On general stability for switched positive linear systems with bounded time-varying delays

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Abstract—This paper focuses on the problem of a common linear copositive Lyapunov function(CLCLF) existence for discrete-time switched positive linear systems(SPLSs) with bounded time-varying delays. In particular, applying system matrices, a special class of matrices are constructed in an appropriate manner. Our results reveal that the existence of a common copositive Lyapunov function can be related to the Schur stability of such matrices. A simple example is provided to illustrate the implication of our results.

Keywords—Common linear copositive Lyapunov functions; Positive systems; Switched systems; Delays.

I. Introduction

SWITCHED systems Switched systems are hybrid dynamical systems composed of subsystems with their own parameterizations subject to a rule orchestrating the switching law between the various subsystems. On the other hand, an additional frequent and inherent constraint in dynamical systems, is the nonnegativity of the states. Many physical systems in the real world involve variables that have nonnegative sign: population levels, absolute temperature, concentration of substances, and so on. Such systems are referred to as *positive systems* which means that any trajectory of the system starting at an initial state in the positive orthant remains forever (see [1], [2]). This feature makes analysis and synthesis of positive systems a challenging and interesting job (see, for example, [3], [8], [5], [6], [7], [4], [9] and references therein).

In the last decades, the importance of switched positive linear systems (SPLSs) has been highlighted by many researchers because of finding broad application in communication systems [8], formation flying [5], and other areas. It should be noted that, although positive systems had been many recent studies in the control engineering and mathematics literature, there are still many open questions relating to SPLSs. Thus, this observation has led to great interest in the stability of such systems under arbitrary switching regimes. A key result in this connection is that stability of such systems is equivalent to the existence of a common Lyapunov function [10]. Generally speaking, three classes of Lyapunov function naturally suggest themselves for LSPSs: common quadratic Lyapunov functions, common diagonal Lyapunov functions, and CLCLFs. For continuous-time SPLSs, the authors of [11] and independently Dvid Angeli, posed a conjecture that the existence of common quadratic Lyapunov function can be determined by testing the Hurwitz-stability of an associated convex set of matrices. Gurvits, Shorten and Mason [12] proved that this conjecture is

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true for pairs of second order systems and is false in general. In the paper [13], a necessary and sufficient was derived for the existence of common diagonal Lyapunov function for the systems with irreducible system matrices. It is well known that traditional Lyapunov functions may give conservative stability conditions for SPLSs as they fail to take account that the trajectories are naturally constrained to the positive orthant. Therefore, it is natural to adopt CLCLFs to deal with the stability of SPLSs [4]. Moreover, work discussed in [14], [15] provided a method for determining whether or not a given SPLSs is stable. Such an approach is based upon determining verifiable conditions for a CLCLF. For discrete-time SPLSs, switched copositive Lyapunov function method was proposed in [6], some necessary and sufficient conditions for the existence of such a function has been established.

Up to now, the studies on CLCLFs pay little attention on delayed SPLSs. Based on this observation, this paper focuses on the CLCLF for discrete-time SPLSs with bounded time-varying delays. By showing the intersection existence of a finite number of convex cones that arise in connection with CLCLFs, we object to determine some tractable conditions for the existence of such a function for delayed SPLSs. The organization of this paper is as follows. Section II gives the mathematical background and notations necessary. Section III is dedicated to derive some checkable necessary and sufficient conditions for the existence of a CLCLF for discrete-time SPLSs with bounded time-varying delays. A simple example is provided to illustrate the main results of this paper in Section IV, and some concluding remarks are presented in Section V.

