International Journal of Business, Human and Social Sciences
ISSN: 2517-9411
Vol:3, No:1, 2009

Comparative Analysis of the Stochastic and Parsimonious Interest Rates Models on Croatian Government Market

Zdravka Aljinović, Branka Marasović, and Blanka Škrabić

Abstract**—**The paper provides a discussion of the most relevant aspects of yield curve modeling. Two classes of models are considered: stochastic and parsimonious function based, through the approaches developed by Vasicek (1977) and Nelson and Siegel (1987). Yield curve estimates for Croatia are presented and their dynamics analyzed and finally, a comparative analysis of models is conducted.

Keywords**—**the term structure of interest rates, Vasicek model, Nelson-Siegel model, Croatian Government market.

I. Introduction

EVALUATION of the yield curve is standard on financial markets in developed countries. A well-evaluated yield curve reflects and forecasts the condition of an economy and is an important factor in decision making of all participants on the financial market and beyond it.

Croatian financial market doesn't have an accessible yield curve, representing the price based on supply and demand of Croatian national currency funds, for a time tenor longer than one year. Yield curve for a long term investment horizon is necessary in most financial model structuring, especially in valuating financial assets and liabilities independently of the valuation method.

There are many methods which have been used to model the term structure of interest rates. In this paper two groups of models are considered: stochastic and parsimonious. From these groups the Vasicek model (1977) and the Nelson-Siegel model (1987) are chosen. A comparative analysis of models is conducted and finally, yield curve estimates for Croatia are presented and their dynamics analyzed.

The paper is organized in five sections. After this introductory section, in the second section the Vasicek model, as one of the most known and widely used stochastic models is presented. The Nelson-Siegel model, the commonly used parsimonious model for developing yield curve in financial practice, is exposed in the third section. In the fourth section the parameters of previously presented models and corresponding yield curves on Croatian Government market are estimated. Finally, some concluding remarks are given.

Z. Aljinović is with the University of Split, Faculty of Economics, Matice hrvatske 31, Split 21000, Croatia (phone: +385-21-430-644; fax: +385-21-430-701; e-mail: zdravka.aljinovic@ efst.hr).

B. Marasović is with the University of Split, Faculty of Economics, Matice hrvatske 31, Split 21000, Croatia (phone: +385-21-430-697; fax: +385-21-430-701; e-mail: branka.marasovic@ efst.hr).

B. Škrabić is with the University of Split, Faculty of Economics, Matice hrvatske 31, Split 21000, Croatia (phone: +385-21-430-778; e-mail: blanka.skrabic@ efst.hr).

II. The Vasicek Model

A. Stochastic Models Basics

The term structure of interest rates has a long history of traditional theories. The spread in the seventies of inherently risky instruments (variable rate securities and options) posed critical questions for traditional theories, based on the hypotheses of certainty, and gave rise to a golden decade of revolution in the methods of financial analysis on the basis of probability theory. The "new term structure" model, as one of the key products of this golden decade, was developed in this framework: continuous time, perfect and frictionless markets, diffusion processes, no-arbitrage condition i.e. prices constrained by the hypotheses that equivalent assets (or portfolios) in terms of cash flows and other characteristics must earn the same return.

Any asset has many characteristics: the issuer (with a certain probability of default), the coupon flow, the maturity date T, the taxation, etc. To isolate maturity from other dimensions, we shall focus on payment-certain (default free), zero-coupon unit discount bond with no taxes or transaction costs, i.e. a security with price $P(t,T)$ at time t, promising, with probability 1, a payment of one money unit at maturity time $T > t$. Assuming continuous time, it can be written:

$$P(t,T)e^{(T-t)R(t,T)} = 1 \tag{1}$$

where $R(t,T)$ is the interest rate (the rate of return to maturity) of the bond. This relation can be transformed into:

$$R(t,T) = -\frac{\ln P(t,T)}{T-t}. \tag{2}$$

For fixed t, the shape of $R(t,T)$ as T increases determines the term structure of interest rates or, what is the same thing, the yield curve. From (2) it is clear that the term structure can be a rising or falling function of T.

