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A Hyperbolic Characterization of Projective Klingenberg Planes

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***Abstract*—In this paper, the notion of Hyperbolic Klingenberg plane is introduced via a set of axioms like as Affine Klingenberg planes and Projective Klingenberg planes. Models of such planes are constructed by deleting a certain number m of equivalence classes of lines from a Projective Klingenberg plane. In the finite case, an upper bound for m is established and some combinatoric properties are investigated.

***Keywords*—Hyperbolic planes, Klingenberg planes, Projective planes.

I. Introduction

When one mentions the plane geometries, it reminds of affine planes, projective planes and hyperbolic planes. The property that differs these planes from others is given by the relation of being parallel on the set of lines.

In affine planes, only one parallel line can be drawn to a line from a point not lying on this given line (Euclid's famous 5th postulate, [8]). In projective planes all lines intersect, that is we cannot mention of parallel lines. In hyperbolic planes, exactly k parallel lines ($k \geq 2$) to a given line from a point not lying on this given line. In literature, there is a lot of work on these planes.

Geometrical structures which are more general then affine and projective planes are obtained by taking a class of points instead of a point; a class of lines instead of a line and by reorganising the incidence relation [1].

For affine planes, this generalisation can be found in [3], and for projective planes, in [2]. There is no such generalisation in literature for hyperbolic planes. Our main aim in this work is to give such a generalisation for hyperbolic planes and to present this generalisation as a system of axioms.

In this paper incidence structures are defined as in [5] and blocks are called lines. For any point P, (P) denotes the set of lines incident with the point P, $[P]$ the cardinality of (P), and $[P,Q]$ the number of lines joining P and Q. (l), $[l]$, and $[l,d]$ are defined dually.

A *Projective Klingenberg plane* (PK-plane) is an incidence structure $\mathcal{K} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ together with an equivalence relation o on $\mathcal{P}$ and $\mathcal{L}$ (called neighbour relation, and the equivalence class of P (resp. l) is denoted by $< P >$ (resp. $< l >$)) such that

(PK1) $P \o Q \Longrightarrow [P,Q] = 1, \forall P,Q \in \mathcal{P}$

(PK2) $l \o d \Longrightarrow [l,d] = 1, \forall l,d \in \mathcal{L}$

(PK3) There exists a projective plane $\mathcal{K}^*$ (the canonical image of $\mathcal{K}$) and an incidence structure epimorphism $\varphi : \mathcal{K} \longrightarrow \mathcal{K}^*$ such that

$$PoQ \iff \varphi(P) = \varphi(Q), \forall P,Q \in \mathcal{P}$$
$$lod \iff \varphi(l) = \varphi(d), \forall l,d \in \mathcal{L}$$

Axiom (PK3) is equivalent to

(PK3)' Putting $< P > \mathcal{I} < l >$ iff there are Q, d with QoP, dol and $Q\mathcal{I}d$, the equivalence classes with respect to this incidence an ordinary projective plane $\mathcal{K}^*$.

In the above definition $\o$ means "non-neighbouring", and PK-planes are denoted by $\mathcal{K} = (\mathcal{P}, \mathcal{L}, \mathcal{I}, o)$.

A point P is said to be *near a line* l and this is denoted by Pol whenever PoQ for some $Q\mathcal{I}l$. For any element x of $\mathcal{K}$, we denote the neighbour class of x by x-*class*. Detailed information about PK-planes can be found in [2], [4].

One can easily show the following lemma.

Lemma 1.1: Let $\mathcal{K} = (\mathcal{P}, \mathcal{L}, \mathcal{I}, o)$ be a PK-plane. Then

(i) $Pol \iff \exists h \in \mathcal{L}$, such that hol, and $P\mathcal{I}h$

(ii) $hod \iff \exists H_i\mathcal{I}h, \exists D_i\mathcal{I}d \ni H_ioD_i,\ H_1\o H_2,\ D_1\o D_2, h,d \in \mathcal{L},\ H_i, D_i \in \mathcal{P},\ i = 1,2.$

(iii) "$P_1 \in \mathcal{P}, l_1, l_2 \in \mathcal{L}, P_1\mathcal{I}l_1, l_1ol_2$" $\Longrightarrow \exists P_2 \ni P_2\mathcal{I}l_2,\ P_1oP_2.$

When $|\mathcal{P} \cup \mathcal{L}|$ is finite, the geometric structure is called finite. Now, we state a theorem for finite regular PK-planes which can be found in [10]. The original proof of this theorem for Hjelmslev Planes is due to Kleinfeld [12]. Drake and Lenz [6] observed that this proof remains valid for PK-planes:

Theorem 1.1: Let $\mathcal{K} = (\mathcal{P}, \mathcal{L}, \mathcal{I}, o)$be a PK-plane. Then there are natural numbers t and r which are called the parameters of $\mathcal{K}$, with

(i) $| < P > | = | < l > | = t^2, \forall P \in \mathcal{P}, l \in \mathcal{L}$

(ii) $|(P) \cap < l > | = |(l) \cap < P > | = t, \forall P\mathcal{I}l$

(iii) Let r be the order of projective plane $\mathcal{K}^*$. If $t \neq 1$, we have $r \leq t$ (then $\mathcal{K}$ is called *proper* and we have $t = 1$ iff $\mathcal{K}$ is an ordinary projective plane).

(iv) $[P] = [l] = t(r + 1), \forall P \in \mathcal{P}, \forall l \in \mathcal{L}$

(v) $|\mathcal{P}| = |\mathcal{L}| = t^2(r^2 + r + 1).$

II. Hyperbolic Klingenberg Planes

A projective or affine Klingenberg plane (PK-, AK-plane) is a generalization of ordinary projective plane where two points may also be multiply joined or not joined at all (see[3]). Now we can give a definition for Hyperbolic-Klingenberg plane (HK-plane) and it is given as a generalization of an ordinary projective plane, too.

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A *hyperbolic plane* is a geometric structure such that

(A1) There are at least two points on each line.

(A2) Two distinct points lie on one and only one line.

(A3) There exist at least four points, no three of which are collinear.

(A4) Through each point X not on a line l there pass at least two lines not meeting (parallel to) l.

(A5) If a subset S of the points contains all points on the lines through pairs of distinct points of S , then the subset S contains all points of the geometric structure.(see [7],[9],[11],[13]).

A *Hyperbolic Klingenberg plane* (HK-plane) $\mathcal{H}$ is a system $(\mathcal{P},\mathcal{L},\mathcal{I},\|,o)$, where $(\mathcal{P},\mathcal{L},\mathcal{I})$ is an incidence structure and o is an equivalence relation on $\mathcal{P}\cup\mathcal{L}$ (called neighbouring) such that no element of $\mathcal{P}$ (point) is neighbour to any element of $\mathcal{L}$ (line), $\|$ is an equivalence relation on $\mathcal{L}$ (called *parallelism*) and $\mathcal{H}$ satisfies the following axioms for all $P,Q\in\mathcal{P}$ and $g,h\in\mathcal{L}$:

(HK1) $P\not{o}Q \Longrightarrow [P,Q]=1, \forall P,Q\in\mathcal{P}$

(HK2) $l\in\mathcal{L}\Longrightarrow \exists P\mathcal{I}l, Q\mathcal{I}l, P\not{o}Q$

(HK3) There exists at least four pairwise non-neighbour points, no three of which are collinear.

(HK4) For each point-line pair (P,l), $P\not{o}l$ there are at least two non-neigh-bouring lines through P parallel to l.

(HK5) There exists an hyperbolic plane $\mathcal{H}^*=(\mathcal{P}^*,\mathcal{L}^*,\mathcal{I}^*)$ and an incidence structure epimorphism

$$\varphi:\mathcal{H}\longrightarrow\mathcal{H}^*$$

such that

$$\begin{aligned} PoQ &\iff \varphi(P)=\varphi(Q), \forall P,Q\in\mathcal{P} \\ lod &\iff \varphi(l)=\varphi(d), \forall l,d\in\mathcal{L}\end{aligned}$$

and if $[g,h]=0$ then $\varphi(g)\|\varphi(h)$.

Various models for hyperbolic planes such as Poincare models (see [9]), Sandler's models (see [13]) and the extension of Sandler's models (see [11]) have been developed. It is well known that if a line is deleted from a projective plane then the remaining substructure forms an affine plane. Graves [7], Kaya-Özcan [11] and Sandler [13] have given examples of hyperbolic planes obtained by deletion from projective planes. Sandler had shown that if three non-concurrent lines are deleted from a projective plane then the remaining structure forms a hyperbolic plane in the sense of Graves [7]. Kaya-Özcan [11] has extended Sandler's construction and showed that if m lines no three of which are concurrent are deleted from a projective plane then the remaining structure forms an hyperbolic plane. If $\mathcal{K}$ is a PK-plane and l is a line, by deleting all lines neighbour to l and all points near to l, the remaining structure forms an affine-Klingenberg plane. Now we will adopt the method of [11] to obtain an HK-plane from a PK-plane.