II. Notation and Background

Throughout, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}^n(\mathbb{R}^n_{0,+}, \mathbb{R}^n_+)$ stands for the $n-$dimensional real (nonnegative, positive) vector space and $\mathbb{R}^{n \times n}$ is the space of $n \times n$ matrices. For A in $\mathbb{R}^{n \times n}$, a_{kl} denotes the element in the (kl) position of A. $A \succeq 0 (\preceq 0)$ means that all elements of matrix A are nonnegative (nonpositive) and $A \succ 0 (\prec 0)$ means that all elements of matrix A are positive (negative). The notion $A > 0 (< 0)$ means that A is a symmetric positive (negative) definite matrix. $A^T(A^{-1})$ represents the transpose (inverse) of matrix A. Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{N}_0 = \{0\} \bigcup \mathbb{N}$. $\lambda(A)$ represents the eigenvalue of A, and $\rho(A)$ denotes the spectral radius of A. Also, when referring to the linear switched systems, stability shall be used to denote uniform asymptotic stability under arbitrary switching signals.

A convex cone in $\mathbb{R}^n$ is a set $\Omega \in \mathbb{R}^n$ such that, for any $x, y \in \Omega$ and any $\alpha \geq 0, \beta \geq 0, \alpha x + \beta y \in \Omega$. The convex

cone is said to be open (closed) if it is open (closed) with respect to the usual Euclidean topology on $\mathbb{R}^n$. For an open convex cone Ω, $\bar{\Omega}$ denotes the closure of Ω.

Lemma 1: [16] Let Ω_1, Ω_2 be open convex cones in $\mathbb{R}^n$. Suppose that $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \{0\}$. Then there is a vector $v \in \mathbb{R}^n$ such that

$$v^T x < 0 \quad for\ all \quad x \in \Omega_1$$

and

$$v^T x > 0 \quad for\ all \quad x \in \Omega_2.$$

Next, we shall now recall some basic facts about positive systems.

The discrete-time linear time-invariant system

$$\Sigma_A : x(k+1) = Ax(k), \quad x_0 = x(0)$$

is said to be positive if $x_0 \succeq 0$ implies that $x(k) \succeq 0$ for all $k \in \mathbb{N}$. See [4] for a description of the basic theory and several applications of positive linear systems. A well known result is that the system Σ_A is positive if and only if the matrix A satisfies $A \succeq 0$. In addition, A matrix A is said to be *Schur* if and only if spectral radius $\rho(A) < 1$. A classic result shows that the positive system Σ_A is stable if and only if A is a Schur matrix. Before proceeding, we collect some results for a matrix $A \succeq 0$ to be Schur, which are relevant for the work of this paper.

Lemma 2: [7] Let $A \succeq 0$ in $\mathbb{R}^{n \times n}$. Then A is a Schur matrix if and only if there exists a vector $v \succ 0$ in $\mathbb{R}^n$ with $(A - I)v \prec 0$.

Lemma 3: Let $A \succeq 0$ in $\mathbb{R}^{n \times n}$ be Schur, then $\det(A - I) \neq 0$.

The following result is easily derived from [17]

Lemma 4: Consider matrices A, B with $A \succeq B \succeq 0$, if A is Schur, then B is also Schur.

The CLCLF approach lies on the following fact.

Definition 1: For discrete time-invariant SPLS

$$\Sigma_{A_i} : x(k+1) = A_i x(k), x_0 = x(0) \succeq 0, i = 1, \cdots, m,$$

the function $V(x) = v^T x$ is said to be a CLCLF if and only if $V(x) > 0$ and $\dot{V}(x) < 0$ for all $x \succ 0$, or, equivalently, $v \succ 0$ and $(A_i - I)^T v \prec 0$.

III. MAIN RESULTS

The discrete-time switched linear system with bounded time-varying delays given by

$$x(k+1) = A^{(0)}(k)x(k) + \sum_{l=1}^{p} A^{(l)}(k)x(k - h^{(l)}(k)),$$
$$x(0) = \varphi(k) \succeq 0, k = -h, \cdots, 0, \tag{1}$$

where $x(k)$ are the state vectors, $k \in \mathbb{N}_0$, $A^{(l)}(k) \in \{A_1^{(l)}, \cdots, A_m^{(l)}\}, A_i^{(l)} \succeq 0$ in $\mathbb{R}^{n \times n}$. $h^{(l)}(k) \in \mathbb{N}_0$ are the time-variant delays satisfying $0 \leq h^{(l)}(k) \leq h^{(l)}$ with constant $h^{(l)} \in \mathbb{N}_0$ and $l \in \mathcal{P} = \{0, \cdots, p\}$. $i \in \mathcal{I} = \{1, \cdots, m\}$ is the index set, m is the number of the subsystems, $\varphi : \{-h, \cdots, 0\} \to \mathbb{R}_{0,+}$ is the vector-valued initial function, and $h = \max\{h^{(l)} | l \in \mathcal{P}\}$. Also, write $\mathcal{H} = \{0, 1, \cdots, h\}$.