One concept of great importance in this model is the spot interest rate (very short term interest rate) [1]. The spot interest rate is the yield of instantaneously maturing discount bond. Hence denoting the spot rate at time t by $r(t)$ we have

$$r(t)=\lim_{T\to t} R(t,T). \qquad (3)$$

The spot rate of interest is the rate of return investors can earn over the next very short interval of time. It is assumed that $r(t)$ is a stochastic process, subject to two requirements. First, it is assumed that $r(t)$ follows a Markov process. Under this assumption, the future development of the spot rate given its present value is independent of the past development that has led to the present level. The probability distribution of the segment $\{r(\tau),\tau \geq t\}$ is thus completely determined by the value of $r(t)$. Second, it is assumed that $r(t)$ does not change value by an instantaneous jump. Processes which satisfy these two requirements are called diffusion processes.

Finally, the spot rate diffusion process $r(t)$ is described by the Ito stochastic differential equation

$$dr(t)=f(t,r(t))dt+g(t,r(t))dz(t) \qquad (4)$$

where $f(t,r(t))$ is the drift coefficient, $g^2(t,r(t))$ is the diffusion coefficient and $\{z(t)\}$ is a standard Wiener process.

Most of the diffusion models for interest rates are based on the no-arbitrage principle and they are characterized by similar assumptions and properties. Let us state the main assumptions on the bond market:

A.1) The single variable that determines the state of the economy at time t is the spot rate $r(t)$, defined by relation (3);

A.2) The spot rate follows a diffusion process, which is described by the Ito SDE (4);

A.3) The market is efficient: there are no transactions costs, information is available to all investors simultaneously and every investor acts rationally (prefers more wealth to less, and uses all available information).

Assumption A.3) implies that investors have homogeneous expectations and that no profitable riskless arbitrage is possible.

Assumption A.1) implies that the price of a discount bond $P(t,T)$ will be determined solely by the spot interest rate over its term,

$$P(t,T)=P(t,T,r(t)). \qquad (5)$$

It is also assumed that the function $P(t,T,r(t))$ is differentiable with continuous partial derivatives P_t, P_r, P_{rr}.

Using Ito's lemma and according to postulated assumptions the basic equation for pricing of discount bonds in a market described by assumptions A.1), A.2) and A.3) is obtained:

$$\frac{\partial P}{\partial t}+(f+gq)\frac{\partial P}{\partial r}+\frac{1}{2}g^2\frac{\partial^2 P}{\partial r^2}-rP=0, \qquad (6)$$

where the quantity $q(t,r)$ can be interpreted as the increase in expected instantaneous rate of return per additional unit of risk, and it is usually called the market price of risk.

The price at time t, $P(t,T,r)$, of a zero-coupon bond maturing with unit value at time T, will be obtained using equation (6) subject to the boundary condition

$$P(T,T,r)=1. \qquad (7)$$

The term structure of interest rates is then readily obtained by equation (2).

It is important to emphasize that the equation (6) is, actually, the general valuation equation which, together with appropriate boundary condition, can be used to price any interest rate sensitive (IRS) asset. Coupon bonds can be priced by equation (6) if we regard each as a portfolio of discount issues, one for each coupon payment and one for the terminal payment of the bond.

B. Vasicek's Specializations of the Basic Model

Vasicek [6] specialize the previous basic model according to the following assumptions.

It is assumed that the market price of risk is a constant, $q(t,r(t)) = q$, independent of the calendar time and of the level of the spot rate. Second, it is assumed that the spot rate follows the *Ornstein-Uhlenbeck process*,

$$dr(t)=\alpha(\gamma-r(t))dt+\rho dz(t), \qquad (8)$$

with $\alpha > 0$, corresponding to the choice

$f(t,r(t))=\alpha(\gamma-r(t)), g(t,r(t))=\rho$ in equation (4).

The Ornstein-Uhlenbeck process with $\alpha > 0$ is sometimes called the elastic random walk. It is a Markov process with normally distributed increments, which, in contrast to the random walk (the Wiener process) possess a stationary distribution. The drift coefficient $\alpha(\gamma - r)$ represents a force that keeps pulling the process towards its long-term mean γ with magnitude proportional to the deviation of the process from the mean. The stochastic element, which has a constant variance ρ^2, causes the process to fluctuate around the level γ in an erratic, but continuous fashion.

