International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:5, No:5, 2011

Theory of Fractions in College Algebra Course

Alexander Y. Vaninsky

Abstract**—**The paper compares the treatment of fractions in a typical undergraduate college curriculum and in abstract algebra textbooks. It stresses that the main difference is that the undergraduate curriculum treats equivalent fractions as equal, and this treatment eventually leads to paradoxes and impairs the students' ability to perceive ratios, proportions, radicals and rational exponents adequately. The paper suggests a simplified version of rigorous theory of fractions suitable for regular college curriculum.

Keywords**—**Fractions, mathematics curriculum, mathematics education, teacher preparation

I. Introduction

FRACTIONS are with us from elementary school to college. The regular college curriculum usually introduces the concept of a fraction by considering one or more parts in a whole that is divided into a greater number of equal parts. Thus, a typical demonstration of the fraction $\frac{1}{6}$ is one slice of a pizza that is divided into six equal slices. Though further in the course students go much farther in studying and using fractions, the basic concept remains essentially the same. Definitions like "A fraction is a ratio of two numbers" simply exchange one undefined term, fraction, for another—in this case, ratio—and so do not add much to students' mathematical understanding of fractions.

This paper is aimed to improve the situation. It is in line with publications by [8] that encourages "to develop a realistic alternative to the teaching and learning of fractions", Philipp and [5] explaining "the way in which teachers view algebra … as a language for generalizing arithmetic", Gordon (2008) emphasizing the importance of conceptual understanding in College Algebra, and Wu (2011) suggesting to teach fractions using the notion of equivalence classes. Its main goal is to present a simplified version of rigorous theory of fractions suitable for regular college curriculum.

II. Limitations of the Simplified Concept of a Fraction

The simplified concept of a fraction that students customarily learn, although helpful at the initial stage, has a negative impact later. Thus, while conceptualizing the fraction as "one slice out of six" is intuitive and easy, attempting to understand the fraction $\frac{-1}{-6}$ in the same way causes difficulties. Another problem arises in a claim of equality for these two fractions: $\frac{1}{6}=\frac{-1}{-6}$ A lack of knowledge of the mathematical theory underlying fractions impairs the ability of students to understand more comprehensive applications of fractions that they encounter later. Rational exponents provide an example of this potential misunderstanding. Students who consider two fractions such as $\frac{1}{3}$ and $\frac{2}{6}$ equal may arrive at different results when they work with them as rational exponents: $(-8)^{\frac{1}{3}}=\sqrt[3]{(-8)^1}=-2$, but $(-8)^{\frac{2}{6}}=\sqrt[6]{(-8)^2}=2$. This is a paradox, because equal quantities should be interchangeable. (We will consider it below in this paper as an example of necessary restrictions on the fraction representatives.)

Baker (2006) offers an additional example of a fraction-related paradox. When introducing elementary operations with imaginary numbers, the author faced a problem: different representations of a negative fraction can lead to different results. For example, although the fractions $-\frac{1}{4},\frac{-1}{4}$ and $\frac{1}{-4}$ are considered equal, their square roots in terms of imaginary numbers are not all the same: $\sqrt{-\frac{1}{4}}=\sqrt{\frac{-1}{4}}=i\frac{1}{2}$, but $\sqrt{\frac{1}{-4}}=\frac{1}{2i}=-i\frac{1}{2}$. This is another example that requires restriction on the definition and feasible operations with fractions. As shown below in this paper, taking a square root of a negative fraction, we should consider it as an additive inverse of a positive fraction, thus eliminating ambiguity. It should be mentioned also that if we want to stay in the framework of a unique result of taking the radical (" the principal value of the square root of a negative number") then we should deny the multiplication law: $\sqrt{ab}\neq\sqrt{a}\sqrt{b}$ if we allow negative values of a or b.

The rules for the addition and comparison of fractions present another source of confusion. Students learn to add fractions by using the least common denominator—a process that they usually study as a definition of the rule for adding fractions rather than as merely a useful technique. Formulating or applying the general rule for comparing fractions usually

A. Y. Vaninsky is with Hostos Community College of The City University of New York, Bronx, NY 10451 USA (phone: +1-718-319-7930; fax: +1-718-518-6607; e-mail: avaninsky@hostos.cuny.edu).

confuses students as well. Another perplexing matter is the question, Is $\frac{\sqrt{2}}{\pi}$ a fraction?

The other question is this: what is ratio? And what is the difference between a fraction and a ratio? Furthermore, what is a proportion? Should proportions be defined in terms of fractions or in terms of ratios?

A curriculum that does not help students gain a rigorous understanding of the theory of fractions does not permit an adequate statement of a problem that calls for solving proportions. A typical definition of a proportion in a college curriculum is something like this: "A proportion is an equation that states that two equivalent ratios are equal." However, this definition raises a question: What is the difference between equivalent and equal ratios? Students who are able to make mechanical use of the cross-multiplication rule for solving proportions are frequently confused about the theory underlying this rule.

These and other problems underscore the need for a college curriculum to present the mathematical theory of fractions in a more rigorous way, while preserving the standard framework of the regular curriculum as well.

III. Fraction Theory Informally Discussed

What are fractions? Conceptually, fractions are shares of a whole. But this general notion needs refining, since not every share of a whole is considered to be a fraction. For example, the length of a side of a square is a share of the length of its diagonal. But this share, which is equal to $\frac{1}{\sqrt{2}}$, is not considered to be a fraction, for reasons that are outside the scope of this paper. Refining the concept of a fraction as a share of a whole requires examining the undefined notion of equal parts of a whole. A fraction is a share of a whole obtained by considering one or several of its equal parts. The fraction-bar notation signifies this fact: a fraction $\frac{m}{n}$ means that a whole has been divided into n equal parts, of which m are under consideration. In mathematics this basic concept is extended by allowing for consideration of zero parts of the whole, a number of parts that is greater than the whole, or a number of parts that have been removed from the whole. This extension corresponds to zero, improper fractions, or negative fractions, respectively.

