

# Numerical solution of Volterra integro-differential equations of fractional order by Laplace decomposition method

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**Abstract**—In this paper, the Laplace decomposition method is developed to solve linear and nonlinear fractional integro-differential equations of Volterra type. The fractional derivative is described in the Caputo sense. The Laplace decomposition method is found to be fast and accurate. Illustrative examples are included to demonstrate the validity and applicability of presented technique and comparison is made with exacting results.

**Keywords**—integro-differential equations, Laplace transform, fractional derivative, Adomian polynomials, Padé approximants.

## I. INTRODUCTION

**I**N this paper, we study the Laplace decomposition method for a special kind of nonlinear fractional integro-differential equation

$$D^\alpha y(t) = p(t)y(t) + g(t) + \lambda \int_0^t k(t, \tau)F(y(\tau))d\tau, t \in [0, 1], \quad (1)$$

with the initial conditions

$$y^{(i)} = \delta_i, i = 0, 1, 2, \dots, n-1, n-1 < \alpha \leq n, n \in N, \quad (2)$$

where  $g \in L^2([0, 1])$ ,  $p \in L^2([0, 1])$ ,  $k \in L^2([0, 1]^2)$  are known functions,  $y(t)$  is the unknown function,  $D^\alpha$  is the Caputo fractional differential operator of order  $\alpha$ . Such equations arise in the mathematical modeling of various physical phenomena, such as heat conduction in materials with memory. Moreover, these equations are encountered in combined conduction, convection and radiation problems[1], [2].

Most of nonlinear fractional integro-differential equations do not have exact analytic solution, so approximation and numerical technique must be used. There are only a few of techniques for the solution of fractional integro-differential equations, since it is relatively a new subject in mathematics.

Recently, several numerical methods to solve fractional differential equations and fractional integro-differential equations have been given such as variational iteration method[3], homotopy perturbation method[4], Adomian decomposition method[5], [6], homotopy analysis method[7], collocation method[8], [9] wavelet method[10], [11] and other method[12].

The Laplace decomposition method is a numerical algorithm to solve nonlinear ordinary, partial differential equations.

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Khuri[13] used this method for the approximate solution of a class of nonlinear ordinary differential equations. The numerical technique basically illustrates how the Laplace transform can be used to approximate the solution of the nonlinear differential by manipulating the decomposition method which was first introduced by Adomian. To the best of authors knowledge no attempt have been made to exploit this method to solve nonlinear fractional integro-differential equation. Our aim in this paper is to apply this technique to fractional integro-differential equation.

A Padé approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function. The  $[L/M]$  Padé approximant to a formal power series  $y(t) = \sum_{i=0}^{\infty} a_i t^i$  is given by:

$$\left[ \frac{L}{M} \right] = \frac{P_L(t)}{Q_M(t)} = \frac{p_0 + p_1 t + \dots + p_L t^L}{1 + q_1 t + \dots + q_M t^M}. \quad (3)$$

The two polynomials in the numerator and denominator of (1) have no common factor. This means that the formal power series

$$y(t) = \frac{P_L(t)}{Q_M(t)} + O(t^{L+M+1}).$$

In this case Padé approximant  $[L/M]$  is unique determined.

In this paper, we applied Laplace decomposition method to solve nonlinear Volterra integro-differential equation of fractional order.

The paper organized as follows: In section 2, we introduce some necessary definitions and properties of the fractional calculus theory and Laplace transform. In section 3, we construct our method to approximate the solution of the fractional integro-differential equation(1). Numerical examples are given in Section4.

## II. BASIC DEFINITIONS

In this section, we give some definitions and properties of the fractional calculus and Laplace transform.

**Definition 1** A real function  $f(t)$ ,  $t > 0$ , is said to be in the space  $C_\mu$ ,  $\mu \in R$ , if there exists a real number  $p > \mu$ , such that  $f(t) = t^p h_1(t)$ , where  $f_1(t) \in C(0, \infty)$ , and it is said to be in space  $C_\mu^n$  if and only if  $f^{(n)} \in C_\mu$ ,  $n \in N$ .

