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On Positive Definite Solutions of Quaternionic Matrix Equations

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Abstract—The real representation of the quaternionic matrix is definited and studied. The relations between the positive (semi)define quaternionic matrix and its real representation matrix are presented. By means of the real representation, the relation between the positive (semi)definite solutions of quaternionic matrix equations and those of corresponding real matrix equations is established.

Keywords—Matrix equation, Quaternionic matrix, Real representation, positive (semi)definite solutions.

I. Introduction

In the study of quaternionic quantum mechanics and some other applications of quaternions [1], [2], [3], one often encounters the problem of solutions of quaternionic linear equations. Because of noncommutativity of quaternions, solving quaternionic linear equations is more difficult. In papers[4], [5], [6], by means of a complex representation and a companion vector, the authors have studied quatemionic linear equations and presented a Cramer rule for quaternionic linear equations and an algebraic algorithm for the least squares problem, respectively, in quaternionic quantum theory. In the paper[8], by means of a real representation of the quaternionic matrix, we gave an iterative algorithms for the least squares problem in quaternionic quantum theory.

How to find positive (semi)definite solution of quaternionic matrix equations is also an important problem in quaternionic quantum theory. However, to our best knowledge, the problem has not been studied for its difficulty.

In this paper, we will pay attention to positive (semi)definite solutions of quaternionic matrix equations by means of a real representation of the quaternionic matrix and establish the relation between this problem and the corresponding problem in the real number field. Because the latter has been studied wildly, we may apply the existing results to the former.

Let $\mathbf{R}$ denote the real number field, $\mathbf{Q}=\mathbf{R}\oplus\mathbf{R}i\oplus\mathbf{R}j\oplus\mathbf{R}k$ the quaternion field, where $i^2=j^2=k^2=-1, ij=-ji=k$. For any quaternion $a=a_1+a_2i+a_3j+a_4k$ where $a_j\in\mathbf{R}$, the conjugate of a is $\bar{a}=a_1-a_2i-a_3j-a_4k$. For any quaternion matrix A, $A^T, \bar{A}$ and A^H denote the transpose, conjugate and conjugate transpose of A over quaternion field, respectively. $\mathbf{F}^{m\times n}$ denotes the set of $m\times n$ matrices on a field $\mathbf{F}$. For $A\in\mathbf{Q}^{n\times n}$, A is unitary if $A^HA=AA^H=I$ and Hermitian if $A^H=A$. For any Hermitian matrix $A\in\mathbf{Q}^{n\times n}$, A is positive (semi)definite if $x^HAx>0(\geq 0)$ for any nonzero vector $x\in\mathbf{Q}^n$.

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II. Real representation

In this section, we will give the definition of the real representation and study the relation between the positive (semi)define quaternionic matrix and its real representation matrix.

Let $A_l\in\mathbf{R}^{m\times n}(l=1,2,,3,4)$. The real representation matrix is defined[7] in the form

$$A^R\equiv\begin{pmatrix}A_1 & -A_2 & -A_3 & -A_4\\ A_2 & A_1 & -A_4 & A_3\\ A_3 & A_4 & A_1 & -A_2\\ A_4 & -A_3 & A_2 & A_1\end{pmatrix}\in\mathbf{R}^{4m\times 4n}. \tag{1}$$

The real matrix A^R is uniquely determined by quaternion matrix $A=A_1+A_2i+A_3j+A_4k\in\mathbf{Q}^{m\times n}$, and it is said to be a real representation matrix of quaternion matrix A.

Let I_t be $t\times t$ identity matrix and define

$$P_t=\begin{pmatrix}I_t & 0 & 0 & 0\\ 0 & -I_t & 0 & 0\\ 0 & 0 & I_t & 0\\ 0 & 0 & 0 & -I_t\end{pmatrix}, \tag{2}$$

$$Q_t=\begin{pmatrix}0 & -I_t & 0 & 0\\ I_t & 0 & 0 & 0\\ 0 & 0 & 0 & I_t\\ 0 & 0 & -I_t & 0\end{pmatrix}, \tag{3}$$

$$S_t=\begin{pmatrix}0 & 0 & 0 & -I_t\\ 0 & 0 & I_t & 0\\ 0 & -I_t & 0 & 0\\ I_t & 0 & 0 & 0\end{pmatrix}, \tag{4}$$

$$R_t=\begin{pmatrix}0 & 0 & -I_t & 0\\ 0 & 0 & 0 & -I_t\\ I_t & 0 & 0 & 0\\ 0 & I_t & 0 & 0\end{pmatrix}. \tag{5}$$

Then it is easy verify the following properties.