Mathematically, fractions originate from integer numbers that allow addition, subtraction, and multiplication, but not division. Division of integers is not always possible, and this fact is inconvenient for many practical applications. Fractions resolve the problem. They extend the set of integer numbers **Z** = {…,–3, –2, –1, 0, 1, 2, 3,…}to rational numbers, which inherit all of the integers' properties and allow division as well.

The extension technique that this paper presents may be applied not only to integers but also to any mathematical structures that allow addition, subtraction, and multiplication. Such structures are called *rings*. In abstract algebra, rings are extended to *fields of quotients*, the sets of abstract fractions. The last allow division. An example of a ring is a set of polynomials. It can be extended to the set of algebraic fractions called rational expressions—a field of quotients generated by polynomials. More details, examples, and proofs may be found in [3] and [7]. This paper concerns itself only with integer numbers and fractions originating from them. These fractions are called rational numbers.

Mathematically, fractions are ordered pairs of integer numbers: (a, b), (m, n), (p, q), …, subject to specific axioms of equivalency, order, and operations. In what follows, it is useful to keep in mind that the first element in a pair is a numerator, the second is a denominator, and pairs representing fractions are customarily written with the fraction-bar notation: $\frac{a}{b}$, $\frac{m}{n}$, and $\frac{p}{q}$, respectively. The axioms that convert the pairs into fractions are as follows.

Axiom 1. The second element in a pair is not zero.

Axiom 2. Two pairs (m, n) and (p, q) are equivalent if $mq = np$.

Axiom 3. Addition and multiplication of pairs are performed as follows:

$$(m, n)+ (p, q) = (mq + pn, mn);\ (m, n) \cdot (p, q) = (mp, nq). \quad (1)$$

Axiom 4. A pair (m, n) is positive if $m \cdot n > 0$. A pair (m, n) is greater than a pair (p, q) if their difference is positive:

$$(m, n) > (p, q) \text{ if } (m, n) – (p, q) > 0. \quad (2)$$

Combining axioms 3 and 4 gives a rule of fraction comparison:

$$(m, n) > (p, q) \text{ if } (m, n) – (p, q) = (mq – np, nq) > 0, \quad (3)$$

that is true if and only if

$$(mq – np) \cdot nq > 0. \quad (4)$$

The last expression is the fraction comparison rule in a general form. For practical applications, it can be simplified. Thus, if fractions (m, n) and (p, q) are positive, with $m, n, p, q > 0$, then $(m, n) > (p, q)$ if and only if $mq > np$. This is simply a cross-multiplication rule.

Axiom 1 states that a denominator of a fraction cannot be zero. Axiom 2 clarifies whether two expressions, like $\frac{1}{2}$ and $\frac{2}{4}$, are *equal fractions* or just two different numerals for *one fraction*. Abstract algebra accepts the second answer: these are simply two numerals that represent same fraction. This idea may be justified in the following way. Fractions are related to shares of a whole. In the example above, "one part out of two" is $\frac{1}{2}$, while "two parts out of four" is $\frac{2}{4}$. Both $\frac{1}{2}$ and $\frac{2}{4}$ actually represent equal shares of the whole, or "one-half," and thus, it is reasonable to consider them as merely two equivalent numerals.

ISSN: 2517-9934
Vol:5, No:5, 2011

Axiom 2 formalizes this reasoning by using implicitly the well-known cross-multiplication rule to define *equivalent pairs*. In fraction-bar notation, the pairs $\frac{m}{n}$ and $\frac{p}{q}$ are equivalent if $mq = np$. Students know this rule, but the standard curriculum usually presents it as a technique rather than a definition. By contrast, in abstract algebra, pairs satisfying the condition of axiom 2 are called *equivalent*, rather than *equal*. It is a whole set of equivalent pairs—an *equivalence class*—that is considered as one fraction. Two fractions are equal if their equivalence classes are the same. Particular pairs located in one equivalence class are called *representatives of a fraction*. For example, a fraction $\frac{1}{2}$ is the following equivalence class:

$$\frac{1}{2} = \{\ldots,(-3, -6), (-2, -4), (-1, -2), (1, 2), (2, 4), (3, 6), \ldots\}, \quad (5)$$

with pairs (–3, –6), (–2, –4), (–1, –2), (1, 2), (2, 4), (3, 6) being its representatives.

College algebra curriculum typically presents neither ordered pairs of integer numbers nor equivalence classes. The only notion that it usually considers is a *fraction*, and the only notation that it customarily uses to express this notion is fraction-bar notation. It commonly calls $\frac{1}{2}$ and $\frac{2}{4}$ *equal fractions*, although mathematically speaking, they are merely *equivalent representatives*. Abstract algebra reserves the word *equal* for equivalence classes, which are sets of equivalent pairs. Consider, for example, an equivalence class for a fraction $\frac{2}{4}$. It contains the pair (2, 4) — an obvious representative—and all pairs (m, n) that are equivalent to (2, 4). These include all pairs (m, n) such that $2n = 4m$, $n \neq 0$. This equation can be rewritten as $n = 2m$, $m \neq 0$. Assigning to m all integer numbers except zero, or …, –3, –2, –1, 1, 2, 3, …, produces an equivalence class for the fraction $\frac{2}{4}$:

$$\frac{2}{4} = \{\ldots,(-3, -6), (-2, -4), (-1, -2), (1, 2), (2, 4), (3, 6),\ldots\}. \quad (6)$$

This equivalence class is exactly the same as that for the fraction $\frac{1}{2}$, as given by formula (5). The coincidence of the two equivalence classes indicates that the fractions $\frac{1}{2}$ and $\frac{2}{4}$ are *equal*. The ordered pairs (1, 2) and (2, 4), even if they are written with fraction-bar notation as $\frac{1}{2}$ and $\frac{2}{4}$, are *equivalent*. Unfortunately, a standard college curriculum does not suggest separate names for a fraction as an equivalence class and as a class representative. A rigorous teaching of fractions should stress this difference. In this paper below we use the word "fraction" as a name of an equivalence class, unless opposite is mentioned explicitly or contextually.