Let $\mathcal{K}=(\mathcal{P},\mathcal{L},\mathcal{I},o)$ be an infinite PK-plane and $l_i\in\mathcal{L}$, $i=1,2,...,m$ denote pairwise non-neighbour lines such that no three of them are concurrent, $\mathcal{K}_m=(\mathcal{P}_m,\mathcal{L}_m,\mathcal{I},o)$ be substructure obtained from $\mathcal{K}$ removing all lines l_i, together with the points which are near to l_i, for $i=1,2,..m$, and $m\geq 3$ is any natural number. In symbols

$\mathcal{P}_m=\{P\in\mathcal{P}\ |P\not{o}Q,\ Qol_i,\ i=1,2,...,m\}$

$\mathcal{L}_m=\mathcal{L}\backslash\{d\in\mathcal{L}\ |\exists i,\ dol_i,\ i=1,2,...,m\}$

In this paper we accept in any substructure $\mathcal{K}_m$, parallelism is defined with

$$\text{“}l_1\|l_2\text{”} :\Longleftrightarrow: \text{“}l_1 \ and \ l_2 \ intersect \ on \ removed \ points.\text{”}$$

Lemma 2.1: If $\mathcal{K}$ is infinite, then for any substructure $\mathcal{K}_m$ of $\mathcal{K}$,

(i) $\mathcal{K}_m^*=\varphi(\mathcal{K}_m)$ is a hyperbolic plane.

(ii) There are at least two pairwise non-neighbour points on each line.

(iii) There exists at least four pairwise non-neighbour points, no three of which are collinear.

(iv) If $\exists i$ such that dol_i then d is a removed line.

Proof: The lines $\varphi(l_i)=l_i^*, i=1,2,...,m$ form a set of m lines such that no three of them are concurrent in the projective plane $\mathcal{K}^*$. Since the lines l_i are removed from $\mathcal{K}$ together with points near l_i, the lines $\varphi(l_i)=l_i^*$ are removed from $\mathcal{K}^*$ together with their points. We denote the remaining substructure by $\mathcal{K}_m^*$. It is easy to show that $\mathcal{K}_m^*$ is forms a hyperbolic plane (This is shown in [11] for finite structure under some assumptations).

Together with (i) and Lemma 1.1, (ii), (iii) and (iv) are obtained easily. ∎

Theorem 2.1: If $\mathcal{K}$ is any infinite PK-plane then $\mathcal{K}_m$ is an HK-plane.

Proof: (HK2), (HK3) and (HK5) are obtained from Lemma 2.1. (HK1) is obtained from (PK1). Since at least three pairwise non-neighbour points are removed from each line in $\mathcal{K}$, there exist at least three pairwise non-neighbour lines which are through a point P and parallel to l if $P\not{o}l$. ∎

III. FINITE HK-PLANES

Let $\mathcal{K}$ be a finite PK plane with parameters t,r in the sense of Theorem 1.1. Then there are $r+1$ pairwise non-neighbour points on each line of $\mathcal{K}$. Let $\mathcal{K}_m^r$ be a substructure such that $m\leq r+2$ which is obtained by removing the pairwise non-neighbour lines l_i, $i=1,2,...,m$ no three of which are concurrent together with the points near l_i. Since no three of the pairwise non-neighbour lines l_i are concurrent, any $m-1$ of them intersect the remaining removed lines at pairwise non-neighbour points in $\mathcal{K}$ and therefore $m-1\leq r+1$ which gives the restriction $m\leq r+2$. The following lemma, which gives the basic combinatorical properties of $\mathcal{K}_m^r$, can be shown by easy computations:

Lemma 3.1: Following properties are valid in any structure $\mathcal{K}_m^r$, $m\leq r+2$:

(i) Two non-neighbour points of $\mathcal{K}_m^r$ are on exactly one line.

(ii) Through each point of $\mathcal{K}_m^r$ there pass exactly

$$t(r+1)$$

lines and there pass exactly

$$r+1$$

pairwise non-neighbour lines of $\mathcal{K}_m^r$.

(iii) There are exactly

$$t^2(r^2+r+1-m)$$

lines and there are exactly

$$r^2+r+1-m$$

pairwise non-neighbour lines in $\mathcal{K}_m^r$.

(iv) There are exactly

$$t^2r^2+\frac{t^2}{2}(m-1)(m-2r-2)$$

points and there are exactly

$$r^2+\frac{1}{2}\left(m-1\right)\left(m-2r-2\right)$$

pairwise non-neighbour points in $\mathcal{K}_m^r$.

A point of $\mathcal{K}$ is called a *corner point* if it is an intersection point of any two non-neighbour lines in the set of removed lines. A point of $\mathcal{K}^*$ as a point class is called a *proper corner point* if it is an intersection class of any two non-neighbour lines in the set of removed lines from $\mathcal{K}$.

