

# Maxwell-Cattaneo Regularization of Heat Equation

F. Ekoue, A. Fouache d'Halloy, D. Gigon, G Plantamp, and E. Zajdman

**Abstract**—This work focuses on analysis of classical heat transfer equation regularized with Maxwell-Cattaneo transfer law. Computer simulations are performed in MATLAB environment. Numerical experiments are first developed on classical Fourier equation, then Maxwell-Cattaneo law is considered. Corresponding equation is regularized with a balancing diffusion term to stabilize discretizing scheme with adjusted time and space numerical steps. Several cases including a convective term in model equations are discussed, and results are given. It is shown that limiting conditions on regularizing parameters have to be satisfied in convective case for Maxwell-Cattaneo regularization to give physically acceptable solutions. In all valid cases, uniform convergence to solution of initial heat equation with Fourier law is observed, even in nonlinear case.

**Keywords**—Maxwell-Cattaneo heat transfers equations, fourier law, heat conduction, numerical solution.

## I. INTRODUCTION

THE intention in this paper is to use Maxwell-Cattaneo law instead of usual Fourier law in heat equation and to numerically show whether in this way heat equation can be regularized with respect to classical one [1]. Linear Fourier law writes

$$q = -K \text{grad } \theta \quad (1)$$

where  $K > 0$  is the conduction parameter [2]. Fourier law induces a parabolic equation for temperature field  $\theta$ . Any initial disturbance in a material body is propagated instantly due to parabolic nature of the equation [3]. To eliminate this unphysical feature, Maxwell-Cattaneo law is one of various modifications of Fourier law and takes in simplest case the form

$$(1 + \tau \partial_t)q = -\text{grad}(\theta) \quad (2)$$

$\tau > 0$  represents the thermal relaxation time which must be very small (of the order of picoseconds for most metals),  $\tau \partial_t q$  is the thermal inertia which avoids the phenomenon of infinite propagation [4]-[6]. Liquid material parameters will be used for numerical resolution, as for gaseous materials and most solids the solutions seem unrealistic [7]. The analysis is built

on the principle of singular equations [8]. It means that out of each new equation, the previous equation can be recovered when the added correction terms are going to zero. It is intended here to discuss in what respect the solutions of modified equations are themselves converging to "simple" solutions obtained when added correction terms are zero. It will be numerically shown that, despite very uniform convergence, there exists parameter domain where solutions can have unphysical transient. This implies that regularizing Maxwell-Cattaneo scheme does not create new spurious solutions provided new regularizing parameters are correctly adjusted.

## II. NUMERICAL RESOLUTION OF MAXWELL-CATTANEO EQUATION WITH FLUX DIFFUSION TERM

To numerically solve heat equation with modified flux conduction (2), it is necessary to add a new corrective term  $\sigma \partial_x^2 q$  representing flux diffusion term in order to make discretizing process asymptotically stable. The new equation also allows a more realistic approach. Therefore, the Maxwell-Cattaneo equation with flux diffusion has been digitized.

$$\begin{cases} \partial_t \theta = -\nabla q \\ \tau(\partial_t q - \sigma \Delta q) = -q - K \nabla \theta \end{cases} \quad (3)$$

This system can be written as:

$$\tau \partial_t^2 \theta + \partial_t \theta - K \partial_x^2 \theta - \tau \sigma \partial_x^2 (\partial_t \theta) = 0 \quad (4)$$

Classical approximations for the finite difference method are used

$$\partial_t \theta = \frac{1}{dt} (\theta_i^{j+1} - \theta_i^j) \quad (5)$$

$$\partial_x^2 \theta = \frac{1}{dx^2} (\theta_{i+1}^j - 2\theta_i^j + \theta_{i-1}^j) \quad (6)$$

$$\partial_t^2 \theta = \frac{1}{dt^2} (\theta_i^{j+1} - 2\theta_i^j + \theta_i^{j-1}) \quad (7)$$

$$\partial_t (\partial_x^2 \theta) = \frac{1}{dt dx^2} (\theta_{i+1}^j - 2\theta_i^j + \theta_{i-1}^j - \theta_{i+1}^{j-1} + 2\theta_i^{j-1} - \theta_{i-1}^{j-1}) \quad (8)$$