**Definition 2** The fractional derivative  $D^\alpha$  of  $f(t)$  in the Caputo's sense is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \quad (4)$$

for  $n-1 < \alpha \leq n, n \in N, t > 0, f(t) \in C^n_1$ .

**Definition 3** The Laplace transform of a function  $f(t), t > 0$  is defined as

$$\mathcal{L}[f(t)] = F(s) = \int_0^{+\infty} e^{-st} f(t) dt,$$

where  $s$  can be either real or complex. It has following properties:

**Lemma 1:** Laplace Transform of an Integral: If  $F(s) = \mathcal{L}[f(t)]$  then

$$\mathcal{L}\left[\int_0^t f(\tau) d\tau\right] = \frac{F(s)}{s}. \quad (5)$$

**Definition 4** Given two functions  $f$  and  $g$ , we define, for any  $t > 0$ ,

$$(f * g)(t) = \int_0^t f(x)g(t-x)dx. \quad (6)$$

The function  $f * g$  is called the convolution of  $f$  and  $g$ .

**Theorem 2:** The convolution theorem

$$\mathcal{L}[f * g] = \mathcal{L}[f(t)] \cdot \mathcal{L}[g(t)]. \quad (7)$$

**Theorem 3:** The Laplace transform  $\mathcal{L}[f(t)]$  of the Caputo derivative is defined as [14]

$$\mathcal{L}[D^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad n-1 < \alpha \leq n \quad (8)$$

### III. ANALYSIS OF THE METHOD

According to Laplace decomposition method we apply Laplace transform first on both sides of (1)

$$\begin{aligned} \mathcal{L}[D^\alpha y(t)] &= \mathcal{L}[p(t)y(t)] + \mathcal{L}[g(t)] \\ &+ \mathcal{L}\left[\lambda \int_0^t k(t, \tau) F(y(\tau)) d\tau\right]. \end{aligned} \quad (9)$$

Using the differentiation property of Laplace transform (8) we get

$$\begin{aligned} s^\alpha \mathcal{L}[y(t)] - c &= \mathcal{L}[p(t)y(t)] + \mathcal{L}[g(t)] \\ &+ \mathcal{L}\left[\lambda \int_0^t k(t, \tau) F(y(\tau)) d\tau\right], \end{aligned} \quad (10)$$

where  $c = \sum_{k=0}^{m-1} s^{\alpha-k-1} y^{(k)}(0)$ , and

$$\begin{aligned} \mathcal{L}[y(t)] &= \frac{c}{s^\alpha} + \frac{1}{s^\alpha} \mathcal{L}[p(t)y(t)] + \frac{1}{s^\alpha} \mathcal{L}[g(t)] \\ &+ \frac{1}{s^\alpha} \mathcal{L}\left[\lambda \int_0^t k(t, \tau) F(y(\tau)) d\tau\right]. \end{aligned} \quad (11)$$

The second step in Laplace decomposition method is that we represent solution as an infinite series given below

$$y(t) = \sum_{n=0}^{\infty} y_n. \quad (12)$$

The nonlinear operator is decomposed as

$$Ny = F(y(t)) = \sum_{n=0}^{\infty} A_n(y) \quad (13)$$

where  $A_n$  is the Adomian polynomials[15] of  $y_0, y_1, y_2, \dots, y_n, \dots$  that are given by

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ F\left(\sum_{i=0}^{\infty} \lambda^i y_i\right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \dots$$

For the nonlinear function  $Ny = F(y)$  the first Adomian polynomials are given by

$$\begin{aligned} A_0 &= F(y_0), \\ A_1 &= y_1 F^{(1)}(y_0), \\ A_2 &= y_2 F^{(1)}(y_0) + \frac{1}{2!} y_1^2 F^{(2)}(y_0), \\ A_3 &= y_3 F^{(1)}(y_0) + y_1 y_2 F^{(2)}(y_0) + \frac{1}{3!} y_1^3 F^{(3)}(y_0) \\ &\vdots \\ A_n &= \sum_{v=1}^n c(v, n) F^{(v)}(y_0). \end{aligned}$$

The first index of  $c(v, n)$  is the order of derivatives from 1 to  $n$ , and the second is the order of the Adomian polynomial. The  $c(v, n)$  are products (or sums of products) of  $v$  components of  $f$  whose subscripts sum to  $n$ , divided by the factorial of the number of repeated subscripts.