Obviously, the switched linear system (1) is positive with $\varphi(k) \succeq 0, A_i^{(l)} \succeq 0$ [9]. Before presenting our derivation, we shall make some preliminary works.

Now, for $g \in \mathcal{H}, l \in \mathcal{P}$, let $\bar{x}(k) = [x^T(k), x^T(k-1), \cdots, x^T(k-h)]^T$ in $\mathbb{R}^{n(h+1)}$ and

$$\bar{A}(k) = \begin{bmatrix} \bar{A}^{(0)}(k) & \cdots & \bar{A}^{(h-1)}(k) & \bar{A}^{(h)}(k) \\ I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & I_n & 0 \end{bmatrix} \tag{2}$$

with

$$\bar{A}^{(g)}(k) = \begin{cases} A^{(l)}(k), & h^{(l)}(k) = g; \\ 0, & h^{(l)}(k) \neq g. \end{cases} \tag{3}$$

Then, the switched system (4) without delays equivalent to (1) has the form

$$\bar{x}(k+1) = \bar{A}(k)\bar{x}(k), \tag{4}$$

where $\bar{A}(k) \in \{\bar{A}_1, \cdots, \bar{A}_m\}$ with

$$\bar{A}_i = \begin{bmatrix} \bar{A}_i^{(0)} & \cdots & \bar{A}_i^{(h-1)} & \bar{A}_i^{(h)} \\ I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & I_n & 0 \end{bmatrix} \tag{5}$$

It easy to see that $\bar{A}_i \succeq 0$ in $\mathbb{R}^{n(h+1) \times n(h+1)}$, and the switched system (4) is thus positive.

Particularly, for later discussion, we should give some relevant notations. We use $\Psi_{n(h+1),m}$ to denote the set of all mapping $\varphi : \{1, \cdots, n(h+1)\} \to \mathcal{I}$. Define the matrix $\bar{A}_\varphi (\bar{A}_1, \cdots, \bar{A}_m)$ by

$$\bar{A}_\varphi (\bar{A}_1, \cdots, \bar{A}_m) = \left(\bar{A}_{\varphi(1)}^{(1)}, \cdots, \bar{A}_{\varphi(n)}^{(n)} \right). \tag{6}$$

Obviously, for $1 \leq j \leq n(h+1)$, the matrices $\bar{A}_\varphi (\bar{A}_1, \cdots, \bar{A}_m) \succeq 0$ are in $\mathbb{R}^{n(h+1) \times n(h+1)}$, whose the jth column $\bar{A}_{\varphi(j)}^{(j)}$ is the jth column of one of the $\bar{A}_1, \cdots, \bar{A}_m$. In addition, all matrices formed in such way are denoted as $\mathscr{A}(\bar{A}_1, \cdots, \bar{A}_m) = \{\bar{A}_\varphi (\bar{A}_1, \cdots, \bar{A}_m) | \varphi \in \Psi_{n(h+1),m}\}$, it is easy to see that $\{\bar{A}_1, \cdots, \bar{A}_m\} \subset \mathscr{A}(\bar{A}_1, \cdots, \bar{A}_m)$.

Moreover, the following key role lemma will be used for our objective.

Lemma 5: For all $i \in \mathcal{I}$, let matrices $\bar{A}_i \succeq 0$ in $\mathbb{R}^{n(h+1) \times n(h+1)}$ be Schur, and $\Omega_{\bar{A}_i} = \{v \succ 0 | (\bar{A}_i - I)^T v \prec 0\}$ with $v \in \mathbb{R}^{n(h+1)}$ such that $\cap_{i=1}^m \Omega_{\bar{A}_i} = \{0\}$. Then there exist m positive definite diagonal matrices D_i in $\mathbb{R}^{n(h+1) \times n(h+1)}$ such that $\sum_{i=1}^m (\bar{A}_i - I)D_i = 0$.