Then (4) becomes:

$$\begin{aligned} \theta_i^{j+1} = & \frac{s+Kr}{p+1} \cdot \theta_{i+1}^j + \frac{2p+1-2s-2Kr}{p+1} \cdot \theta_i^j + \frac{s+Kr}{p+1} \cdot \theta_{i-1}^j \\ & - \frac{s}{p+1} \cdot \theta_{i+1}^{j-1} + \frac{2Kr-p}{p+1} \cdot \theta_i^{j-1} - \frac{s}{p+1} \cdot \theta_{i-1}^{j-1} \end{aligned} \quad (9)$$

with  $s = \tau \sigma / dx^2$ ,  $p = \tau / dt$ ,  $r = dt / dx^2$ .

$\theta_i^j$  is the value of temperature at point  $(i; j; i$  is varying from 1 to  $N$ , according to the  $x$  coordinate on the interval  $[a; b]$  and  $j$  is varying from 1 to  $M$ , according to the time  $t$  on the interval

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$[0; t_{max}]$ . There is an inconsistency in  $i-1$  when  $i = 1$  and in  $i+1$  when  $i = N$ , making it necessary to set boundary conditions at these points.

Dirichlet's boundary conditions:

$$\theta(t, a) = \theta(t, b) = 0 \quad (10)$$

Neumann's boundary conditions:

$$K \partial_x \theta(t, a) = K \partial_x \theta(t, b) = 0 \quad (11)$$

Another inconsistency is in  $j-1$  at initial time ( $t = 0$ ) is also visible. It is therefore necessary to fix an initial condition at this time. Moreover, the temperature at time  $j+1$  depends on temperature at time  $j$ , therefore the temperature has been set at the initial time  $t=0$ . For all studied models, initial temperature will be equal to a Gaussian on an interval  $[a, b]$ .

$$\theta(t = 0, x) = e^{(-3(x-0.5(a+b))^2)} \quad (12)$$

#### A. Figures

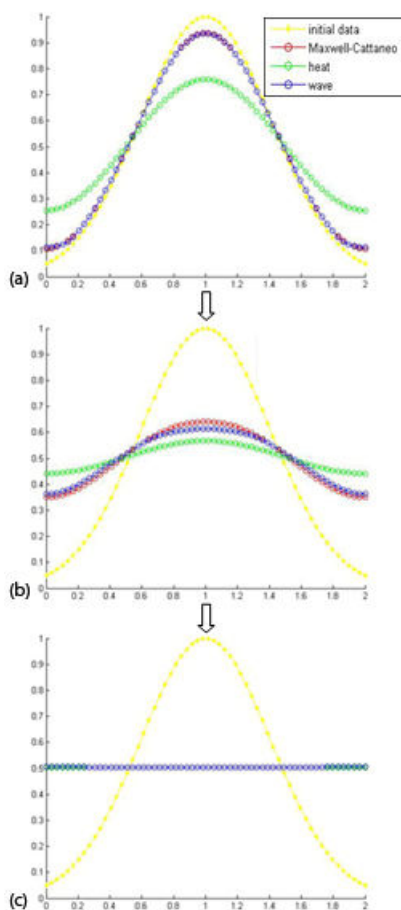


Fig. 1 (a) The beginning of the simulation involving the three equations (heat, wave, Maxwell-Cattaneo) at the Neumann's conditions, (b) Middle of the simulation, (c) End of the simulation  $t = t_{max}$   
 $K = 0.75$ ,  $\tau = 0.2$ ,  $\sigma = 0.1$

#### B. Interpretations

As expected, simulations show that thermal conductivity  $K$  has an impact on heat propagation velocity. Moreover, as seen on Figs. 1 (a), (b), (c), thermal relaxation time  $\tau$  influences the number of curve oscillations and impacts heat propagation velocity. Indeed, when  $\tau$  tends to 0, the results of Fourier equation are recovered. As seen from Fig. 1 (b), temperature stabilizes uniformly faster with (parabolic) heat equation than with (hyperbolic) wave equation and Maxwell-Cattaneo equation. Temperature diffuses less rapidly for Maxwell-Cattaneo simulation, which is realistic as heat does not propagate with an infinite speed. In addition, the diffusive term  $\sigma$  damps the oscillations created by thermal relaxation time  $\tau$ , which allows Maxwell-Cattaneo model to get closer to physical reality. When  $\sigma$  tends to 0, Maxwell-Cattaneo simulation approximates the wave equation. When  $\tau$  tends to 0, the simulation approximates Fourier heat equation. At the end of simulation, the temperature stabilizes uniformly toward a mean value  $\theta = 0.5$ , a value directly depending on space interval. It is observed that on the interval  $[0; t]$ , the heat does not propagate in the same way depending on the models. It is only during time interval  $[t; t_{max}]$  that asymptotic convergence toward an average value  $\theta$  is observed in all models. Fourier heat equation is therefore a singular limit of a regular equation.