According to given assumptions in this model, the expression for the value P of the zero-coupon bond at time t, with maturity at time T follows:

$$P(t,T,r(t))=exp\left[\frac{1}{\alpha}\left(1-e^{-\alpha(T-t)}\right)\left(R_\infty-r(t)\right)-(T-t)R_\infty-\frac{\rho^2}{4\alpha^3}\left(1-e^{-\alpha(T-t)}\right)^2\right] \qquad (9)$$

III. The Nelson-Siegel Model

The commonly used model for developing yield curve in financial practice is the Nelson-Siegel model [5]. Nelson and Siegel introduced a simple, parsimonious model that is flexible enough to represent the range of shapes generally

associated with yield curves: monotonic, humped and S shaped.
A class of functions that readily generates the typical yield curve shapes is that associated with solutions to differential or difference equations. If the instantaneous forward rate at maturity *T*, *f(t,T)*, is given by the solution to a second-order differential equation with real and unequal roots, it is of the form:

$$f(t,T)=\beta_0+\beta_1 e^{-\frac{T-t}{\tau_1}}+\beta_2 e^{-\frac{T-t}{\tau_2}} \quad (10)$$

where τ_1 and τ_2 are time constants associated with the equation, and β_0, β_1 and β_2 are determined by initial conditions.

Now, zero-coupon rates *R(t)* can be calculated by averaging the corresponding instantaneous forward rates:

$$R(t,T)=\frac{1}{T-t}\int_t^T f(x,T)dx \quad (11)$$

A more parsimonious model that can generate the same range of shapes is given by the equation solution for the case of equal roots:

$$f(t,T)=\beta_0+\beta_1 e^{-\frac{T-t}{\tau}}+\beta_2 \frac{T-t}{\tau} e^{-\frac{T-t}{\tau}} \quad (12)$$

By substituting (12) into (11) and integrating, it is obtained:

$$\begin{aligned} R(t,T) &= \frac{1}{T-t}\int_t^t f(x,T)dx = \frac{1}{T-t}\int_t^T\left(\beta_0+\beta_1 e^{-\frac{T-x}{\tau}}+\beta_2\frac{T-x}{\tau}e^{-\frac{T-x}{\tau}}\right)dx \\ &= \frac{1}{T-t}\left(\beta_0(T-t)-\beta_1\tau \quad e^{-\frac{T-t}{\tau}}+\beta_1\tau+\beta_2\tau\left(-\frac{T-t}{\tau}e^{-\frac{T-t}{\tau}}-e^{-\frac{T-t}{\tau}}+1\right)\right) \end{aligned} \quad (13)$$

After a simple rearrangement of this expression, the yield to maturity is given by:

$$R(t,T)=\beta_0+(\beta_1+\beta_2)\frac{1-e^{-\frac{T-t}{\tau}}}{\frac{T-t}{\tau}}-\beta_2 e^{-\frac{T-t}{\tau}}. \quad (14)$$

So, the forward and zero-coupon yield curves are functions of four parameters: β_0, β_1, β_2 and τ.

It can be seen that

$$\lim_{T\to\infty} R(t,T)=\beta_0, \quad (15)$$

and β_0 corresponds to zero-coupon rates for very long maturities.
At the short end of the curve it is:

$$\lim_{T\to t} R(t,T)=\beta_0+\beta_1 \quad (16)$$

which implies that the sum of parameter values β_0 and β_1 should be equal to the level of the shortest interest rates.

Here, there are also some other features of the parameters that we should consider. If β_1 is negative, the forward curve will have a positive slope and vice versa. Also, if β_2, as being the identifier of the magnitude and the direction of the hump, is positive, a hump will occur at τ whereas, if it is negative, a U-shaped value will occur at τ. Thus, it can be concluded that parameter τ which is positive, specifies the position of the hump or U-shape on the entire curve. Consequently, Nelson and Siegel propose that with appropriate choices of weights for these three components, it is possible to generate a variety of yield curves based on forward rate curves with monotonic and humped shapes [5].