Proposition 2.1. *Let* $A,B\in\mathbf{Q}^{m\times n}, C\in\mathbf{Q}^{n\times s}, \alpha\in\mathbf{R}$. *Then*

(a). $(A+B)^R=A^R+B^R, (\alpha A)^R=\alpha A^R$, $(AC)^R=A^RC^R$,

(b). $Q_m^2=R_m^2=S_m^2=-I_{4m}, Q_m^T=-Q_m$, $R_m^T=-R_m, S_m^T=-S_m$,

(c). $R_mQ_m=S_m, Q_mS_m=R_m, S_mR_m=Q_m$,

(d). $Q_mR_m=S_m^T, S_mQ_m=R_m^T, R_mS_m=Q_m^T$,

(e). $Q_m^TA^RQ_n=Q_mA^RQ_n^T=A^R$, $R_m^TA^RR_n=R_mA^RR_n^T=A^R$, $S_m^TA^RS_n=S_mA^RS_n^T=A^R$

(f). $(A^*)^R = (A^R)^T, (A^T)^R \neq (A^R)^T, (A^\dagger)^R = (A^R)^\dagger$,

(g). $\begin{pmatrix} A & M \\ N & D \end{pmatrix}^R = \Pi_1 \begin{pmatrix} A^R & M^R \\ N^R & D^R \end{pmatrix} \Pi_2$,
where Π_1 and Π_2 are permutation matrices.

(h). $x^T Q_n x = x^T R_n x = x^T S_n x = 0$
for any vector $x \in \mathbf{R}^{4n}$

Theorem 2.2. *For any matrix $Y \in \mathbf{R}^{4m\times 4n}$, Y is a real representation matrix if and only if*

$$Q_m^T Y Q_n = R_m^T Y R_n = S_m^T Y S_n = Y.$$

Proof. *Necessity.* It is obvious in terms of (e) in proposition 2.1.

Sufficiency. Let

$$\hat{Y} \equiv (Y + Q_m^T Y Q_n + R_m^T Y R_n + S_m^T Y S_n)/4.$$

Then $\hat{Y} = Y$. Partition Y as $Y = (Y_{ij})_{4\times 4}$, where Y_{ij}s are $m \times n$ matrices. By direct computation, we have

$$\hat{Y} = Y = \begin{pmatrix} \hat{Y}_1 & -\hat{Y}_2 & -\hat{Y}_3 & -\hat{Y}_4 \\ \hat{Y}_2 & \hat{Y}_1 & -\hat{Y}_4 & \hat{Y}_3 \\ \hat{Y}_3 & \hat{Y}_4 & \hat{Y}_1 & -\hat{Y}_2 \\ \hat{Y}_4 & -\hat{Y}_3 & \hat{Y}_2 & \hat{Y}_1 \end{pmatrix},$$

where

$$\begin{aligned} \hat{Y}_1 &= (Y_{11} + Y_{22} + Y_{33} + Y_{44})/4, \\ \hat{Y}_2 &= (Y_{21} - Y_{12} + Y_{43} - Y_{34})/4, \\ \hat{Y}_3 &= (Y_{24} + Y_{31} - Y_{13} - Y_{42})/4, \\ \hat{Y}_4 &= (Y_{32} + Y_{41} - Y_{23} - Y_{14})/4. \end{aligned}$$

So Y is the real representation matrix of quaternionic matrix $\hat{Y}_1 + \hat{Y}_2 i + \hat{Y}_3 j + \hat{Y}_4 k$.

The following result may be verified directly.