### III. NUMERICAL RESOLUTION OF NONLINEAR MAXWELL-CATTANEO EQUATION

Here Maxwell-Cattaneo heat transfer equation is considered for nonlinear Fourier law. Namely the system takes the form:

$$\begin{cases} \partial_t \theta = -\nabla q \\ \tau(\partial_t q - \sigma \Delta q) = -q - \nabla f(\theta) \\ f(\theta) = K\theta + \mu\theta|\theta|^\beta \end{cases} \quad (13)$$

$f(\theta)$  is assumed to be monotone,  $K > 0$ ,  $\mu > 0$  and  $\beta \in [1; 2]$ .

This system can be written as,

$$\tau \partial_t^2 \theta + \partial_t \theta - \tau \sigma \partial_x^2 (\partial_t \theta) - K \partial_x^2 \theta - \mu \partial_x^2 \theta |\theta|^\beta = 0 \quad (14)$$

As for previous case (9), the numerical study of this one dimension model requires to discretize the partial derivatives with finite difference method.

$$\begin{aligned} \theta_i^{j+1} = & \frac{s+Kr}{p+1} \cdot \theta_{i+1}^j + \frac{\mu r}{p+1} \cdot \theta_{i+1}^j \cdot |\theta_{i+1}^j|^\beta \\ & + \frac{2p+1-2s-2Kr}{p+1} \cdot \theta_i^j - \frac{2\mu r}{p+1} \cdot \theta_i^j \cdot |\theta_i^j|^\beta \\ & + \frac{s+Kr}{p+1} \cdot \theta_{i-1}^j + \frac{\mu r}{p+1} \cdot \theta_{i-1}^j \cdot |\theta_{i-1}^j|^\beta \\ & - \frac{s}{p+1} \cdot \theta_{i+1}^{j-1} + \frac{2s-p}{p+1} \cdot \theta_i^{j-1} - \frac{s}{p+1} \cdot \theta_{i-1}^{j-1} \end{aligned} \quad (15)$$

As before, the initial condition is set at  $t = 0$  for  $\theta_i^{j-1}$ , and the boundary conditions at  $i = 1$  and  $i = N$  for  $\theta_{i-1}^j$  and  $\theta_{i+1}^j$ .

## A. Figures

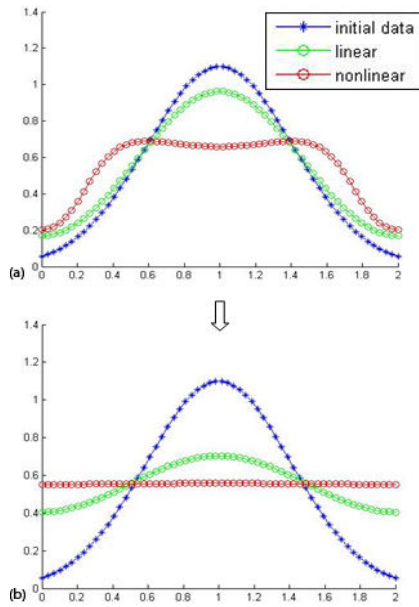


Fig. 2 (a) Simulation of Maxwell-Cattaneo equation with and without ( $\mu=0$ ) nonlinear function, (b) simulation at time  $t = t_1$  ( $0 < t_1 < t_{max}$ ) when the non linear equation are stabilized.