IV. The Results - Yield Curve Development in Croatia

In developing a HRK risk free yield curve, the data of prices of Government bonds (Zagreb Stock Exchange), Treasury bills data (Ministry of Finance) and the data of interest rates from Money Market Zagreb are used. Using nonlinear least squares methods [7] applied to the prices of bonds of different maturities trading at a given point in time, the parameters of the Vasicek and Nelson-Siegel model for each month from January 2006 to August 2008 are estimated (Tables I and II). The obtained parameter values are now inserted into the price formula (9) (and "term structure formula" (2)) for Vasicek i.e. into the formula (14) for Nelson-Siegel model. Appropriate yield curves for a seven years time horizon are given on Fig. 1 and Fig. 2.

It can be seen that resulting yield curves for both models are very similar. Mostly, upward slope trend dominates and in some cases there is a gap that can be explained by liquidity premium on Treasury bond instruments.

Looking for a long time horizon, all curves are increasing. Also, in observed period, the trend of rising interest rates is obvious. In Table III, there are results of one year interest rates estimated by Vasicek and Nelson-Siegel model.

At the end of this analysis we would like to give an answer on very important question in the paper: Which model is better? Although, in our previous researches in this matter, [2] and [3], we preferred stochastic models in this case it is obtained, by alienation coefficient, that parsimonious Nelson-Siegel model is better in empirical values approximation (Table III).

TABLE I
PARAMETERS OF THE VASICEK MODEL

	R∞	r(t)-R∞	α	ρ
Jan-06	0,0387	-0,0011	22,1535	61,7120
Feb-06	0,0377	-0,0013	24,4286	72,4829
Mar-06	0,0388	-0,0009	37,1487	136,3253
Apr-06	0,0408	-0,0011	27,0516	61,4113
May-06	0,0414	-0,0009	26,9794	51,4615
Jun-06	0,0431	-0,0012	20,7702	35,2502
Jul-06	0,0482	-0,0212	0,9556	0,0517
Aug-06	0,0448	-0,0011	21,9366	30,2674
Sep-06	0,1394	-1,1125	0,0930	-0,0012
Oct-06	0,0467	-0,0024	8,7756	1,3818
Nov-06	0,0500	-0,0234	1,0608	0,0564
Dec-06	0,0883	-0,5299	0,1013	-0,0001
Jan-07	0,1069	-0,3037	0,2337	-0,0124
Feb-07	0,0885	-0,2899	0,1712	-0,0030
Mar-07	0,0867	-0,2970	0,1737	-0,0027
Apr-07	0,0659	-0,2595	0,1132	0,0004
May-07	0,0894	-0,2724	0,1845	-0,0044
Jun-07	0,1057	-0,2544	0,2376	-0,0137
Jul-07	0,1123	-0,2415	0,2412	-0,0168
Aug-07	0,0707	-0,2799	0,1161	0,0006
Sep-07	0,1002	-0,2643	0,2126	-0,0073
Oct-07	0,2330	-1,0004	0,1692	-0,0219
Nov-07	0,1205	-0,2132	0,2789	-0,0305
Dec-07	0,0639	0,0000	1,6231	-0,5598
Jan-08	0,0694	0,0000	1,1646	-0,3499
Feb-08	0,0628	-0,0042	2,0774	-0,4093
Mar-08	0,0563	-0,0002	199,9524	200,0834
Apr-08	0,0486	-0,0147	0,3216	0,0136
May-08	0,0549	-0,0002	123,5527	217,7333
Jun-08	0,0562	-0,0002	200,0670	193,4691
Jul-08	0,0561	-0,0004	66,6771	291,2541
Aug-08	0,0569	-0,0003	83,2739	221,2728