It is trivial that any corner point-class corresponds to a proper corner point under φ.

Lemma 3.2: Any line class (or any line) of $\mathcal{K}_m^r$ contains at most $\frac{m}{2}$ or $\frac{m-1}{2}$ pairwise non-neighbour corner points in $\mathcal{K}$, according to m is even or odd, respectively.

Proof: Let l be a line of $\mathcal{K}_m^{r*}$ and assume that it contains s proper corner points in K^*. For each proper corner point, there exist exactly two removed non-neighbour lines of $\mathcal{K}$ on that point by the definition of proper corner point. Since no three of the removed non-neighbour lines are concurrent, there exist exactly $2s$ distinct lines with this property. Then $2s \leq m$ because the total number of such lines is m. Furthermore, if m is an odd number clearly $2s \leq m$ implies $2s \leq m-1$. Then proof is completed by considering the property that any corner point-class corresponds to proper corner point under φ which is given just before Lemma 3.2. ■

The following corollaries are immediate:

Corollary 3.1: Any line class of $\mathcal{K}_m^r$ contains at most

$$t^2\frac{m}{2}$$

or

$$t^2\frac{m-1}{2}$$

corner points in $\mathcal{K}$, according to m is even or odd, respectively.

Corollary 3.2: Any line class of $\mathcal{K}_m^r$ passing through s non-neighbour corner points in $\mathcal{K}$, has exactly

$$t^2(r+1+s-m)$$

points in $\mathcal{K}_m^r$. Any line of $\mathcal{K}_m^r$ has exactly

$$r+1+s-m$$

pairwise non-neighbour points.

Now we state a lemma which can be proved easily by using Lemma 3.2.

Lemma 3.3: If any line of $\mathcal{K}_m^r$ considered as a line of $\mathcal{K}$ contains s corner points then the number of deleted pairwise non-neighbour points from this line is $m-s$ and $m-s \geq \frac{m}{2}$ if m is even, $m-s \geq \frac{m+1}{2}$ if m is odd.

Corollary 3.3: Let n denote the minimum number of pairwise non-neighbour corner point on a line class of $\mathcal{K}_m^r$, and k denote the number of pairwise non-neighbour points on a line in $\mathcal{K}_m^r$. Then

$$r+1+n-m \leq k \leq r-\frac{1}{2}(m-2)$$

if m is even and

$$r+1+n-m \leq k \leq r-\frac{1}{2}(m-1)$$

if m is odd.

Proof: Let l be a line of $\mathcal{K}_m^r$. l considered as a line of $\mathcal{K}$ has at least n pairwise non-neighbour corner points and therefore it intersects at least $m-n$ pairwise non-neighbour removed line at pairwise non-neighbour points in $\mathcal{K}$. But all of these points of intersection are deleted and therefore l contains at least

$$r+1-(m-n)$$

pairwise non-neighbour points in $\mathcal{K}_m^r$. We have

$$r+1-m+n \leq k$$

since at most $m-n$ pairwise non-neighbour points can be deleted from a line. On the other hand if l , considered as a line of $\mathcal{K}$ contains s pairwise non-neighbour corner points then the number of pairwise non-neighbour points deleted from l is $m-s$. But in the case where m is an even number

$$m-s \geq \frac{m}{2}$$

by Lemma 3.3. Hence the number of pairwise non-neighbour points on l in $\mathcal{K}_m^r$ is less than $r+1-\frac{m}{2}$, that is,

$$k \leq r+1-\frac{m}{2}=r-\frac{m-2}{2}$$

. Similarly, if m is an odd number, then

$$m-s \geq \frac{m+1}{2}$$

by Lemma 3.3. Consequently

$$k \leq r+1-\frac{m+1}{2}=r-\frac{m-1}{2}.$$

■

Proposition 3.1: Let n be the minimum number of pairwise non-neighbour corner point on a line of $\mathcal{K}_m^r$. If

$$3 \leq m \leq r + n + \frac{1}{2}(1 - \sqrt{4r+5})$$

then $\mathcal{K}_m^r$ is a HK-plane.