## B. Interpretations

With nonlinear diffusion function, the heat diffuses and stabilizes faster at an average temperature  $\theta$  different from 0.5 as seen on Figs. 1 (c) and 2 (b), respectively corresponding to converging time for linear and nonlinear Maxwell-Cattaneo laws. The value of  $\mu$  has an impact on heat propagation velocity; mainly the larger  $\mu$  is, the faster the heat diffuses. Like diffusive term  $\sigma$ , the value of  $\mu$  affects the oscillations created by thermal relaxation time  $\tau$  by damping them. When  $\mu$  tends to 0, the results of linear Maxwell-Cattaneo equation are recovered.

## IV. NUMERICAL RESOLUTION OF MAXWELL-CATTANEO EQUATION WITH CONVECTIVE TERM

For completion and as it may often occur in industrial applications, a convective effect has been added to Maxwell-Cattaneo equation in the form of a first order space derivative term corresponding to a displacement of supporting fluid medium

$$\begin{cases} \partial_t \theta + v \partial_x \theta = -\partial_x q \\ \tau(\partial_t q - \sigma \partial_x^2 q) = -q - \partial_x f(\theta) \end{cases} \quad (16)$$

where  $v$  is liquid velocity.

Use of classical approximations for finite difference method gives here;

$$\begin{cases} \frac{1}{\Delta t}(\theta_i^{j+1} - \theta_i^j) + \frac{v}{\Delta x}(\theta_i^j - \theta_{i-1}^j) = -\frac{1}{\Delta x}(q_i^j - q_{i-1}^j) \\ \tau \left( \frac{1}{\Delta t}(q_i^{j+1} - q_i^j) - \frac{\sigma}{\Delta x^2}(q_{i+1}^j - 2q_i^j + q_{i-1}^j) \right) = -q_i^{j+1} - \frac{K}{\Delta x}(\theta_i^j - \theta_{i-1}^j) \end{cases} \quad (17)$$

$$\Leftrightarrow \begin{cases} \theta_i^{j+1} = \left(1 - \frac{v \Delta t}{\Delta x}\right) \theta_i^j + \frac{v \Delta t}{\Delta x} \theta_{i-1}^j - \frac{\Delta t}{\Delta x} (q_i^j - q_{i-1}^j) \\ q_i^{j+1} = \frac{\tau \sigma \Delta t}{\Delta x^2 (\tau + \Delta t)} q_{i+1}^j + \frac{\tau}{(\tau + \Delta t)} \left(1 - \frac{2 \sigma \Delta t}{\Delta x}\right) q_i^j \\ \quad + \frac{\tau \sigma \Delta t}{\Delta x^2 (\tau + \Delta t)} q_{i-1}^j - \frac{K \Delta t}{\Delta x (\tau + \Delta t)} (\theta_i^j - \theta_{i-1}^j) \\ \quad - \frac{\mu \Delta t}{\Delta x (\tau + \Delta t)} (\theta_i^j |\theta_i^j|^\beta - \theta_{i-1}^j |\theta_{i-1}^j|^\beta) \end{cases} \quad (18)$$

with  $\frac{\mu \Delta t}{\Delta x (\tau + \Delta t)} (\theta_i^j |\theta_i^j|^\beta - \theta_{i-1}^j |\theta_{i-1}^j|^\beta)$  the nonlinear term, equal to zero if  $\mu=0$ .

$\theta_i^j$  is the value of temperature and  $q_i^j$  is the value of flux at point  $(i; j)$ ;  $i$  is varying from 1 to  $N$ , according to the  $x$  coordinates on the interval  $[a; b]$  and  $j$  is varying from 1 to  $M$ , according to the time  $t$  on the interval  $[0; t_{max}]$ .

Here again there are some inconsistencies in  $i-1$  for  $i=1$  and in  $i+1$  for  $i=N$  for  $q$  and  $\theta$ . It is therefore necessary to set new boundary conditions.

$$\Rightarrow q_0^j = q_{N+1}^j = 0 \text{ (Neumann)} \quad (19)$$

$$\Rightarrow \theta_0^j = \theta_1^j \text{ and } \theta_{N+1}^j = \theta_N^j \quad (20)$$