TABLE II
PARAMETERS OF THE NELSON-SIEGEL MODEL

	τ	β0	β1	β2
Jan-06	0,0302	3,8850	-2,4578	3,0486
Feb-06	0,0507	3,7109	-2,9127	4,3265
Mar-06	0,0121	3,8939	-3,2260	0,9485
Apr-06	0,0111	4,0577	-3,2269	0,8181
May-06	0,0140	4,1134	-2,7011	1,6791
Jun-06	0,0118	4,2950	-3,0528	0,9100
Jul-06	1,1877	4,6174	-1,7579	2,3778
Aug-06	0,0005	4,4371	188,3011	-247,0262
Sep-06	3,4260	3,8658	-0,4698	4,3195
Oct-06	0,1230	4,7124	-2,0365	0,0000
Nov-06	0,4362	4,9756	-2,4684	0,0000
Dec-06	0,9043	5,0292	-1,7553	0,0000
Jan-07	4,1888	12,4548	-8,9284	-10,8475
Feb-07	1,8035	5,0511	-1,2910	0,0000
Mar-07	8,9999	1,8999	1,5473	6,7423
Apr-07	9,0000	-9,2121	12,7835	21,3252
May-07	0,6318	5,0954	-1,0125	-3,3801
Jun-07	0,3241	5,2151	0,0069	-6,1146
Jul-07	0,2847	5,5260	0,7612	-6,4450
Aug-07	1,5413	5,3798	-1,6655	0,4196
Sep-07	0,4302	5,6102	-0,8193	-4,8425
Oct-07	0,3307	5,5015	1,9311	-7,6594
Nov-07	0,2155	5,5739	2,1638	-9,2468
Dec-07	0,2647	5,7343	1,0704	-6,2919
Jan-08	0,2664	5,7728	1,7171	-6,4618
Feb-08	8,9999	-2,1133	7,1423	13,1928
Mar-08	1,2667	5,8839	-1,2191	1,4972
Apr-08	1,2781	5,8245	-1,4663	1,9419
May-08	0,0045	5,4767	-3,2694	-0,0060
Jun-08	0,0011	5,5429	-5,7632	-0,0123
Jul-08	0,0033	5,5874	-2,6779	0,0058
Aug-08	0,0072	5,6763	-2,3722	0,0004

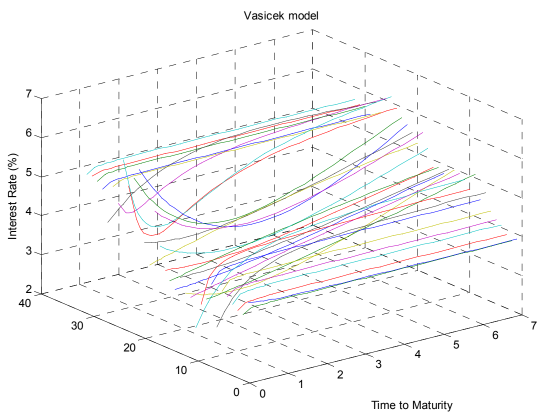

Fig. 1 Vasicek HRK yield curves (01.2006 – 08.2008)

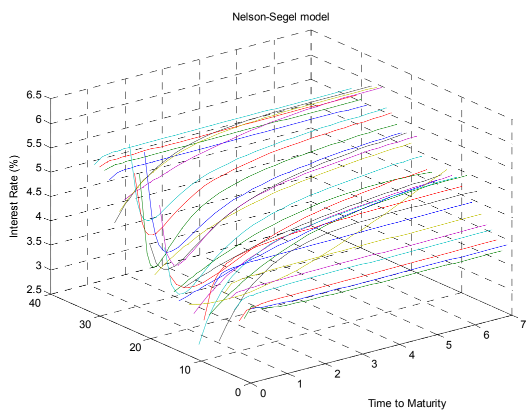

Fig. 2 Nelson-Siegel HRK yield curves (01.2006 – 08.2008)

