

Exterior Calculus: Economic Growth Dynamics

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Abstract—Mathematical models of dynamics employing exterior calculus are mathematical representations of the same unifying principle; namely, the description of a dynamic system with a characteristic differential one-form on an odd-dimensional differentiable manifold leads, by analysis with exterior calculus, to a set of differential equations and a characteristic tangent vector (vortex vector) which define transformations of the system. Using this principle, a mathematical model for economic growth is constructed by proposing a characteristic differential one-form for economic growth dynamics (analogous to the action in Hamiltonian dynamics), then generating a pair of characteristic differential equations and solving these equations for the rate of economic growth as a function of labor and capital. By contracting the characteristic differential one-form with the vortex vector, the Lagrangian for economic growth dynamics is obtained.

Keywords—Differential geometry, exterior calculus, Hamiltonian geometry, mathematical economics.

I. INTRODUCTION

TO construct mathematical models of complex economic systems, some economists employ Hamiltonian mechanics, thermodynamics and statistics ([1], [2], [3], [4] and [5]); recent thermodynamic developments use differential forms. In the present investigation, exterior calculus and its' main tool (differential forms) are used to construct a mathematical model of economic growth dynamics. This approach makes use of a principle previously used ([6], [7], [8], and [9]) to construct mathematical models for Hamiltonian mechanics, geometric optics, irreversible thermodynamics, Black hole dynamics, classical electromagnetism, classical string mechanics, and Navier-Stokes dynamics. The principle states that:

Mathematical models of dynamics employing exterior calculus are mathematical representations of the same unifying principle; namely, the description of a dynamic system with a characteristic differential one-form on an odd-dimensional differentiable manifold leads, by analysis with exterior calculus, to a set of differential equations and a characteristic tangent vector which define transformations of the system [8]. The origin of this principle is Arnold's [6] use of differential forms to define Hamiltonian geometry.

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As a prelude to applications of this principle, section II contains a discussion of differential forms. Then, in section III, it is shown how differential one-forms are used to develop a model for a dynamic system. With this preparation, the model for dynamics on a differential one-form is applied to economic growth dynamics (section IV). This model allows computation of the rate of economic growth as a function of the capital and labor; these results are entirely dependent on the use of this differential geometric approach.

II. DIFFERENTIAL ONE-FORMS

The exterior derivative of a scalar function f (a differential one-form $\mathbf{d}f$) has the same effect on f as the exact differential df in conventional calculus; namely, it represents an infinitesimal change in a function f induced by an arbitrary displacement of a point. However, df is already a scalar, whereas $\mathbf{d}f$ must be contracted with a tangent vector \mathbf{v} to become a scalar. The operation of contraction, denoted by $\mathbf{d}f(\mathbf{v})$, thus removes the arbitrariness in the direction of the displacement, where this direction is the same as that of the tangent vector \mathbf{v} (tangent vectors and the exterior derivative operator are denoted by boldface symbols and a boldface \mathbf{d} , respectively). In this setting, consider an n -dimensional differentiable manifold M with n local coordinates x^k . At every point of M ,

- there exists a basis set of tangent vectors $\{\partial/\partial x^k\}$ for an n -dimensional vector space of tangent vectors \mathbf{v} belonging to tangent space TM_x and
- there exists a basis set of differential one-forms $\{\mathbf{d}x^k\}$ for an n -dimensional vector space of differential one-forms $\mathbf{d}f$ on tangent space TM_x .

The tangent bundle $TM(=\cup TM_x)$ and cotangent bundle $T^*M(=\cup T^*M_x)$ where T^*M_x is the dual of TM_x , have the natural structure of a differential manifold of dimension $2n$ with local coordinates $\{x^k, \mathbf{d}x^k(\mathbf{v})\}$ and $\{x^k, \mathbf{d}f(\partial/\partial x^k)\}$, respectively. Differential one-form $\mathbf{d}S$ on T^*M_x is defined by the contraction $\mathbf{d}S(\xi) = \mathbf{d}f(\mathbf{v})$ where $\xi \in T(T^*M_x)$; hence,

$$\mathbf{d}S = \mathbf{d}f\left(\frac{\partial}{\partial x^k}\right)\mathbf{d}x^k \quad (1)$$