Substituting (12) and (13) into (11), we will get

$$\begin{aligned} \mathcal{L}\left[\sum_{n=0}^{\infty} y_n\right] &= \frac{c}{s^\alpha} + \frac{1}{s^\alpha} \mathcal{L}[g(t)] + \frac{1}{s^\alpha} \mathcal{L}\left[p(t) \sum_{n=0}^{\infty} y_n\right] \\ &+ \frac{1}{s^\alpha} \mathcal{L}\left[\lambda \int_0^t k(t, \tau) \sum_{n=0}^{\infty} A_n(y) d\tau\right]. \end{aligned} \quad (14)$$

Matching both sides of (14) yields the following iterative algorithm:

$$\mathcal{L}[y_0] = \frac{c}{s^\alpha} + \frac{1}{s^\alpha} \mathcal{L}[g(t)], \quad (15)$$

$$\mathcal{L}[y_1] = \frac{1}{s^\alpha} \mathcal{L}[p(t)y_0] + \frac{1}{s^\alpha} \mathcal{L}\left[\lambda \int_0^t k(t, \tau) A_0(y) d\tau\right], \quad (16)$$

$$\mathcal{L}[y_2] = \frac{1}{s^\alpha} \mathcal{L}[p(t)y_1] + \frac{1}{s^\alpha} \mathcal{L}\left[\lambda \int_0^t k(t, \tau) A_1(y) d\tau\right]. \quad (17)$$

In general, the recursive relation is given by

$$\mathcal{L}[y_{n+1}] = \frac{1}{s^\alpha} \mathcal{L}[p(t)y_n] + \frac{1}{s^\alpha} \mathcal{L}\left[\lambda \int_0^t k(t, \tau) A_n(y) d\tau\right]. \quad (18)$$

Applying inverse Laplace transform to (15-18), so our required recursive relation is given below

$$y_0(t) = H(t) \quad (19)$$

$$\begin{aligned} y_{n+1}(t) &= \mathcal{L}^{-1}\left[\frac{1}{s^\alpha} \mathcal{L}[p(t)y_n]\right] + \\ &\mathcal{L}^{-1}\left[\frac{1}{s^\alpha} \mathcal{L}\left[\lambda \int_0^t k(t, \tau) A_n(y) d\tau\right]\right], \quad n \geq 0, \end{aligned} \quad (20)$$

where  $H(t)$  is a function that arises from the source term and the prescribed initial conditions. The initial solution is important, the choice of (19) as the initial solution always leads to noise oscillation during the iteration procedure. The modified Laplace decomposition method[16] suggests that the function  $H(t)$  defined above in (19) be decomposed into two parts:

$$H(t) = H_1(t) + H_2(t).$$

Instead of iteration procedure (19) and (20), we suggest the following modification

$$y_0(t) = H_1(t), \quad (21)$$

$$y_1(t) = H_2(t) + \mathcal{L}^{-1} \left[ \frac{1}{s^\alpha} \mathcal{L} [p(t)y_0] \right] + \mathcal{L}^{-1} \left[ \frac{1}{s^\alpha} \mathcal{L} \left[ \lambda \int_0^t k(t, \tau) A_0(y) d\tau \right] \right], \quad (22)$$

$$y_{n+1}(t) = \mathcal{L}^{-1} \left[ \frac{1}{s^\alpha} \mathcal{L} [p(t)y_n] \right] + \mathcal{L}^{-1} \left[ \frac{1}{s^\alpha} \mathcal{L} \left[ \lambda \int_0^t k(t, \tau) A_n(y) d\tau \right] \right]. \quad (23)$$

The solution through the modified Laplace decomposition method high depend on the choice of  $H_1(t)$  and  $H_2(t)$ . We will show how to suitably choose  $H_1(t)$  and  $H_2(t)$  by examples.