Two problems arise in relation to axiom 2. First, is the equivalence rule well defined? In other words, is it true that every pair is included in one and only one equivalence class? Second, are the results of operations independent of the choice of fraction representatives? Axiom 2 imposes conditions on all definitions, operation rules, and theorems. All of them should hold when a pair (m, n) is changed for an equivalent pair (p, q) such that $mq = np$. In particular, the addition and multiplication rules, when applied to equivalent pairs, should lead to equivalent results. The comparison rule should also give the same result irrespective of the representatives used.

Usually, these problems resolve themselves, as shown below. At the same time, however, in some applications of fractions, they do not: different representatives give different results. In such cases, it is essential to state clearly for which representatives of the equivalence class a specific definition, operation, or theorem is applicable. An example is the definition of a rational exponent, which is applicable only to fraction representatives in lowest terms.

Axiom 3 states the rules for the addition and multiplication of fractions. These rules are well defined, as the discussion below will show. Being *well defined* means that the rules are independent from the choice of representatives and are applicable to equivalence classes. From the rules of operations stated by axiom 3, it follows that fractions represented by pairs (m, 1) replicate integer numbers with respect to addition and multiplication. Such fractions are *isomorphic* to the integers. This observation allows for consideration of fractions (m, 1) as integer numbers m. Thus, the fraction $\frac{2}{1}$ may be considered as the integer 2. It is important to note that a fraction is actually *not* an integer number, and mathematics instructors should respect the confusion of students who cannot add or multiply a fraction and an integer. The students' confusion is legitimate: rigorously speaking, the required operations cannot be performed directly unless an integer number m is changed for a fraction (m, 1) or $\frac{m}{1}$ in fraction-bar notation.

It is important to observe that the rule of addition in axiom 3 does not refer to the least common denominator (LCD). Using the LCD is simply a useful technique that permits working with smaller numbers in exchange for doing more mental math. For instance, compare the process of adding the two fractions $\frac{1}{3}$ and $\frac{1}{5}$ by directly applying axiom 3 with the process of adding the same two fractions by using the LCD. In the first case,

$$\frac{1}{3} + \frac{1}{5} = \frac{1\times 5 + 1\times 3}{3\times 5} = \frac{8}{15}.$$

In the second case, the LCD of (3, 5) equals 15, so

$$\frac{1}{3} + \frac{1}{5} = \frac{1\times 5}{3\times 5} + \frac{1\times 3}{5\times 3} = \frac{5}{15} + \frac{3}{15} = \frac{5+3}{15} = \frac{8}{15}.$$

In this example, directly applying axiom 3 turns out to be easier than working with the LCD, while leading to the same result. Axiom 3 justifies a simple rule of addition, similar to the cross-multiplication rule: "To add fractions, cross-multiply and then divide the result by the product of denominators." Consideration of this simple rule elicits a question: Might it not be reasonable to introduce the addition of fractions in basic mathematic courses through instruction based on axiom 3? This question deserves special discussion.

Axiom 4 gives a criterion of positivity that is not usually stated in standard algebra curriculum. Because a fraction has infinitely many different representatives, being able to determine whether it is positive on the basis of only one representative is important. The rule is simple: a fraction is positive if its numerator and denominator have the same signs. The conclusion does not depend on a specific representative. For example, a fraction represented by $\frac{1}{2}$ or, equivalently, by $\frac{-1}{-2}$, is positive, because $1 \cdot 2 > 0$ and $(-1) \cdot (-2) > 0$.

The second part of axiom 4 applies axiom 3 to the subtraction of fractions to allow for the comparison of fractions. For example, comparing two fractions represented by $\frac{1}{2}$ and $\frac{-1}{-3}$, respectively, for the purpose of determining which of them is greater, involves computing the difference of the fractions by using the representatives:

$$\frac{1}{2} - \frac{-1}{-3} = \frac{1\times(-3)-(-1)\times 2}{2\times(-3)} = \frac{-1}{-6}.$$

The product of the numerator and denominator is $(-1) \cdot (-6) = 6 > 0$. This product is positive, and thus, by axiom 4, $\frac{1}{2}$ is greater than $\frac{-1}{-3}$. Using formula (4) leads directly to the same conclusion. Computing the value of $(mq - np) \cdot nq,$ as required by this formula, yields

$$(1 \cdot (-3) - 2 \cdot (-1)) \cdot (2 \cdot (-3)) = 6 > 0.$$

The result is positive, so $\frac{1}{2} > \frac{-1}{-3}$. This result also follows from the observation that $\frac{-1}{-3}$ and $\frac{1}{3}$ are equivalent because $(-1) \cdot (3) = 1 \cdot (-3) = -3$, and $\frac{1}{2}$ is evidently greater than $\frac{1}{3}$ —an intuitive idea related to the size of a pizza slice. The answer to the question of whether the rule of positivity is well defined is yes: neither positivity nor the result of fraction comparison depends on the representatives chosen.