III. DYNAMICS

In Arnold's treatment [6] of Hamiltonian mechanics and in the present case of economic growth as a dynamic system, a temporal coordinate x^0 is introduced as an additional local coordinate for M , TM and T^*M , thereby changing TM and T^*M into odd-dimensional manifolds. As a result, an additional term $\mathbf{d}f(\partial/\partial x^0)\mathbf{d}x^0$ is added to (1), where $\mathbf{d}f(\partial/\partial x^0)\mathbf{d}x^0$ is defined as a function of all $(2n + 1)$ coordinates; hence, $\mathbf{d}f(\partial/\partial x^0)\mathbf{d}x^0$ describes the phase flow on this extended cotangent bundle. Using b_k for $\mathbf{d}f(\partial/\partial x^k)$ and $\Omega \mathbf{d}x^k$ for $\mathbf{d}f(\partial/\partial x^0)\mathbf{d}x^0$, the equation for $\mathbf{d}S$ becomes

$$\mathbf{d}S = b_k \mathbf{d}x^k + \Omega(x^0, \dots, x^n, b_1, \dots, b_n) \mathbf{d}x^0 \tag{2}$$

In Hamiltonian mechanics b_k, Ω and x^0 are represented by the momenta, Hamiltonian and time, respectively, but for the example discussed in section IV, other variables will play the role of b_k, Ω and x^0 , as well as of S and x^k . Hence, for the remainder of this section the geometry of extended phase space is presented in a general setting that not only applies to Hamiltonian mechanics (which defines this geometry), geometric optics, irreversible thermodynamics, black hole dynamics, and Navier-Stokes dynamics, but also to economic growth dynamics.

The general procedure begins by taking the exterior derivative of $\mathbf{d}S$ to get the following differential two-form:

$$\mathbf{d}\omega = \mathbf{d}b_k \wedge \mathbf{d}x^k + \left[\left(\frac{\partial \Omega}{\partial x^k} \right) \mathbf{d}x^k + \left(\frac{\partial \Omega}{\partial b_k} \right) \mathbf{d}b_k + \left(\frac{\partial \Omega}{\partial t} \right) \mathbf{d}t \right] \wedge \mathbf{d}t \tag{3}$$

where $\omega \equiv \mathbf{d}S$. If x^k and b_k are to describe mappings of the temporal coordinate onto the direction of the system phase flow, then (a) x^k and b_k must be functions of x^0 alone and (b) the following contraction must be satisfied at each point (b_k, x^k, x^0) of the transformation:

$$\mathbf{d}\omega(\xi, \eta) = 0 \tag{4}$$

where the tangent vector ξ is given by

$$\xi = \left(\frac{db_k}{dx^0} \right) \frac{\partial}{\partial b_k} + \left(\frac{dx^k}{dx^0} \right) \frac{\partial}{\partial x^k} + \frac{\partial}{\partial x^0} \tag{5}$$

and η is an arbitrary vector. $\mathbf{d}\omega$ is a mapping of a pair of

vectors onto an oriented surface; if the contraction $\mathbf{d}\omega(\xi, \eta) = 0$, then the mapping is defined only if the coordinates db_k/dx^0 and dx^k/dx^0 of ξ have the values

$$dx^k/dx^0 = -(\partial\Omega/\partial b_k) \text{ and } db_k/dx^0 = (\partial\Omega/\partial x^k) \tag{6}$$

By substituting the coordinate values from (6) into (5) the vortex vector \mathbf{R} is obtained, as given by

$$\mathbf{R} = \left(\frac{\partial \Omega}{\partial x^k} \right) \frac{\partial}{\partial b_k} - \left(\frac{\partial \Omega}{\partial b_k} \right) \frac{\partial}{\partial x^k} + \frac{\partial}{\partial x^0} \tag{7}$$

The foregoing discussion leads to the following two points: first, contraction of $\mathbf{d}S$ with the vortex vector, now called \mathbf{R} , gives

$$\mathbf{d}S(\mathbf{R}) = -b_k (\partial\Omega/\partial b_k) + \Omega \tag{8}$$

where $\mathbf{d}S(\mathbf{R})$ is the Lagrangian on extended tangent space $(x^k, dx^k/dx^0, x^0)$. Secondly, note that for (4) (where the exterior derivative of a characteristic differential one-form is contracted on a pair of tangent vectors and set equal to the unique scalar zero), the analysis refers to vortex tubes which do not end. For vortex tubes which end in an elementary volume, $\mathbf{d}\omega(\xi, \eta)$ is set equal to a unique scalar other than zero. A previous application [8] of the present model to the source dependent Maxwell equations illustrates the difference in procedure required for such vortex tubes.