Proof: Conversely, we show that if there exist no some positive definite diagonal matrices D_i in $\mathbb{R}^{n(h+1) \times n(h+1)}$ such that $\sum_{i=1}^m (\bar{A}_i - I)D_i = 0$, then at least one nonzero vector $v \succeq 0$ in $\mathbb{R}^{n(h+1) \times n(h+1)}$ belongs to the set $\cap_{i=1}^m \bar{\Omega}_{\bar{A}_i}$.

First of all, suppose that there exist no some positive definite diagonal matrices D_i such that $\sum_{i=1}^m (\bar{A}_i - I)D_i = 0$. Then for any vector $w \succ 0$ in $\mathbb{R}^{n(h+1)}$, $\sum_{i=1}^m (\bar{A}_i - I)D_i w = 0$ is not true. This further implies that $\sum_{i=1}^m (\bar{A}_i - I)w_i \neq 0$ with some $w_i = D_i w \succ 0$ in $\mathbb{R}^{n(h+1)}$. As $\bar{A}_i \succeq 0$ are Schur matrices, from Lemma 2, we thus find

$$\left\{ \sum_{i=1}^m (\bar{A}_i - I)w_i | w_i \succ 0 \right\} \cap \mathbb{R}_+^{n(h+1)} = \emptyset. \tag{7}$$

For simplicity, we use Ω to denote the set $\left\{\sum_{i=1}^{m}(\bar{A}_i - I)w_i | w_i \succ 0\right\}$. From (7), it is easy to show that $\bar{\Omega} \cap \mathbb{R}_{0,+}^{n(h+1)} = \{0\}$. Applying Lemma 1, we know that there exists a vector $v \in \mathbb{R}^{n(h+1)}$ such that

$$v^T x < 0 \quad \text{for all} \quad x \in \Omega$$

and

$$v^T x > 0 \quad \text{for all} \quad x \in \mathbb{R}_+^{n(h+1)}.$$

From the later inequality, it is easy to show that $v \succeq 0$. In this case, for $w_i \succ 0$, the front inequality can be rewritten as

$$\sum_{i=1}^{m} v^T (A_i - I)w_i < 0. \tag{8}$$

Due to $v \succeq 0$(nonzero) and $w_i \succ 0$, (8) means that, for all $i \in \mathcal{I}$, $(\bar{A}_i - I)^T v \preceq 0$ have to hold. Finally, we find a nonzero $v \succeq 0$ in $\mathbb{R}^{n(h+1) \times n(h+1)}$ such that $v \in \cap_{i=1}^{m} \bar{\Omega}_{\bar{A}_i}$. This completes the proof. ∎

Based on the preliminary results above, we present the following Theorem for the SPLS (1).

Theorem 1: For all $l \in \mathcal{P}$, let $A_1^{(l)}, \cdots, A_m^{(l)} \succeq 0$ in $\mathbb{R}^{n \times n}$, then the following statements are equivalent. (i)

(i) For any $\bar{A} \in \mathscr{A}(\bar{A}_1, \cdots, \bar{A}_m)$, $\bar{A}$ is a Schur matrix.
(ii) The discrete-time SPLS *(1)* has a CLCLF.
(iii) The discrete-time SPLS

$$\bar{x}(k+1) = \bar{A}(k)\bar{x}(k), \quad \bar{x}_0 = \bar{x}(0) \succeq 0, \tag{9}$$

has a CLCLF, where $\bar{A}(k) \in \mathscr{A}^*(\bar{A}_1, \cdots, \bar{A}_m)$ in $\mathbb{R}^{n \times n}$ with $\mathscr{A}^*(\bar{A}_1, \cdots, \bar{A}_m) \subseteq \mathscr{A}(\bar{A}_1, \cdots, \bar{A}_m)$, the number of elements in $\mathscr{A}^*(\bar{A}_1, \cdots, \bar{A}_m)$ is denoted by m^* and satisfies $m \le m^* \le m^{n(h+1)}$.