#### IV. NUMERICAL EXAMPLES

In order to show the effectiveness of the Laplace decomposition method for solving Volterra integro-differential equations of fractional order, we present some examples. All the results are calculated by using the symbolic calculus software Mathematica.

**Example1** Consider the following linear fractional integro-differential equation[8]:

$$y^{3/4}(t) = \frac{6t^{9/4}}{\Gamma(13/4)} + \left( \frac{-t^2 e^t}{5} \right) y(t) + \int_0^t e^t \tau y(\tau) d\tau \quad (24)$$

with the initial condition

$$y(0) = 0 \quad (25)$$

and the the exact solution is  $y(t) = t^3$ . First, we apply the Laplace transform to both sides of (24)

$$\mathcal{L}[y^{3/4}(t)] = \mathcal{L} \left[ \frac{6t^{9/4}}{\Gamma(13/4)} \right] + \mathcal{L} \left[ \left( \frac{-t^2 e^t}{5} \right) y(t) \right] + \mathcal{L} \left[ \int_0^t e^t \tau y(\tau) d\tau \right]$$

Using the property of Laplace transform (8) and the initial conditions (25), we get

$$s^{3/4} \mathcal{L}[y(t)] = \mathcal{L} \left[ \frac{6t^{9/4}}{\Gamma(13/4)} \right] + \mathcal{L} \left[ \left( \frac{-t^2 e^t}{5} \right) y(t) \right] + \mathcal{L} \left[ \int_0^t e^t \tau y(\tau) d\tau \right]$$

and

$$\mathcal{L}[y(t)] = \frac{1}{s^{3/4}} \mathcal{L} \left[ \frac{6t^{9/4}}{\Gamma(13/4)} \right] + \frac{1}{s^{3/4}} \mathcal{L} \left[ \left( \frac{-t^2 e^t}{5} \right) y(t) \right] + \frac{1}{s^{3/4}} \mathcal{L} \left[ \int_0^t e^t \tau y(\tau) d\tau \right].$$

Substituting (12) into (13) into above equation, we have

$$\mathcal{L} \left[ \sum_{n=0}^{\infty} y_n \right] = \frac{1}{s^{3/4}} \mathcal{L} \left[ \frac{6t^{9/4}}{\Gamma(13/4)} \right] + \frac{1}{s^{3/4}} \mathcal{L} \left[ \left( \frac{-t^2 e^t}{5} \right) \sum_{n=0}^{\infty} y_n \right] + \frac{1}{s^{3/4}} \mathcal{L} \left[ \int_0^t e^t \tau \sum_{n=0}^{\infty} y_n(\tau) d\tau \right]. \quad (26)$$

Match both side of (26), we have the following relation:

$$\mathcal{L}[y_0] = \frac{1}{s^{3/4}} \mathcal{L} \left[ \frac{6t^{9/4}}{\Gamma(13/4)} \right] \quad (27)$$

$$\mathcal{L}[y_1] = \frac{1}{s^{3/4}} \mathcal{L} \left[ \left( \frac{-t^2 e^t}{5} \right) y_0 \right] + \frac{1}{s^{3/4}} \mathcal{L} \left[ \int_0^t e^t \tau y_0(\tau) d\tau \right] \quad (28)$$

$$\mathcal{L}[y_{n+1}] = \frac{1}{s^{3/4}} \mathcal{L} \left[ \left( \frac{-t^2 e^t}{5} \right) y_n \right] + \frac{1}{s^{3/4}} \mathcal{L} \left[ \int_0^t e^t \tau y_n(\tau) d\tau \right]. \quad (29)$$

Applying inverse Laplace transform to (27-29) we get

$$y_0 = t^3$$

$$y_1 = \mathcal{L}^{-1} \left[ \frac{1}{s^{3/4}} \mathcal{L} \left[ \left( \frac{-t^2 e^t}{5} \right) t^3 \right] \right] + \mathcal{L}^{-1} \left[ \frac{1}{s^{3/4}} \mathcal{L} \left[ \int_0^t e^t \tau \cdot \tau^3 d\tau \right] \right] = 0$$

$$y_{n+1} = 0.$$

Therefore, the solution is obtained to be

$$y(t) = \sum_{n=0}^{\infty} y_n = t^3.$$

The results are better than the results of [8].