IV. Fractions in Abstract Algebra

A presentation of the main definitions and theorems related to the rigorous theory of fractions follows. Our objective in this section is to restrict the use of abstract algebra to the needs of rigorous presentation of the theory of fractions in college algebra course. As before, the discussion is restricted to the rational numbers—that is, fractions formed from integer numbers.

Definition 1. Fractions are equivalence classes of ordered pairs of integers satisfying axioms 1–4.

Pairs of integers representing the equivalence classes are customarily written in the fraction-bar notation—that is, as $\frac{m}{n}$, instead of as (m, n). Whether we mean a fraction or its representative will follow from the context in each specific case. It is important to note that when we refer to a fraction—say, $\frac{1}{2}$—we are actually referring to infinitely many of its representatives: $\ldots, \frac{-3}{-6}, \frac{-2}{-4}, \frac{-1}{-2}, \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots$. Two fractions are *equal* if the sets of their representatives are the same. Representatives of a fraction are *equivalent*.

A special sign should be used to indicate equivalency—for instance, "~": $\frac{-3}{-6} \sim \frac{1}{2}$. However, for simplicity, college algebra customarily calls representatives *equal* and uses the equals sign: $\frac{-3}{-6} = \frac{1}{2}$. The theorems below justify operations with fractions.

Theorem 1. Each ordered pair of integers belongs to one and only one equivalence class. The operations of addition and multiplication and the ordering rule are well defined. Addition and multiplication are commutative and associative. Multiplication is distributive with respect to addition.

Both [3] and Shifrin (1996) offer a proof of this theorem, which is of principal importance. It states that any representative can be selected to compare fractions or to perform operations. Addition and multiplication follow the usual laws of algebraic operations. For example, if we need to add the fractions $\frac{1}{2}$ and $\frac{1}{3}$, we can select any of their respective representatives for the addition. The result will always belong to the same equivalence class—the equivalence class corresponding to the fraction $\frac{5}{6}$. Thus, if we take $\frac{-3}{-6}$ and $\frac{3}{9}$ as representatives of the fractions $\frac{1}{2}$ and $\frac{1}{3}$, respectively, and add them according to axiom 3, the result will be

$$\frac{-3}{-6} + \frac{3}{9} = \frac{-3\times 9+3\times(-6)}{-6\times 9} = \frac{-45}{-54}.$$

The last expression is equivalent to $\frac{5}{6}$, because $(-45) \cdot 6 = (-54) \cdot 5 = 270$. Students in college algebra would customarily say that they can simplify the fraction $\frac{-45}{-54}$ to $\frac{5}{6}$ by dividing both its numerator and its denominator by (-9).

In a similar manner, we can use any of the representatives of fractions for comparison. For example, $\frac{1}{2}$ is greater than $\frac{1}{3}$

according to the simplified rule for axiom 4 because $1 \cdot 3 > 2 \cdot 1$. But will the result be the same if we take different representatives of each fraction? As before, consider $\frac{-3}{-6}$ and $\frac{3}{9}$ as representatives of $\frac{1}{2}$ and $\frac{1}{3}$, respectively. By applying axiom 4 again—this time in a general form—we get $[(-3) \cdot 9 - (-6) \cdot 3] \cdot [(-6) \cdot 9] = 648 > 0$.

So $\frac{-3}{-6}$ is greater than $\frac{3}{9}$ — the same result as before.

Generally speaking, fractions can be formed not only from integers, but also from many other mathematical objects. Depending on the nature of these objects, fractions may or may not possess some specific properties. The integer numbers do not just form a ring. They form a structure called an *ordered integral domain*, which has some specific properties. A set of equivalence classes of pairs satisfying axioms 1–4 extends it to an ordered field of quotients. Fractions composed of integers—the fractions considered in this paper—inherit all the properties of this field. The following properties are essential here: (a) additive and multiplicative identities exist and are unique, (b) each element has a unique additive inverse, and (c) each element, except the additive identity, has a multiplicative inverse. Clarifications of the identities and inverses are given below. Proofs can be found in Bland (2002) or Shifrin (1996).

The next theorem states the main properties of the fractions formed from integer numbers. Recall that such fractions are called *rational numbers* and are the only fractions that this paper considers.

Theorem 2. The properties of fractions (rational numbers) are as follows:

a. A fraction representative $\frac{m}{n}$ can be chosen with a positive or a negative denominator.

b. The representatives $\frac{m}{n}$ and $\frac{km}{kn}$ $(k \neq 0)$ are equivalent and thus represent the same fraction.

c. The sum of $\frac{m}{n}$ and $\frac{p}{n}$ equals $\frac{m+p}{n}$.

d. The subtraction rule is $\frac{m}{n} - \frac{p}{q} = \frac{mq-np}{nq}$.

e. The additive identity representatives are $\frac{0}{q}$, $q \neq 0$.

f. The multiplicative identity representatives are $\frac{q}{q}$, $q \neq 0$.

g. The additive inverse representatives of $\frac{m}{n}$ are $\frac{-m}{n}$ or $\frac{m}{-n}$, $n \neq 0$. They are denoted as $-\frac{m}{n}$.

h. The multiplicative inverse representatives of $\frac{m}{n}$, $m \neq 0$, are $\frac{n}{m}$. They are denoted as $\left(\frac{m}{n}\right)^{-1}$.

i. The division rule is $\frac{m}{n} \div \frac{p}{q} = \frac{mq}{np}$, $p \neq 0$.

j. Fractions having a representative $\frac{m}{1}$ are isomorphic to the integers.