## A. Figures

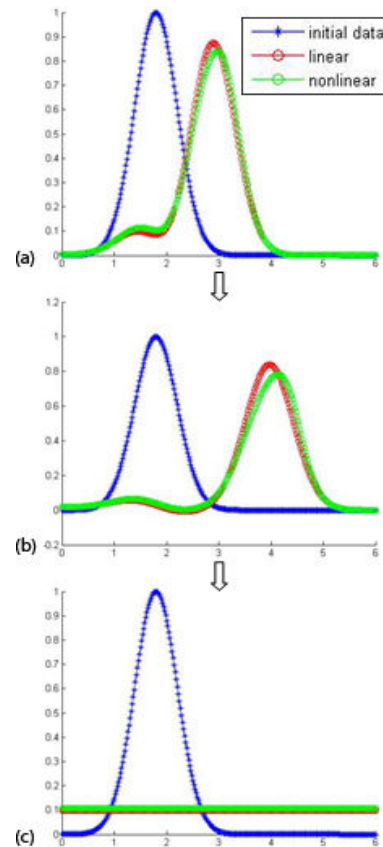


Fig. 3 (a) Evolution of the simulation of linear and non-linear Maxwell-Cattaneo's equation with liquid velocity, (b) Middle of the simulation, (c) End of the simulation  $t = t_{max}$   
 $K=1, \tau=0.3, \sigma=0.4, \mu=2, \beta=2, v=10$

### B. Interpretations

For a positive velocity  $v$ , the heat spreads from left to right attenuating itself over time, see Fig. 3. As in previous models, the heat stabilizes uniformly at an average temperature  $\theta$  after a time  $t_{max}$ . During a time interval  $[t_1; t_2] \subset [0; t_{max}]$ , a “hollow” is observed where the temperature is lower than its final value  $\theta$  and, depending on the parameters, can reach negative values – which is physically impossible. So if Maxwell-Cattaneo law solves the paradox of infinite speed, it just generates another physical inconsistency requiring another analysis. The difficulty to fix the working parameter space is in the determination of exactly acceptable solutions. In very general terms, let  $H(\theta, v, \mu) = 0$  the initial equation (here nonlinear convective heat equation with Fourier law). It has been proposed to regularize this equation into MC  $(\sigma, \tau) * H(\theta, v, \mu) = 0$ , where MC(.,.) stands for Maxwell-Cattaneo heat flux equation. Numerical discrete filter for resolution transforms in turn the equation into  $N(dt, dx) * MC(\sigma, \tau) * H(\theta, v) = 0$ , and despite numerical stability conditions are satisfied for soft convergence, solution  $\theta = \theta(t, x)$  of filtered regularized equation may be unphysical and constraints on parameters of filtering regularizing operators have to be set. Practically however, if negative temperature is unacceptable, it is not clear that the existence of a dip in the solution as shown on Fig. 3 middle is itself physically acceptable. So a necessary condition results from the condition  $\theta(x, t) > 0$  over the full space-time domain, and a sufficient one would be to impose that  $\partial_x \theta(x, t) > 0$  in the same domain, which is difficult to derive analytically. As a remark already observed above, heat propagates faster with nonlinear Maxwell-Cattaneo model than with linear one. The observed “hollow” is slightly reduced with the non linear model – which confirms that  $\mu$  is damping the oscillations.

### V. CONCLUSION

The numerical analysis of a regularization process of usual heat equation with both time and space corrective terms  $\tau \partial_t$  and  $\sigma \partial_x^2$  has been discussed. The simulations are made very easy and workable in a modest MATLAB environment. They are showing that Maxwell-Cattaneo law solves the paradox of heat propagation infinite speed, but it may generate another physical inconsistency. Depending on the value of  $\tau$ , temperature can become negative during heat propagation. Feasibility conditions on  $\tau$  and  $\sigma$  have to be applied for avoiding this phenomenon. Mainly thermal relaxation time has to be very small and diffusive term  $\sigma$  has to be large enough to damp out the oscillations created by  $\tau$ . In this case, it is generally observed that the solutions of considered equations with different parameters are all uniformly converging toward solutions with zero parameters, but time evolution is different. Aside thermal conductivity  $K$  of the medium, thermal relaxation time  $\tau$  and diffusive term  $\sigma$  contribute to generate a finite heat propagation speed which also impacts heat time evolution. It is therefore necessary to impose conditions on time step  $dt$  (or on  $t_{max}$ ) to make it possible to directly address

an industrial problem with valid precautions guaranteeing correct limits of proposed regularization process.

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