**Example2** Consider the following nonlinear integro-differential equation with a difference kernel[10]

$$D^{\frac{6}{5}} y(t) = \frac{5}{2\Gamma(4/5)} t^{\frac{4}{5}} - \frac{t^9}{252} + \int_0^t (t-\tau)^2 [y(\tau)]^3 d\tau, \quad 0 \leq t < 1 \quad (30)$$

with the initial condition

$$y(0) = y'(0) = 0 \quad (31)$$

Applying the Laplace transform to both sides of (30) and using the initial conditions we obtain

$$s^{\frac{6}{5}} \mathcal{L}[y(t)] = \mathcal{L} \left[ \frac{5}{2\Gamma(4/5)} t^{\frac{4}{5}} - \frac{t^9}{252} \right] + \mathcal{L} \left[ \int_0^t (t-\tau)^2 [y(\tau)]^3 d\tau \right].$$

Applying convolution theorem (7), we can get

$$s^{\frac{6}{5}} \mathcal{L}[y(t)] = \mathcal{L} \left[ \frac{5}{2\Gamma(4/5)} t^{\frac{4}{5}} - \frac{t^9}{252} \right] + \mathcal{L}[t^2] \cdot \mathcal{L}[y(t)^3],$$

or equivalently

$$\begin{aligned}\mathcal{L}[y(t)] &= \frac{1}{s^{6/5}} \mathcal{L} \left[ \frac{5}{2\Gamma(4/5)} \cdot \frac{\Gamma(9/5)}{s^{9/5}} - \frac{1}{252} \cdot \frac{362880}{s^{10}} \right] \\ &\quad + \frac{1}{s^{6/5}} \frac{2}{s^3} \cdot \mathcal{L}[y(t)^3], \\ \mathcal{L}[y(t)] &= \frac{2}{s^3} - 1440 \frac{1}{s^{56/5}} + \frac{2}{s^{21/5}} \cdot \mathcal{L}[y(t)^3].\end{aligned}\quad (32)$$

Substituting (12) and (13) into (32) leads to

$$\mathcal{L} \left[ \sum_{n=0}^{\infty} y_n \right] = \frac{2}{s^3} - 1440 \frac{1}{s^{56/5}} + \frac{2}{s^{21/5}} \cdot \mathcal{L} \left[ \sum_{n=0}^{\infty} A_n \right].$$

So we have following relation:

$$\mathcal{L}[y_0] = \frac{2}{s^3}, \quad (33)$$

$$\mathcal{L}[y_1] = -1440 \frac{1}{s^{56/5}} + \frac{2}{s^{21/5}} \cdot \mathcal{L} \left[ \sum_{n=0}^{\infty} A_0 \right], \quad (34)$$

$$\mathcal{L}[y_{n+1}] = \frac{2}{s^{21/5}} \cdot \mathcal{L} \left[ \sum_{n=0}^{\infty} A_n \right], \quad n \geq 1. \quad (35)$$

Taking the inverse Laplace transform of both sides of (33,34), and using the recursive relation (35) gives

$$\begin{aligned}y_0 &= t^2 \\ y_1 &= 0 \\ y_{n+1} &= 0\end{aligned}$$

Therefore, the solution is obtained to be

$$y(t) = \sum_{n=0}^{\infty} y_n = t^2,$$

which is the exact solution.