The proof of theorem 2 that follows uses theorem 1, which ensures the validity of all results for the whole equivalence class once they are proved for selected representatives.

(a) A proof follows from $mn = (-m)(-n)$, which asserts the equivalency of $\frac{m}{n}$ and $\frac{-m}{-n}$. If $n > 0$, then $-n < 0$, and vice versa. Thus, a positive or a negative denominator may be chosen, as desired.

(b) A proof follows from $m(kn) = (km)n = mkn$.

(c) By axiom 3: $\frac{m}{n} + \frac{p}{n} = \frac{mn+np}{nn} = \frac{n(m+p)}{nn} = \frac{m+p}{n}$. The last transformation follows from (b).

(d) A proof follows from

$\left(\frac{m}{n} - \frac{p}{q}\right) + \frac{p}{q} = \frac{mq-np}{nq} + \frac{p}{q} = \frac{(mq-np)q + (nq)p}{(nq)q} = \frac{mqq}{nqq} = \frac{m}{n}$. The last transformation is justified by (b).

(e) For any $\frac{m}{n}$, $\frac{m}{n} + \frac{0}{q} = \frac{mq + n0}{nq} = \frac{mq}{nq} = \frac{m}{n}$, where the last transformation is justified by (b). All representatives of the $\frac{0}{q}$-form are equivalent, because for any two of them, $\frac{0}{q_1}$ and $\frac{0}{q_2}$, $0 \cdot q1 = q2 \cdot 0 = 0$. Moreover, any representative $\frac{p}{q_1}$ that is equivalent to $\frac{0}{q}$ has $p = 0$, because $p \cdot q = q1 \cdot 0 = 0$, and $q \neq 0$.

(f) For any $\frac{m}{n}$, $\frac{m}{n} \times \frac{q}{q} = \frac{mq}{nq} = \frac{m}{n}$, where the last transformation is justified by (b). All representatives of the $\frac{q}{q}$-form are equivalent, because for any two of them, $\frac{q_1}{q_1}$ and $\frac{q_2}{q_2}$, $q1 \cdot q2 = q2 \cdot q1$. Moreover, any representative $\frac{r}{s}$ that is equivalent to $\frac{q}{q}$ has $r = s$, because $r \cdot q = q \cdot s$. The factor $q \neq 0$ can be cancelled on both sides, giving $r = s$.

(g) A proof follows from

$\frac{m}{n} + \frac{-m}{n} = \frac{mn + (-m)n}{nn} = \frac{0}{nn}$ or $\frac{m}{n} + \frac{m}{-n} = \frac{m(-n) + mn}{nn} = \frac{0}{nn}$.

The last term, as a consequence of (*e*), represents the additive identity, as required.

(h) A proof follows from $\frac{m}{n} \times \frac{n}{m} = \frac{mn}{nm}$. As a consequence of (*f*), the last expression is the multiplicative identity. Furthermore, for any representative $\frac{r}{s}$ such that $\frac{m}{n} \times \frac{r}{s} = \frac{q}{q}$, $q \neq 0$, we have $(m{\cdot}r) \cdot q = q \cdot (n{\cdot}s)$. As $q \neq 0$, it can be cancelled on both sides, giving $m{\cdot}r = n{\cdot}s$. The last statement means that any multiplicative inverse $\frac{r}{s}$ is equivalent to $\frac{n}{m}$.

(i) By the definition of division, $\frac{m}{n} \div \frac{p}{q} = \frac{m}{n} \times \left(\frac{p}{q}\right)^{-1}$, where the superscript "–1" stands for the multiplicative inverse. As a consequence of (h), $\left(\frac{p}{q}\right)^{-1} = \frac{q}{p}$, so

$$\frac{m}{n} \div \frac{p}{q} = \frac{m}{n} \times \left(\frac{p}{q}\right)^{-1} = \frac{m}{n} \times \frac{q}{p} = \frac{mq}{np}.$$

(j) The isomorphism of the fractions represented by $\frac{m}{1}$ and the integers represented by *m* follows from the addition and multiplication rules for fractions (axiom 3):

$$\frac{m}{1} + \frac{n}{1} = \frac{m \times 1 + 1 \times n}{1 \times 1} = \frac{m+n}{1};$$

$$\frac{m}{1} \times \frac{n}{1} = \frac{m \times n}{1 \times 1} = \frac{mn}{1}. \therefore$$

Some situations require using a specific, simplest representative. Such a representative is called the representative *in lowest terms*.

Definition 2. A fraction representative is in lowest terms if its numerator and denominator have no common factors other than 1 or –1 and the denominator is positive.

This definition differs from a standard one by the requirement of positivity for the denominator. The proposed definition allows for a proof that ensures that the representative in lowest terms is unique. Otherwise, two fraction representatives could be considered to be in lowest terms—for example, $\frac{1}{2}$ and $\frac{-1}{-2}$.

Theorem 3. The representative in lowest terms (the representative with a positive denominator) is unique.

A proof follows from the uniqueness of the presentation of the natural numbers as a product of powers of prime numbers (see Bland [2002]). Consider the positive fractions first. Let $\frac{m}{n}$ and $\frac{p}{q}$ be two representatives in lowest terms of a positive fraction. Then $m, n, p, q > 0$, and $mq = np$. Presenting each of these as a unique product of powers of prime numbers and noting that the pairs *m* and *n*, and *p* and *q* cannot contain the same factors, we conclude that $mq = np$ is possible only when *m* and *p*, and *n* and *q*, have the same prime factorizations. So, $m = p$ and $n = q$, and the representative in lowest terms is unique. For negative fractions, a proof follows from an observation that each negative fraction is an additive inverse of a unique positive fraction. As such, each negative fraction inherits all of the representatives of the positive fraction, differing only in the sign of each numerator. Thus, the representative in lowest terms also differs only in the sign of the numerator. $\therefore$

The following theorem clarifies the structure of equivalence classes. It states that all representatives of a fraction are simply multiples of the representative in lowest terms.