Corollary 2.3. *For any $Y \in \mathbf{R}^{4m\times 4n}$,*

$$Y + Q_m^T Y Q_n + R_m^T Y R_n + S_m^T Y S_n$$

is a real representation matrix.

By direct verification, we easily obtain the following result.

Theorem 2.4. *For any $V \in \mathbf{R}^{4m\times n}$,*

$$(V, Q_m V, R_m V, S_m V)$$

is a real representation matrix.

The following result follows from (a) and (f) in proposition 2.1.

Theorem 2.5. *$U \in \mathbf{Q}^{m\times m}$ is an unitary matrix if and only if its real representation matrix U^R is an orthogonal matrix.*

Theorem 2.6. *Let $A \in \mathbf{Q}^{n\times n}$ be a Hermitian matrix. Then $\lambda^r(A) \in \mathbf{R}$ and*

$$\lambda^r(A) = \lambda(A^R),$$

where $\lambda^r(A)$ denotes the set of right eigenvalues of A.

Proof. First, $\lambda^r(A) \in \mathbf{R}$. Indeed, for $Ax = x\lambda$ with the unit vector x, we have $\lambda = x^H A x$, $\bar{\lambda} = (x^H A x)^H = x^H A x = \lambda$, and therefore $\lambda \in \mathbf{R}$.

Next, for any $\lambda \in \lambda^r(A)$, there exists a quaternionic vector x such that $Ax = x\lambda$, and hence

$$A^R x^R = x^R \mathrm{diag}(\lambda, \lambda, \lambda, \lambda),$$

which implies $A^R x^R(:,1) = \lambda x^R(:,1)$, i.e., $\lambda \in \lambda(A^R)$.

For any $\lambda \in \lambda(A^R)$, there exists a real vector u such that $A^R u = \lambda u$. Because of $A^R R_n u = \lambda R_n u, A^R Q_n u = \lambda Q_n u$ and $A^R S_n u = \lambda S_n u$, we have

$$\begin{aligned} A^R(u, Q_n u, R_n u, S_n u) &= \lambda(u, Q_n u, R_n u, S_n u) \\ &= (u, Q_n u, R_n u, S_n u)\mathrm{diag}(\lambda, \lambda, \lambda, \lambda) \end{aligned}$$

It follows from theorem 2.4 that $(u, Q_n u, R_n u, S_n u)$ is the real representation of some quaternionic vector. Denote this quaternionic vector by u_1, then we have $A^R u_1^R = u_1^R \lambda^R$, and $Au_1 = u_1\lambda$, i.e., $\lambda \in \lambda^r(A)$.

As we know, a Hermitian quaternionic matrix A is positive (semi)definite if and only if $\lambda^r(A) > 0 (\geq 0)$. Therefore, we can easily obtain the following results.

Corollary 2.7. *Let $A \in \mathbf{Q}^{n\times n}$ be Hermitian. Then A is positive (semi)definite if and only if A^R is positive definite(semidefinite).*

Corollary 2.8. *Let $A, B \in \mathbf{Q}^{n\times n}$ be Hermitian. Then $A - B$ is positive (semi)definite if and only if $A^R - B^R$ is positive (semi)definite.*

Theorem 2.9. *Let $A \in \mathbf{R}^{4n\times 4n}$ be symmetric. Then A is positive (semi)definite if and only if $A + Q_n^T A Q_n + S_n^T A S_n + R_n^T A R_n$ is positive (semi)definite.*

Proof. The necessity is obvious. Next, we prove the sufficiency. For any $x \in \mathbf{R}^{4n}$, $(I, Q_n^T, R_n^T, S_n^T)^T$ is full column rank, and therefore there exists $y \in \mathbf{R}^{4n}$ such that

$$\begin{pmatrix} I \\ Q_n \\ R_n \\ S_n \end{pmatrix} y = \begin{pmatrix} x \\ x \\ x \\ x \end{pmatrix}.$$

Due to $(0 \leq) 0 < y^T(A + Q_n^T A Q_n + S_n^T A S_n + R_n^T A R_n)y$

$$= y^T(I, Q_n^T, R_n^T, S_n^T) \begin{pmatrix} A & & & \\ & A & & \\ & & A & \\ & & & A \end{pmatrix} \begin{pmatrix} I \\ Q_n \\ R_n \\ S_n \end{pmatrix} y$$

$= 4x^T A x$, we have $x^T A x > 0 (\geq 0)$, i.e.,$A > 0 (\geq 0)$.