**Example3** Consider the following equation

$$y^\alpha(t) = 1 + \int_0^t y'(\tau)y(\tau)d\tau, \quad 0 \leq t < 1, \quad 0 < \alpha \leq 1 \quad (36)$$

with the initial condition  $y(0) = 0$ . Applying the Laplace transform to both sides of (36) gives

$$\mathcal{L}[y^\alpha(t)] = \mathcal{L}[1] + \mathcal{L} \left[ \int_0^t y'(\tau)y(\tau)d\tau \right],$$

so that

$$s^\alpha \mathcal{L}[y(t)] = \frac{1}{s} + \mathcal{L} \left[ \int_0^t y'(\tau)y(\tau)d\tau \right],$$

or equivalently

$$\mathcal{L}[y(t)] = \frac{1}{s^{\alpha+1}} + \frac{1}{s^\alpha} \mathcal{L} \left[ \int_0^t y'(\tau)y(\tau)d\tau \right].$$

Substituting the series assumption for  $y(t)$  and the Adomian polynomials for  $y'y$  as given above in (12) and (13) respectively, we obtain

$$\mathcal{L} \left[ \sum_{n=0}^{\infty} y_n \right] = \frac{1}{s^{\alpha+1}} + \frac{1}{s^\alpha} \mathcal{L} \left[ \int_0^t \sum_{n=0}^{\infty} A_n d\tau \right].$$

So we can get the following relation:

$$\mathcal{L}[y_0] = \frac{1}{s^{\alpha+1}} \quad (37)$$

$$\mathcal{L}[y_{n+1}] = \frac{1}{s^\alpha} \mathcal{L} \left[ \int_0^t A_n d\tau \right], \quad n \geq 0. \quad (38)$$

Taking the inverse Laplace transform of both sides of (37) and (38) gives

$$\begin{aligned}y_0 &= \frac{t^\alpha}{\Gamma(1+\alpha)} \\ y_n &= \mathcal{L}^{-1} \left[ \frac{1}{s^\alpha} \mathcal{L} \left[ \int_0^t A_{n-1} d\tau \right] \right], \quad n \geq 1\end{aligned}$$

The general form of the approximation  $y(t)$  is given by

$$y(t) = \sum_{k=0}^n C_k t^{(2k+1)\alpha}, \quad (39)$$

where the coefficients are given by

$$\begin{aligned}C_0 &= \frac{1}{\Gamma(1+\alpha)} \\ C_1 &= C_0 C_0 \alpha \frac{\Gamma(2\alpha)}{\Gamma(1+3\alpha)} \\ C_2 &= (C_0 C_1 3\alpha + C_0 C_1 \alpha) \frac{\Gamma(4\alpha)}{\Gamma(1+5\alpha)} \\ C_3 &= (C_0 C_2 5\alpha + C_1 C_1 3\alpha + C_2 C_0 \alpha) \frac{\Gamma(6\alpha)}{\Gamma(1+7\alpha)} \\ C_4 &= (C_0 C_3 7\alpha + C_1 C_2 5\alpha + C_2 C_1 3\alpha + C_3 C_0 \alpha) \frac{\Gamma(8\alpha)}{\Gamma(1+9\alpha)} \\ C_5 &= (C_0 C_4 9\alpha + C_1 C_3 7\alpha + C_2 C_2 5\alpha + C_1 C_3 3\alpha + C_0 C_4 \alpha) \frac{\Gamma(10\alpha)}{\Gamma(1+11\alpha)} \\ &\vdots \\ C_n &= (C_0 C_n (2n-1)\alpha + C_1 C_{n-1} (2n-2)\alpha + \dots + C_0 C_n \alpha) \frac{\Gamma(2n\alpha)}{\Gamma(1+(2n+1)\alpha)}\end{aligned}$$

To consider the behavior of solution of solution for different value of  $\alpha$ , we will take advantage of the formula (39) available for  $0 < \alpha \leq 1$ , and consider the following two special cases: First order case: Setting  $\alpha = 1$  in (39), we obtain the approximate solution in a series form as

$$y(t) = t + \frac{1}{6}t^3 + \frac{1}{30}t^5 + \frac{17}{2520}t^7 + \frac{29}{22680}t^9 + \frac{431}{2494800}t^{11}.$$

The [5/5] Padé approximants gives

$$\left[ \frac{5}{5} \right] = \frac{-139/3780t^5 + 19/18t^3 + t}{1 + 8/9t^2 - 55/252t^4}$$

A Comparison between the exact and the approximate solutions at 10 points is demonstrated for  $n = 6$  in Table1. From Table1, it can be found that the obtained approximate solutions are very close to the exact solution.