Theorem 4. Let $\frac{p}{q}$ be a representative of a fraction and $\frac{m}{n}$ be the representative in lowest terms of the same fraction. Then $\frac{p}{q} = \frac{km}{kn}$ for some integer $k \neq 0$.

A proof is as follows. Because $\frac{m}{n}$ and $\frac{p}{q}$ represent same fraction, they are equivalent; that is, $mq = np$. Consider $m \neq 0$ first. In this case, $p \neq 0$ as well. Furthermore, $q = np/m = n(p/m)$, so *p* is a multiple of *m* because *m* and *n* have no common factors except 1 or –1. Thus, $p = k_1 m$ for some integer k_1. Likewise, $p = mq/n = m(q/n)$, so $q = k_2 n$. Substituting expressions for *p* and *q* into the equivalence equality gives us $mq = m(k_2 n) = np = n(k_1 m)$. This implies $m(k_2 n) = n(k_1 m)$, or $mnk_2 = mnk_1$. If $m \neq 0$, then the product *mn* can be cancelled on both sides, giving us $k_1 = k_2 = k$. In this case, $p = km$ and $q = kn$, as required. If $m = 0$, then $p = 0$ as well. Because $\frac{m}{n}$ is a representative of the additive identity in lowest terms, it is $\frac{0}{1}$, and all the other representatives are $\frac{0}{q}$, $q \neq 0$, as given by theorem 2(e). In this case, we have $\frac{p}{q} = \frac{0}{q} = \frac{q \cdot 0}{q \cdot 1}$, and the theorem holds with $k = q \neq 0$. $\therefore$

Part (c) of theorem 2 gives rise to the following practical rule for the addition of fractions. If the representatives of two fractions have the same denominators, then to add them, we can simply add the numerators. For example, the representatives of the fraction $\frac{1}{2}$ are $\{\ldots, \frac{-3}{-6}, \frac{-2}{-4}, \frac{-1}{-2}, \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots\}$, and the representatives of the fraction $\frac{1}{3}$ are $\{\ldots, \frac{-3}{-9}, \frac{-2}{-6}, \frac{-1}{-3}, \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \ldots\}$. By observation, we can find two representatives with the same denominators—for example, $\frac{3}{6}$ and $\frac{2}{6}$, respectively. By applying theorem 2, part (c), we obtain

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{3+2}{6} = \frac{5}{6}.$$

The standard curriculum usually presents this useful rule without proof, merely as a guideline to a more sophisticated rule using the least common denominator (LCD). Actually, it is a theorem, rather than simply a utilitarian rule of operation. On the one hand, the advantage of the LCD-based technique is that it allows for lowering the values of the numbers involved in the operation of addition. On the other hand, this technique requires more mental work.

Theorem 5. Let $\frac{m}{n}$ and $\frac{p}{q}$ be representatives of two fractions, and an integer s be a common multiple of n and q; that is, $s = k_1n = k_2q$. Then

$$\frac{m}{n}+\frac{p}{q}=\frac{mk_1+pk_2}{s}. \tag{7}$$

Parts (b) and (c) of theorem 2 support a proof:

$$\frac{m}{n}+\frac{p}{q}=\frac{k_1m}{k_1n}+\frac{k_2p}{k_2q}=\frac{k_1m}{s}+\frac{k_2p}{s}=\frac{mk_1+pk_2}{s}.$$

The first transformation is an application of theorem 2(b), and the second, an application of 2(c). ∴

As follows from the proof, it is not necessary to use the *least* common denominator: any multiple of both denominators is sufficient. But the lower the multiple, the smaller the numbers involved in the operation. The preceding discussion suggests an interesting question for discussion in a classroom: Which technique should computers use to add fractions—that based on the LCD or that based on axiom 3? It is likely that computers use direct definition first and then simplify the answer using some sort of the Euclidean division algorithm. In support of this opinion, in may be mentioned that prime factorization of integer numbers is computationally difficult problem. Its complexity, in particular, underlies modern cryptography, Scheinerman (2006).

V. Ratios

Ratios generalize fractions in two dimensions. First, they allow for several, rather than just two, elements. Second, the elements may be real numbers, not necessarily integers, as with fractions. Conceptually, while a fraction is a share of a whole, a ratio is a set of relationships among the parts composing the whole. The parts may be of different nature. A typical example of a ratio is a recipe. For example, a recipe that reads: For serving two persons take 2 pound rabbit, ½ cup flour, 1 tablespoon butter, and 1cup red wine, may be written as the ratio 2 persons : 2 lb : ½ cup : 1 tablespoon : 1 cup. This ratio tells us that depending on the number of persons served, the amounts of each product should be increased or decreased proportionally.

Mathematically, ratios are ordered sequences of real numbers (n-tuples) satisfying the following axioms.

Axiom 1R. No zero elements in a sequence.

Axiom 2R. (Cross - Multiplication Rule) Two sequences $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are equivalent:

$$(a_1, a_2, \ldots, a_n) \sim (b_1, b_2, \ldots, b_n) \text{ if } a_ib_j = a_jb_i, \tag{8}$$

for all $1 \le i, j \le n$.

As previously for fractions, a definition of a ratio is this:

Definition 1R. Ratios are equivalence classes of ordered sequences of real numbers satisfying axioms 1R and 2R.