III. Positive semidefinite solutions of quaternionic matrix equation

In this section, we discuss the relation between the positive (semi)definite solutions of quaternionic matrix equation

$$AXA^H = B \tag{6}$$

and those of real matrix equation

$$A^R U (A^R)^T = B^R, \tag{7}$$

where $A \in \mathbf{Q}^{m\times n}, B \in \mathbf{HQ}^{m\times m}$.

First, we give the relation between the general solutions of quaternionic matrix equation

$$AXC = E \tag{8}$$

and those of real matrix equation

$$A^R U (C^R)^T = E^R, \tag{9}$$

where $A \in \mathbf{Q}^{m\times n}, C \in \mathbf{Q}^{p\times q}, E \in \mathbf{Q}^{m\times q}$. The following result is a special case of the corresponding result of [7].

Lemma 3.1. *Quaternionic matrix equation (8) has a solution $X \in \mathbf{Q}^{n\times p}$ if and only if real matrix equation (9) has a solution $U \in \mathbf{Q}^{4n\times 4p}$, in which case,*

$$X = \frac{1}{16}(I_n, iI_n, jI_n, kI_n)(U + Q_n^T U Q_p + R_n^T U R_p + S_n^T Q S_p)\begin{pmatrix} I_p \\ -iI_p \\ -jI_p \\ -kI_p \end{pmatrix}$$

is a quaternionic matrix solution of (8). Furthermore, if (9) has an unique solution, then (8) has also an unique solution.

Theorem 3.2. *Given $A \in \mathbf{Q}^{m\times n}, B \in \mathbf{HQ}^{m\times m}$. Then*

1. (6) has a positive (semi)definite solution if and only if real matrix equation (7) has a positive (semi)definite solution.

2. When (6) has a positive (semi)definite solution, the general expression of this solution is

$$X = \frac{1}{16}(I_n, iI_n, jI_n, kI_n)(U + Q_n^T U Q_p + R_n^T U R_p + S_n^T Q S_p)\begin{pmatrix} I_p \\ -iI_p \\ -jI_p \\ -kI_p \end{pmatrix},$$

where U is a positive (semi)definite solution of (7). Furthermore, if U is the maximal(minimal) solution of (7), then the corresponding solution X is also the maximal(minimal) solution of (6).

Proof. If (7) has a positive (semi)definite solution U, then from Theorem 2.9 and Theorem 3.1, we know that

$$\hat{U} \equiv \frac{1}{4}(U + Q^T U Q + S^T U S + R^T U R)$$

is also a positive (semi)definite solution of (7). Let $\hat{U}$ be the real representation matrix of quaternionic matrix X. It follows from Corollary 2.7 that X is a positive (semi)definite solution of (6).

If U_1 and U_2 are positive (semi)definite solutions of (7) satisfying $U_1 \geq U_2$, then

$$\frac{1}{4}(U_1 + Q^T U_1 Q + S^T U_1 S + R^T U_1 R) \geq \frac{1}{4}(U_2 + Q^T U_2 Q + S^T U_2 S + R^T U_2 R),$$

and both are positive (semi)definite solutions of (7). Let them be the real representation matrices of quaternionic matrces X_1 and X_2, respectively. Then from Corollary 2.8, we have X_1 and X_2 are positive (semi)definite solutions of (6) satisfying $X_1 \geq X_2$.

Theorem 3.2 establishs the relation between positive (semi)definite solutions of quaternionic matrix equations (6) and those of corresponding real matrix equations (7). For the latter, there have been many good theoretical results and numerical methods, which may be applied to the former.

IV. Conclusion

In the paper, we only take a simple but common equation (6) as an example. Our idea is applied to more complicated linear quaternionic matrix equations.

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