beta^{\prime}(b, a)\right)$.

## Proof

We have

$$
\begin{gathered}
\beta^{\prime}(f(b), a)=\sum_{i, j} m_{i, j} f\left(b_{i}\right) a_{j}=\sum_{i} f\left(b_{i}\right) \sum_{j} m_{i, j} a_{j} \\
=\sum_{i} f\left(b_{i} \sum_{j} m_{i, j} a_{j}\right)=f\left(\sum_{i} b_{i} \sum_{j} m_{i, j} a_{j}\right) \\
=f\left(\beta^{\prime}(b, a)\right)
\end{gathered}
$$

In particular, for the trace function, $\beta^{\prime}(\operatorname{Tr}(b), a)=$ $\operatorname{Tr}\left(\beta^{\prime}(b, a)\right)$.

Theorem 4: For any linear $B$-code $C$, and for $A$ Frobenius, if char $A$ does not divide $m$, then

$$
\operatorname{Tr}\left(l_{\beta^{\prime}}(C)\right)=l_{\beta}\left(C_{\mid A}\right)
$$

## Proof

We show that $\operatorname{Tr}\left(l_{\beta^{\prime}}(C)\right) \subseteq l_{\beta}\left(C_{\mid A}\right)$. Let $a=\operatorname{Tr}(b)=$ $\left(\operatorname{Tr}\left(b_{i}\right)_{1 \leq i \leq n}\right)$, where $b \in l_{\beta^{\prime}}(C)$. Then

$$
\begin{equation*}
\forall c \in C, \beta^{\prime}(b, c)=0 . \tag{5}
\end{equation*}
$$

Let $a^{\prime} \in \operatorname{Res}(C)$. We have

$$
\beta\left(a, a^{\prime}\right)=\beta(\operatorname{Tr}(b), a)=\sum_{i=1}^{n} \beta\left(\operatorname{Tr}\left(b_{i}\right), a_{i}^{\prime}\right) .
$$

Then, by lemma 4 we have $\beta\left(a, a^{\prime}\right)=\sum_{i=1}^{n} \operatorname{Tr}\left(\beta\left(b, a^{\prime}\right)\right)$. It follows from (5) that $\beta\left(a, a^{\prime}\right)=0$.

Conversely, we show the inverse inclusion. By lemma 3, this is equivalent to showing $l_{\beta}\left(\operatorname{Tr}\left(l_{\beta^{\prime}}(C)\right)\right) \subseteq C_{\mid A}$. Suppose that this is not the case. Let $u \in l_{\beta}\left(\operatorname{Tr}\left(l_{\beta^{\prime}}(C)\right)\right) \backslash C_{\mid A}$. Then $\exists v \in l_{\beta}\left(C_{\mid A}\right)$ such that $\beta(u, v) \neq 0$. note that we are using the identification $\rho: \hat{A} \simeq A$; thus $\beta(u, v)$ is identified with $\rho(\beta(u, v))$. By lemma $1, \exists \gamma \in B$ such that $\operatorname{Tr}(\gamma \cdot \beta(u, v)) \neq 0$. Using lemma 4 , we have

$$
\beta(u, \operatorname{Tr}(\gamma v))=\operatorname{Tr}(\gamma \beta(u, v)) \neq 0 .
$$

But $\left.\forall x \in \operatorname{Tr}\left(l_{\beta^{\prime}} C\right)\right), \beta(u, x)=0$ and $\gamma v \in l_{\beta^{\prime}}(C)$. So, $\beta(u, \operatorname{Tr}(\gamma v))=0$. A contradiction.

Remark 3: We may also give a version of this theorem by considering $\beta_{B}: B^{2} \rightarrow \hat{B}$ and its restriction $\beta_{B \downarrow A}$ to $A$. Then we may have the following version

$$
\operatorname{Tr}\left(l_{\beta_{B}}(C)\right)=l_{\beta_{B \downarrow A}}\left(C_{\mid A}\right)
$$

## VIII. Conclusion

In this paper, we have studied block codes over finite commutative rings $A$, giving a concept of syndrome in the framework of Pontrjagin duality. Also, an analogue of Delsarte's theorem is proved. We note that a comparison between linear functional-based duality and Pontrjagin duality has been treated for 'projective codes'[5]. It is well known [6] that the ring $A$ has a unique decomposition

$$
A=A_{1} \bigoplus \ldots \bigoplus A_{m}
$$

where each $A_{i}$ is a local ring. This in turn gives a decomposition of the code in terms of 'local codes', which suggests further investigation of codes over local rings ( both for encoding and decoding). With the notation of section VII, if we suppose that the extension $A \subseteq B$ of local rings is Galois [9], with Galois group $G$, then it is easy to see that if a $B$-code is $G$-invariant then $\operatorname{Res}(C)=\operatorname{Tr}(C)$. This result and its converse are proved in the case of finite fields in [8].

As a general conclusion, more examination of particular codes over rings (such as cyclic codes) is possible, with use of Pontrjagin duality.

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