It should be stressed that *no one element* of a ratio can be zero. Also, as opposed to fractions, ratios cannot be ordered, added or multiplied. (Except for the ratios of two elements that may be multiplied, as shown below.)

Ordered sequences of real numbers representing the equivalence classes of ratios are customarily written in the colon notation as $a_1 : a_2 : \ldots : a_n$. Similarly to fractions, the following theorems may be proved:

Theorem 1R. Each ordered sequence belongs to one and only one equivalence class.

Also, similarly to fractions, a notion of a "ratio in lowest terms" may be introduced:

Definition 2R. A ratio representative is in lowest terms if its last element is 1.

Theorems 3 and 4 may be combined in

Theorem 3R. The representative in lowest terms is unique. If $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are equivalent, that is are representatives of the same ratio, then $a_i = rb_i$ for some real number $r \neq 0$ for all $i = 1,2,\ldots,n$.

A proof is based on axiom 2R and is similar to that for fractions. ∴

The last element has been chosen just for the convenience of the presentation of the ratio of two elements. In this case, a ratio can be expressed as just a real number. For example, a ratio (25 : 10) ~ (2.5 : 1) may be presented as 2.5; the second element is kept in mind. Any element of a ratio can be chosen for the normalization to lowest terms. Theorem 3R holds, but should be rephrased appropriately.

It may be noted that a ratio may be presented geometrically as a line passing through a origin in the n-dimensional space with removed origin: $\boldsymbol{R}^n \backslash \{\boldsymbol{0}\}$. Each point of the line (except the origin that has been excluded) corresponds to the ratio representative. The lowest term representative is located in the intersection of the line with a hyper plane $x_n=1$, or, if another coordinate is used in the definition of the ratio in lowest terms, $x_i=1$, $1 \le i \le n$. This presentation of ratios allows for their consideration as elements of projective space $\boldsymbol{PR}^{n-1}$ (author is thankful to Tanvir Prince for this comment).

Ratios of two elements, like $a_1 : a_2$, are of special importance in applications. As follows from the theorem 2R, any such ratio has a unique representative in lowest terms $b_1 : 1$. As the two representatives are equivalent: $a_1 : a_2 \sim b_1 : 1$, the latter allows for consideration of the whole set of equivalent ratios of two elements $a_1 : a_2$ as a real number b_1. A ratio in lowest terms may be considered as a point on a line $y = 1$ in an xy-plane with the origin (0,0) removed. It may be noted that fractions allow for similar geometric interpretation. In this case, a discrete coordinate system in xy-plane should be considered, see Arnon, Nesher and Nirenburg (1999) for details.

In some applications of the ratios of two elements such as chained price indexes or chained rates, multiplication of ratios may be consistently defined as

$(a_1: a_2)\,(c_1: c_2) = a_1c_1 : a_2\, c_2,$ (9)
similar to multiplication of fractions. The result is equal to the product of two real numbers representing the ratios and does not depend on the choice of the representatives. For example, if a car runs 40 miles on 2 gallons of gasoline, and the price of a gallon is 3 dollars, then the product of two ratios (40:2) · (1:3) = (40·1 : 2·3) = 40 : 6 ~ 20 : 3 ~ $\frac{20}{3}$: 1 means that the car runs 20 miles per 3 dollars or $\frac{20}{3}$ miles per dollar.

It should be stressed that in spite of some similarities ratios are not fractions. They are different mathematical objects originating from different real-life problems, comprising different elements (real numbers as contrary to integer numbers for fractions) and satisfying different axioms. Ratios (except for the ratios of two elements) cannot be ordered, added or multiplied.

VI. Answers to the Questions Supplied by the Theory of Fractions

Equipped with the theory of fractions, we are in a position to answer the questions posed at the beginning of this paper. The following discussion revisits and resolves each of those questions in turn.

The fractions $\frac{1}{2}$ and $\frac{2}{4}$ are not two equal fractions but merely different representatives of the same fraction. In other words, they are different numerals for the same quantity.

Rational exponents are defined only for fraction representatives in lowest terms. For instance, $(-8)^{\frac{2}{6}}$ is undefined and should be exchanged for $(-8)^{\frac{1}{3}} = -2$.

For negative fractions, both numerator and denominator are assumed positive with the minus sign associated with the fraction itself. Though different equivalent approaches are available, this one seems to us more intuitive and better perceived. Taking square roots from negative fractions requires that we consider radicands as additive inverses of positive fractions. By doing so, we assign the minus sign neither to the numerator nor the denominator but to the fraction itself, by the definition. As a result, the problem cited by Baker (2006) disappears, because only one answer is possible: $\sqrt{-\frac{1}{4}} = i\frac{1}{2}$, and the expressions $\sqrt{\frac{-1}{4}}$ and $\sqrt{\frac{1}{-4}}$ are undefined.

An expression such as $\frac{\sqrt{2}}{\pi}$ is not a fraction. It is simply an incomplete operation over real numbers. It can be viewed, however, as a representative of the ratio $\sqrt{2} : \pi$. The latter is a ratio of a length of the diagonal of a square with a side $s = 1$ and the length of a semicircle of radius $r = 1$.