TABLE III
ONE YEAR INTEREST RATES AND ALIENATION COEFFICIENTS

	One year interest rate (%)		Alienation coefficient	
	Vasicek	Nelson-Siegel	Vasicek	Nelson-Siegel
Jan-06	3,9004	3,9028	0,3700	0,3699
Feb-06	3,7646	3,7826	0,2871	0,2709
Mar-06	3,8637	3,8664	0,2466	0,2442
Apr-06	4,0480	4,0309	0,2808	0,2577
May-06	4,1184	4,0991	0,4466	0,4175
Jun-06	4,2843	4,2698	0,3915	0,3491
Jul-06	4,0816	4,0119	0,4282	0,4275
Aug-06	4,4415	4,4060	0,4060	0,3707
Sep-06	3,7562	3,9785	0,5868	0,4729
Oct-06	4,4770	4,4619	0,2285	0,2318
Nov-06	3,9789	4,0077	0,1941	0,1939
Dec-06	3,7130	3,9671	0,4931	0,4305
Jan-07	3,3136	3,4061	0,5777	0,5698
Feb-07	3,9215	4,0601	0,8244	0,7875
Mar-07	3,6002	3,7123	0,5897	0,5473
Apr-07	3,8900	3,9874	0,4637	0,4087
May-07	3,8482	3,5845	0,7188	0,6460
Jun-07	4,0471	3,6055	0,8433	0,5285
Jul-07	4,6736	4,1484	0,8864	0,5274
Aug-07	4,1112	4,2438	0,3741	0,3438
Sep-07	4,2533	3,8865	0,5828	0,3931
Oct-07	4,9883	4,0716	0,6847	0,2769
Nov-07	4,7783	4,1513	0,7938	0,2144
Dec-07	4,2772	4,5278	0,6248	0,5522
Jan-08	4,3216	4,6898	0,5469	0,4375
Feb-08	5,0386	5,3274	0,7643	0,8318
Mar-08	5,6139	5,3963	0,3729	0,6788
Apr-08	5,2254	5,2664	0,6055	0,6042
May-08	5,4661	5,4621	0,2569	0,2457
Jun-08	5,6010	5,5363	0,2499	0,2209
Jul-08	5,5965	5,5787	0,3008	0,2854
Aug-08	5,6669	5,6592	0,2086	0,2112

V. CONCLUSION

In the paper two groups of interest rate models, stochastic and parsimonious, are presented through the Vasicek model and the Nelson-Siegel model. Presented models are applied on the Croatian financial market to estimate yield curves for each month from January 2006 to August 2008. The resulting yield curves for both models are very similar, where it must be emphasized that, a little bit unexpected, parsimonious Nelson-Siegel model is better in empirical values approximation. Mostly, upward slope trend dominates and in some cases there is a gap that can be explained by liquidity premium on Treasury bond instruments. Looking for a long time horizon, all curves are increasing. Also, in observed period, the trend of rising interest rates is obvious. Beside this comparison of models, it would be interesting to explore the prognostic features of the models, what is one possible way of future researches.

REFERENCES

[1] Z. Aljinović, "Diffusion processes in modelling the term structure of interest rates", in: *Proceedings of the 7th International Conference on Operational Research KOI'98* (I. Aganović, T. Hunjak, R. Scitovski, Eds.), Croatian OR Society, Osijek, 1999, 53-60.
[2] Z. Aljinović, S. Jovanović, "An application of the Cox, Ingersoll, Ross model to the Croatian Government securities market", in: *Proceedings of the 7th International Conference on Operational Research KOI'98* (I. Aganović, T. Hunjak, R. Scitovski, Eds.), Croatian OR Society, Osijek, 1999, 75-82
[3] Z. Aljinović, B. Šego, "How to evaluate the yield curve in transition economy", in: *Proceedings of the 5th International Conference "Enterprise in Transition"*, Faculty of Economics Split, Split, 2003, 262-264 (extended abstract), full text on CD-ROM, 1152-1165
[4] C.R. Nelson, *The Term Structure of Interest Rates*, New York, Basic Books, 1972
[5] C. R. Nelson, A. F. Siegel, "Parsimonious modeling of yield curves", *Journal of Business*, 60, 474-489, 1987
[6] O.Vasicek, "An equilibrium characterization of the term structure", *Journal of Financial Economics,* 5, 177-188, 1977
[7] P. Venkataraman, *Applied Optimization With MATLAB Programing*, John Wiley & Sons, New York, 2002
[8] J.H. Wood, "Are Yield Curves Normally Upward Sloping? The Term Structure of Interest Rates, 1862-1982", *Federal Reserve Bank of Chicago Economic Perspectives*, Vol.7, July/August, 17-23, 1983.