Proof: This proof will be accomplished by showing ((i)⇔((ii) and ((i)⇔((iii), respectively.

((i)⇔((ii) Sufficiency: Note that the equivalence between SPLS (1) and SPLS (4), the statement ((ii) holds means that SPLS (4) has a CLCLF. Conversely, we shall show that if SPLS (4) does not have a CLCLF, then there exists at least a matrix $\bar{A} \in \mathscr{A}(\bar{A}_1, \cdots, \bar{A}_m)$ not to be Schur.

To this end, set $\Omega_{\bar{A}_i} = \{v \succ 0 | (\bar{A}_i - I)^T v \prec 0\}$ for all $i \in \mathcal{I}, v \in \mathbb{R}^{n(h+1)}$. Firstly, we wish to prove that, under a stronger assumption $\cap_{i=1}^{m} \bar{\Omega}_{\bar{A}_i} = \{0\}$ than the false of statement ((iii), there exists at least a matrix $\bar{A} \in \mathscr{A}(\bar{A}_1, \cdots, \bar{A}_m)$ not to be Schur. With this assumption, by Lemma 5, there exist m positive definite diagonal matrices $D_i \in \mathbb{R}^{n(h+1) \times n(h+1)}$, denoted as $D_i = \text{diag}\{d_i^{(j)}\}$ for all $i \in \mathcal{I}, 1 \le j \le n(h+1)$, such that

$$\sum_{i=1}^{m}(\bar{A}_i - I)D_i = 0. \tag{10}$$

This leads to

$$\det\left(\sum_{i=1}^{m}(\bar{A}_i - I)D_i\right) = 0. \tag{11}$$

According to the basic properties of determinant, by elementary calculation, the left-side determinant of (11) becomes

$$\det\left(\sum_{i=1}^{m}(\bar{A}_i - I)D_i\right)$$
$$= \sum_{\varphi \in \Psi_{n(h+1),m}} \det(\bar{A}_\varphi(\bar{A}_1, \cdots, \bar{A}_m) - I) \prod_{j=1}^{n(h+1)} d_{\varphi(j)}^{(j)}, \tag{12}$$

where the notation of $d_{\varphi(j)}^{(j)}$ corresponds the mapping $\varphi(j) \in \Psi_{n(h+1),m}$.

Now, for any $\varphi \in \Psi_{n(h+1),m}$, consider the determinant $\det(\bar{A}_\varphi(\bar{A}_1, \cdots, \bar{A}_m) - I)$. If all matrices belonging to the set $\mathscr{A}(\bar{A}_1, \cdots, \bar{A}_m)$ are Schur, for all $\varphi(j) \in \Psi_{n(h+1),m}$, due to $h \ge 0$ and $d_{\varphi(j)}^{(j)} > 0$, it follows from Lemma 3 $\det\left(\sum_{i=1}^{m}(\bar{A}_i - I)D_i\right) \ne 0$, which would contradict (11). We thus conclude that, if $\cap_{i=1}^{m} \bar{\Omega}_{\bar{A}_i} = \{0\}$, then at least one matrix $\bar{A} \in \mathscr{A}(\bar{A}_1, \cdots, \bar{A}_m)$ not to be Schur .

Next, we shall improve above result to the case of $\cap_{i=1}^{m} \Omega_{\bar{A}_i} = \{0\}$, i.e., show that, if $\cap_{i=1}^{m} \Omega_{\bar{A}_i} = \{0\}$ holds, then there is at least one matrix $\bar{A} \in \mathscr{A}(\bar{A}_1, \cdots, \bar{A}_m)$ not to be Schur. To this end, discuss from two aspects.

Case 1. $\cap_{i=1}^{m} \bar{\Omega}_{\bar{A}_i} = \{0\}$.

In this case, note that the hypothesis $\cap_{i=1}^{m} \Omega_{\bar{A}_i} = \{0\}$, based on the above argument, the result is straightforward.