A proportion is a statement that two ratio representatives are equivalent. For example, 1 : 2 ~ 4 : x, where the sign "~" stands for equivalency. (Note that in the sense of this definition writing a proportion in terms of fractions, like $\frac{1}{2}$ ~ $\frac{4}{x}$ or $\frac{1}{2} = \frac{4}{x}$, is not correct.) Axiom 2R justifies a method of solution. For example, the two pairs (1, 2) and (4, x) are equivalent if $1 \cdot x = 2 \cdot 4$, so $x = 8$. This rigorous notation and reasoning associated with this definition of proportion are not typically used in college algebra. Instead, it is customary to write $\frac{1}{2} = \frac{4}{x}$ and to consider it as a statement of the equality of the two fractions or ratios. It may be noted also that some textbooks define proportions in terms of fractions. In this case, all the terms of a proportion, by the definition, should be integers. Otherwise, rational equations enter the picture, pushing out proportions. For example, both the rational equation $\frac{2}{3} = \frac{3}{x}$ and proportion in terms of ratios 2 : 3 ~ 3 : x have the solution $x = 3 \cdot 3 / 2 = 4.5$. But as a proportion in terms of fractions, the equation has no solution, because the pair (3, 4.5) is not a fraction representative, and the right hand side of the equation $\frac{3}{x} = \frac{3}{4.5}$ is not a fraction.

VII. The Place of Fractions in a Broader Context

How meaningful is the approach suggested in this paper? How valuable can it be? First, it resolves the paradoxes cited earlier. Second, it provides students with a general point of view on fractions, which, like some other mathematical objects, can be considered as pairs of numbers. These objects include, but are not limited to, complex numbers, vectors, and functions. Thus, this approach paves the way to a connected, unified way of teaching and learning of such objects.

For example, to obtain complex numbers, consider the pairs (m, n) and (p, q), where m, n, p, and q are real numbers. Define the sum and product of the pairs as follows:

$$(m, n) + (p, q) = (m + p, n + q),\ (m, n) \cdot (p, q) = (mp - nq, mq + np) \quad (10)$$

It can be shown that when addition and multiplication are defined in this way, they are commutative and associative, and multiplication is distributive with respect to addition. The pairs $(m, 0)$ and $(p, 0)$ are isomorphic to the real numbers m and p, respectively. A pair (0, 1) has an interesting property:

$$(0, 1) \cdot (0, 1) = (-1, 0). \quad (11)$$

The last result means that the square of (0, 1) is isomorphic to a real number –1. This observation justifies the terminology and notation related to complex numbers:

$$(0, 1) = \boldsymbol{i},\ \boldsymbol{i}^2 = -1, (m, n) = (m, 0) + (0, n) = m + \boldsymbol{i}n. \quad (12)$$

With this construction in mind, students can more easily grasp complex numbers. The "imaginary unit" $\boldsymbol{i}$ becomes simply a specific pair of real numbers without any of the mystery that sometimes accompanies its appearance in college algebra curriculum. It is important to note that in contrast to fractions, complex numbers are not ordered, and they have no equivalence classes.

The approach to fractions that the paper describes also facilitates the use of pairs as an alternative to function notation. We can write (m, n) or (p, q) as $n = f(m)$ or $q = f(q)$, respectively. Addition and multiplication are defined only for pairs with equal first terms:

$$(m, n) + (m, q) = (m, n+q), \ (m, n) \cdot (m, q) = (m, nq). \tag{13}$$

These formulas are just different notations for the well-known ones

$$(f + g)(m) = f(m) + g(m), (f \cdot g)(m) = f(m) \cdot g(m), \tag{14}$$

where $f(m) = n$, $g(m) = q$.

Another topic in college algebra that becomes more accessible through the proposed approach to fractions is vectors. Pairs of real numbers can be viewed as vectors in the plane. Only one operation is possible in this case—vector addition:

$$(m, n) + (p, q) = (m+p, n+q). \tag{15}$$

Vectors are not ordered, nor can they be multiplied by one another. However, vectors can be multiplied by a number:

$$m \cdot (p, q) = (mp, mq). \tag{16}$$

It is important to emphasize and explore the similarities between algebraic and geometric interpretations of vectors in the plane and operations over them. Also, similarities can be established between vectors in the plane and complex numbers. These similarities are of importance in the study of functions of complex variables.

In summary, a consideration of fractions as equivalence classes of ordered pairs of integer numbers can serve as a useful teaching tool, leading students to a better understanding of fractions *per se* and preparing them to study rational expressions, complex numbers, vectors, or functions. When fractions are studied more rigorously much earlier, they have the potential to help students grasp broader contexts of abstract algebra later. Viewing fractions as equivalence classes also helps students build a surer understanding of real numbers, which can be viewed as equivalence classes of rational numbers.

REFERENCES

[1] I. Arnon, P. Nesher, and R. Nirenburg. " What can be learnt about fractions only with computers." In: O. Zaslavsky 23rd Conference of the International Group for the Psychology of Mathematics Education (PME-23, 1999). 2, 33-40.

[2] W. Baker, "The Complexities of Complex numbers." Working paper. Mathematics Journal, Spring 2006, 1-2. Hostos Community College, Bronx, NY.

[3] P. Bland, The Basics of Abstract Algebra. NY: W.H. Freeman, 2002.

[4] S. Gordon, S. " What's Wrong with College Algebra." Primus, 2008, 18(6), 516 – 541.

[5] R. Philipp, B. Schappelle. "Algebra as Generalized Arithmetic: Starting with the Known for a Change. Preview." Mathematics Teacher, 1999, 92(4), 310 – 316.

[6] E. Sheinerman. Mathematics. A Discrete Introduction. 2nd Ed. CA: Thomson Brooks/Cole, 2006.

[7] T. Shifrin. Abstract Algebra: A Geometric Approach. NJ: Prentice Hall, 1996.

[8] L. Streefland, Fractions in Realistic Mathematics Education: A Paradigm of Developmental Research. MA: Kluwer. 1991

[9] H. Wu. "The Mis-Education of Mathematics Teachers." Notices of the American Mathematical Society, 2011, 58(2), 372 – 384.