Proof: (HK1) is obvious since PK1. A line of $\mathcal{K}_m^r$ contains at least

$$r + 1 + n - m$$

pairwise non-neighbour points by Corollary 3.3. Therefore (HK2) is satisfied iff

$$r + 1 + n - m \geq 2$$

i.e.

$$r \geq m - n + 1$$

which is clearly valid since

$$\begin{aligned} r &\geq m - n - \frac{1}{2}(1 - \sqrt{4r+5}) \\ &\geq m - n - \frac{1}{2}(1 - \sqrt{13}) > m - n + 1 \end{aligned}$$

by the hypothesis (since r is the order of some projective plane; $r \geq 2$). (HK3) is trivially satisfied, since there exists two non-neighbour, non-intersecting lines in $\mathcal{K}_m^r$ each of which contains at least two non-neighbour points. For a proof of (HK4) let l be a line of $\mathcal{K}_m^r$ and P be a point $P \not o l$. The number $m - s$ in the Lemma 3.3 is also the number of non-neighbour lines passing through P and not meeting l. Hence there exists at least $\frac{m}{2}$ or $\frac{m+1}{2}$ such lines according to as m is even or odd, respectively. Therefore (HK4) is satisfied iff $m \geq 3$. If we take $\varphi = \varphi|_{\mathcal{K}_m^r}$ then we must only show that $\mathcal{K}_m^{r*}$ is a hyperbolic plane. In [9] it is shown that, if

$$3 \leq m \leq r + n + \frac{1}{2}(1 - \sqrt{4r+5})$$

then $\mathcal{K}_m^{r*}$ is hyperbolic plane. ■

IV. SOME COMBINATORIC PROPERTIES OF $\mathcal{K}_m^r$

Let l be a line of $\mathcal{K}_m^r$ and P a point not on l . If l contains s non-neighbouring corner points then there axists $m - s$ non-neighbour lines parallel to l and passing thorough P . Then from Lemma 3.5 there exist at most $m - s$ and at least $\frac{m}{2}$ or $\frac{m+1}{2}$ pairwise non-neighbour lines parallel to l and passing through P according to m is even or odd, respectively. Therefore the number of non-neighbour lines passing thorough P and parallel to l is independent of the choice of P but choice of l .

The lines of $\mathcal{K}_m^r$ can be classified according to the number of non-neighbour points. Let C_s denote the set of all lines of $\mathcal{K}_m^r$ such that each line of it contains exactly s non-neighbour corner points. Then, from the Corollary 3.2 C_s contains exactly

$$r + s - m + 1$$

non-neighbour points and therefore it contains exactly

$$t^2(r + s - m + 1)$$

points. Thus from the Lemma 3.2, there exist

$$\frac{1}{2}m - k + 1$$

or

$$\frac{1}{2}(m+1) - k$$

line classes in $\mathcal{K}_m^r$ which are

$$C_k, C_{k+1}, \cdots, C_{\frac{m}{2}}$$

or

$$C_k, C_{k+1}, \cdots, C_{\frac{k-1}{2}}$$

according to m is even or odd, respectively.

Lemma 4.1: Number of the non-neighbour lines of $\mathcal{K}_m^r$ parallel to a line l in the class C_s is $m(r-1) - rs$.

Proof: Since $l \in C_s$ contains $r + 1 - (m - s)$ non-neighbour points and except l, r non-neighbour lines pass through each of these points it is obvious that the number of non-neighbour lines parallel to l in $\mathcal{K}_m^r$ is $m(r-1) - rs$. ■

Lemma 4.2: Let P be any point of $\mathcal{K}_m^r$ and p_s denote the number of non-neighbour lines pass through P belong to C_s, and q_s denote the number of all pairwise non-neighbour lines of C_s. Then,

$$\begin{aligned} \sum_{s=k}^{t} p_s &= r + 1 \\ \sum_{s=k}^{t} q_s &= r^2 + r + 1 - m \\ \sum_{s=k}^{t} s \cdot p_s &= \binom{m}{2} \\ \sum_{s=k}^{t} s \cdot q_s &= (r-1)\binom{m}{2} \\ \sum_{s=k}^{t} s^2 \cdot q_s &= \left[r - 1 + \binom{m-2}{2} \right] \cdot \binom{m}{2}, \end{aligned}$$

where t is $\frac{m}{2}$ or $\frac{m-1}{2}$ according to m is even or odd, respectively.

V. CONSEQUENCE AND SOME QUESTIONS

In this paper, the definition of HK-plane is given and it is shown that the structures, obtained by deletion from a PK-plane of neighbour classes of pairwise distinct non-neighbour m lines, no three of them collinear, are HK-planes, under some assumptations. Since any subplane of a PK-plane contains at least three pairwise distinct non-collinear, non-neighbour lines, the structure which constructed by deleting any subplane from the superplane will be HK-plane, if it satisfies HK3.

In the finite case, what is the relation between the parameters of the subplane and of the superplane and what is the upper bund of parameters of subplane?

Is there a way to distinguish the subplane deleted Desarguesian HK-plane from all other HK-planes?

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