Case 2. $\cap_{i=1}^{m} \bar{\Omega}_{\bar{A}_i} \ne \{0\}$.

This implies that there exists some nonzero $v \succeq 0$ belonging to the set $\cap_{i=1}^{m} \bar{\Omega}_{\bar{A}_i} \ne \{0\}$. In this case, recall the assumption $\cap_{i=1}^{m} \Omega_{\bar{A}_i} = \{0\}$, select a vector $\delta = (\delta_1, \cdots, \delta_m)^T \succ 0$ with $\delta_i \in \mathbb{R}$. For all $i \in \mathcal{I}$, define $\bar{A}_i(\delta_i) = \bar{A}_i + \delta_i \mathbf{1}_{n(h+1) \times n(h+1)}$ with $n(h+1) \times n(h+1)$ matrix $\mathbf{1}_{n(h+1) \times n(h+1)}$ consisting of all ones. In addition, denote the set $\{v \succ 0 | (\bar{A}_i + \delta_i \mathbf{1}_{n(h+1) \times n(h+1)} - I)^T v \prec 0\}$ by $\Omega_{\bar{A}_i(\delta_i)}$, it is easy to verify that $\Omega_{\bar{A}_i(\delta_i)}$ are open convex cones. Now, we claim that the intersection of the closure $\cap_{i=1}^{m} \bar{\Omega}_{\bar{A}_i(\delta_i)} = \{0\}$ is always true. If not, i.e., there is a nonzero vector $v^* \succeq 0$ in $\mathbb{R}^{n(h+1)}$ such that $v^* \in \cap_{i=1}^{m} \bar{\Omega}_{\bar{A}_i(\delta_i)} \ne \{0\}$. This yields, for this v^*,

$$v^* \in \bigcap_{i=1}^{m} \bar{\Omega}_{\bar{A}_i(\delta_i)}$$
$$= \bigcap_{i=1}^{m} \{v \succeq 0 | (\bar{A}_i + \delta_i \mathbf{1}_{n(h+1) \times n(h+1)} - I)^T v \preceq 0\} \tag{13}$$
$$= \bigcap_{i=1}^{m} \{v \succeq 0 | (\bar{A}_i - I)^T v \preceq -\delta_i \mathbf{1}_{n(h+1) \times n(h+1)} v\}.$$

As $\delta \succ 0$, we derive from (13) that

$$v^* \in \bigcap_{i=1}^{m} \bar{\Omega}_{\bar{A}_i(\delta_i)} = \bigcap_{i=1}^{m} \{v \succ 0 | (\bar{A}_i - I)^T v \prec 0\} = \bigcap_{i=1}^{m} \Omega_{\bar{A}_i} \ne \{0\}.$$

This immediately contradicts the assumption of $\cap_{i=1}^{m} \Omega_{\bar{A}_i} = \{0\}$.

With the above analysis, for all $i \in \mathcal{I}$, choosing δ_i small enough to ensure that all matrices $\bar{A}_i(\delta_i) \succeq 0$ are Schur, we thus conclude that there is at least one matrix $\bar{A} \in \mathscr{A}(\bar{A}_1(\delta_1), \cdots, \bar{A}_m(\delta_m))$ not to be Schur. Finally, set

$\delta = (\delta_1, \cdots, \delta_m)^T \rightarrow 0$, a limiting argument ensures that this result will also be the case of $\mathscr{A}(\bar{A}_1, \cdots, \bar{A}_m)$.