Fractional order case: In this case we will examine the equation (36). Setting  $\alpha = 1/2$  and  $n = 6$

$$\begin{aligned}y(t) &= 1.1284t^{1/2} + 0.9578t^{3/2} + 0.6504t^{5/2} + 0.6151t^{7/2} \\ &\quad + 0.6039t^{9/2} + 0.8494t^{11/2}.\end{aligned}$$

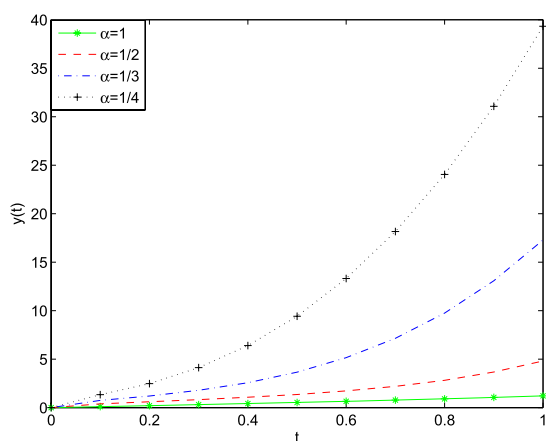


Fig. 1. Comparison of approximate solution for  $\alpha = 1/4, 1/3, 1/2, 1$ .

For simplicity, let  $t^{1/2} = x$  then,

$$y(x) = 1.1284x + 0.9578x^3 + 0.6504x^5 + 0.6151x^7 + 0.6039x^9 + 0.8494x^{11}.$$

Calculating the [5/5] Padé approximants and recalling that  $x = t^{1/2}$ , we get

$$\left[ \frac{5}{5} \right] = \frac{-0.2798t^{5/2} - 0.2529t^{3/2} + 1.1283t^{1/2}}{1 - 1.0729t + 0.0863t^2}$$

Similarly, we can get the results for  $\alpha = 1/3, 1/4$ . The obtained numerical results for  $\alpha = 1/2, 1/3, 1/4$  and  $\alpha = 1$  are summarized in Fig.1. The comparisons show that as  $\alpha \rightarrow 1$ , the approximate solutions tend to  $y(t) = \sqrt{2}\tan(\sqrt{2}t/2)$ , which is the exact solution of the equation in the case of  $\alpha = 1$ .

TABLE I

THE EXACT AND APPROXIMATE SOLUTIONS OF EXAMPLE 3 ( $\alpha = 1$ )

t	Exact solution	Approximate solution	Absolute error
0	0	0	0
0.1	1.001670006759728e-001	1.001670006758852e-001	8.7624e-014
0.2	2.013440870547489e-001	2.013440870107435e-001	4.4005e-011
0.3	3.045825027604838e-001	3.045825011179013e-001	1.6426e-009
0.4	4.110194230146166e-001	4.110194019290210e-001	2.1086e-008
0.5	5.219305154476182e-001	5.219303645940214e-001	1.5085e-007
0.6	6.387957042783595e-001	6.387949568679495e-001	7.4741e-007
0.7	7.633858022505827e-001	7.633829181184960e-001	2.8841e-006
0.8	8.978815374544917e-001	8.978722284210225e-001	9.3090e-006
0.9	1.045043135437502e+000	1.045016794826786e+000	2.6341e-005

## V. CONCLUSION

In this paper, the Laplace decomposition method has been successfully applied to finding the approximate solution of nonlinear fractional integro-differential equation. The method is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of linear and nonlinear

fractional integro-differential equations. It provides more realistic series solutions that converge very rapidly in real physical problems. Finally, the behavior of the solution can be formally determined by using the Padé approximants.

The proposed method can be applied for other nonlinear fractional differential equations, systems of differential and integral equation.

## ACKNOWLEDGMENT

The authors are grateful to the Huaihai Institute of Technology which provided support through Grant No.Z2011151.

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