Necessity: Suppose that the statement ((ii) holds. As SPLS (1) is equivalent to SPLS (4), then SPLS (4) has a CLCLF. By Definition 1, there exists a vector $v \succ 0$ in $\mathbb{R}^{n(h+1) \times n(h+1)}$ such that $(\bar{A}_i - I)^T v \prec 0$ for all $i \in \mathcal{I}$. Let $e^{(j)} = (0, \cdots, 1, \cdots, 0)^T \in \mathbb{R}^{n(h+1)}$ denote the unit vector whose jth element is 1, we get that $(\bar{A}_i^{(j)} - e^{(j)})^T v < 0$ for all $i \in \mathcal{I}, 1 \le j \le n(h+1)$. This implies that $(\bar{A} - I)^T v \prec 0$ for all $\bar{A} \in \mathscr{A}(\bar{A}_1, \cdots, \bar{A}_m)$. Note that the construction given by (6), as $\bar{A}_1, \cdots, \bar{A}_m \succeq 0$ are Schur matrices, it follows from Lemma 2 that all matrices belonging to the set $\mathscr{A}(\bar{A}_1, \cdots, \bar{A}_m)$ must be Schur. Hence, the statement ((i) holds.

((i)⇔((iii) The proof is essentially similar to ((i)⇔((ii).
This completes the proof. ∎

Remark 1: It is worth emphasizing that, in statement ((iii), the subsystem number of (1) is limited to the interval $[m, m^{n(h+1)}]$, which means that we can arbitrarily choose matrices in such range to generate some SPLSs. In addition, note that the construction of the matrices $\bar{A}_\varphi (\bar{A}_1, \cdots, \bar{A}_m)$ given by (6), then it is easy to know that the set $\mathscr{A}(\bar{A}_1, \cdots, \bar{A}_m)$ has $m^{n(h+1)}$ elements, hence, the existence of a CLCLF for SPLS (1) and (9) is equivalent to the Schur stability of $m^{n(h+1)}$ matrices in $\mathscr{A}(\bar{A}_1, \cdots, \bar{A}_m)$.

Remark 2: From Definition 1, the existence of a CLCLF for SPLS (1) is equivalent to the intersection of m convex cones $\cap_{i=1}^m \Omega_{\bar{A}_i} \neq \emptyset$. However, Theorem 1 did not deal with this fact directly. The main idea of Theorem 1 lies in that, through converting a finite number of convex cones to two convex cones which arise in connection with CLCLFs, we show that the non-existence of a common hyperplane tangential between all Ω_{A_i}. This fact leads to the equivalent conditions for the intersection existence of m convex cones, i.e., the existence of a CLCLF. This idea is shown in Corollary 1.

Remark 3: Obviously, if all $\bar{A} \in \mathscr{A}(\bar{A}_1, \cdots, \bar{A}_m)$ are Schur matrices, then for any $\bar{A} \in \mathscr{A}(\bar{A}_1, \cdots, \bar{A}_m)$, the associated discrete time-invariant positive system $\Sigma_{\bar{A}}$ is stable. Moreover, note that the construction of the elements in $\mathscr{A}(\bar{A}_1, \cdots, \bar{A}_m)$, then each subsystem of (1) is also stable. Corollary 2 this implication.

Remark 4: Observe that, for a special case when all $h^{(l)}(k)$ are constant delays for all $l \in \mathcal{P}$, Theorem 1 also holds. Furthermore, select $A^{(l)}(k) = 0$ for all $l \in \mathcal{P}/\{0\}$, SPLS (1) is reduced to the SPLS

$$\Sigma_{A_i^{(0)}} : x(k+1) = A^{(0)}(k)x(k), \quad (14)$$

where $A^{(0)}(k) \in \{A_1^{(0)}, \cdots, A_m^{(0)}\}$ with Schur $A_i^{(0)} \succeq 0, i \in \mathcal{I}$, similar result also holds for such system. Now for SPLS (14), consider a stronger reduced case: there exists a matrix $A_j^{(0)}$ such that $A_j^{(0)} \succeq A_i^{(0)}, j \neq i$, then Corollary 3 can be easily derived from Lemma 4 and Theorem 1.

Remark 5: The main advantage of Theorem 1 lies in that the CLCLF existence can be checked only by system matrices. The case of 2×2 maybe obviously shows this advantage. See Corollary 4.

The following results is straightforward from the above remarks and Theorem 1.

Corollary 1: Let $A_1, \cdots, A_m \succeq 0$ be Schur matrices in $\mathbb{R}^{n \times n}$ and $\Omega_{A_i}, i \in \mathcal{I}$ define as Lemma 5, then the following statements are equivalent. (i)

(i) The discrete-time LSPS *(1)* has a CLCLF.
(ii) The intersection of m convex cones $\cap_{i=1}^m \Omega_{\bar{A}_i}$ is not empty.
(iii) The intersection of two convex cones $\Omega \cap \{x \succeq 0\}$ is empty, where $\Omega = \left\{ \sum_{i=1}^m (\bar{A}_i - I) w_i | w_i \succ 0 \right\}$.

Proof: Together (7) and Definition 1, the result follows immediately. ∎

Corollary 2: If the SPLS *(1)* has a CLCLF. Then each of the following statements is true. (i)

(i) The discrete time-invariant positive system $x(k+1) = \bar{A}x(k), \ \bar{x}_0 = \bar{x}(0) \succeq 0$ is stable with any matrices $\bar{A}$ in $\mathscr{A}(\bar{A}_1, \cdots, \bar{A}_m)$.
(ii) Each subsystem of SPLS *(1)* is stable.

Corollary 3: Consider SPLS *(14)* with Schur $A_i^{(0)}, i \in \mathcal{I}$, if there exists a matrix $A_j^{(0)}$ such that $A_j^{(0)} \succeq A_i^{(0)} \succeq 0, j \neq i$ for all i, then SPLS *(14)* has a CLCLF.

Corollary 4: [15] Given SPLS *(14)* with schur matrices $A_1^{(0)}, A_2^{(0)} \succeq 0$ in $\mathbb{R}^{2 \times 2}$. Then the SPLS *(14)* has a CLCLF if and only if the following conditions are satisfied.

$$\begin{vmatrix} 1 - a_{111} & a_{212} \\ a_{121} & 1 - a_{222} \end{vmatrix} > 0, \quad \begin{vmatrix} 1 - a_{211} & a_{112} \\ a_{221} & 1 - a_{122} \end{vmatrix} > 0.$$

Proof: For 2×2 Schur matrices, as diagonal entries are less than one, the result follows from Theorem 1 immediately. ∎

IV. EXAMPLE

In this section, a simple example is provided to verify technically feasibility and operability of the developed results.

Consider the SPLS with delays (1) given by $A^{(0)}(k) \in \{A_1^{(0)}, A_2^{(0)}\}, A^{(1)}(k) \in \{A_1^{(1)}, A_2^{(1)}\}$ and $h^{(1)}(k) \le 1$ with

$$A_1^{(0)} = \begin{bmatrix} 0.3 & 0.1 \\ 0.3 & 0.2 \end{bmatrix}, \quad A_1^{(1)} = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.3 \end{bmatrix},$$

$$A_2^{(0)} = \begin{bmatrix} 0.1 & 0.1 \\ 0.5 & 0.3 \end{bmatrix}, \quad A_2^{(1)} = \begin{bmatrix} 0.1 & 0.3 \\ 0.4 & 0.2 \end{bmatrix}.$$

It follows from (2)-(5) that

$$\bar{A}_1 = \begin{bmatrix} 0.3 & 0.1 & 0.1 & 0.2 \\ 0.3 & 0.2 & 0.3 & 0.3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.3 \\ 0.5 & 0.3 & 0.4 & 0.2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

In this case, from (6) and Theorem 1, it is easy to see that the set $\mathscr{A}(\bar{A}_1, \bar{A}_2)$ has 2^4 elements. By simply algebra verification , it turns out that all matrices belonging to $\mathscr{A}(\bar{A}_1, \cdots, \bar{A}_m)$ are Schur. Hence, from Theorem 1, we conclude that the SPLS (1) has a CLCLF. This is also the SPLS (9).

V. Conclusions

In this paper, some necessary and sufficient conditions for the existence of a CLCLF have been presented for the discrete-time SPLSs with bounded time-varying delays. These conditions are related to the Schur stability of some matrices and easy to be verified by algebraic approach. These results are likely to help further work in this area. One future work is to extend the results presented here to the continuous-time SPLSs with delays.

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