These results lead to the following proposal for all physical processes assumed to proceed in a characteristic direction: Mathematical models of dynamics employing exterior calculus are mathematical representations of the same unifying principle; namely, the description of a dynamic system with a characteristic differential one-form on an odd-dimensional manifold leads, by analysis with exterior calculus to a set of differential equations and a vortex vector which define transformations of the systems.

IV. ECONOMIC GROWTH DYNAMICS ON A DIFFERENTIAL ONE-FORM

The principle described in sections II and III is now illustrated with applications to economic growth dynamics. In analogy with Hamiltonian dynamics, the present investigation proposes a differential one-form for economic growth dynamics on an odd-dimensional differentiable manifold. It is then shown that the use of exterior calculus predicts a pair of differential equations and a characteristic tangent vector (the vortex vector) for economic growth dynamics. This pair of

equations is solved for the rate of change of economic growth as a function of labor $L^i(t)$ and capital $K_i(t)$. By contracting this differential one-form with the vortex vector, the Lagrangian is obtained.

A. Differential one-form for economic growth dynamics Y ; Dynamics

Using as a starting point the growth function $Y(K_i, L^i, t)$, the differential one-form proposed for economic growth dynamics Y is

$$dS = K_i dL^i - Y dt \tag{9}$$

where S plays the role of the action in Hamiltonian mechanics, $Y(K_i, L^i, t)$ is the growth function (the Omega function, e.g., the Hamiltonian), K_i is the capital, L^i is the labor, and t is the time. $K^i(t)$ (compare to momentum in Hamiltonian mechanics) is conjugate to the "position" variable $L^i(t)$, as indicated by the following conditions for conjugacy:

- (a) $K_i = dS(\partial / \partial L^i) =$ contraction of dS with $\partial / \partial L^i$
 - (b) $K_i = K_i(t)$ and $L^i = L^i(t)$
 - (c) $Y = Y(K_i, L^i, t)$
- (10)

Using the symbol $\omega \equiv dS$, the exterior derivative of dS is

$$d\omega = dK_i \wedge dL^i - \left[\left(\frac{\partial Y}{\partial L^i} \right) dL^i + \left(\frac{\partial Y}{\partial K^i} \right) dK_i + \left(\frac{\partial Y}{\partial t} \right) dt \right] \wedge dt \tag{11}$$

Following the procedure of Story ([7], [8] and [9]), consider the vectors $\xi, \eta \in T(T^*M_L)$, where $T(T^*M_L)$ is the tangent of the cotangent space at point L^i along the L -axis and where vector ξ and arbitrary vector η are

$$\xi = \left(\frac{dK_i}{dt} \right) \frac{\partial}{\partial K^i} + \left(\frac{dL^i}{dt} \right) \frac{\partial}{\partial L^i} + \frac{\partial}{\partial t} \tag{12}$$

$$\eta = \beta_{\kappa_i} \frac{\partial}{\partial K^i} + \beta_{L^i} \frac{\partial}{\partial L^i} + \frac{\partial}{\partial t} \tag{13}$$

Employing the mapping $d\omega : (\xi, \eta) \rightarrow d\omega(\xi, \eta)$, note that this this mapping and the contraction

$$d\omega(\xi, \eta) = 0 \tag{14}$$

are defined only if the coordinates $\frac{dL^i}{dt}$ and $\frac{dK_i}{dt}$ of ξ have the values

$$\frac{dL^i}{dt} = + \left(\frac{\partial Y}{\partial K_i} \right) \quad \text{and} \quad \frac{dK_i}{dt} = - \left(\frac{\partial Y}{\partial L^i} \right) \tag{15}$$

for arbitrary tangent vector η . These equations define the relationship between coordinates $\left(\frac{dK_i}{dt}, \frac{dL^i}{dt}, 1 \right)$ and coordinate values $\left(\frac{-\partial Y}{\partial L^i}, \frac{\partial Y}{\partial K_i}, 1 \right)$ for tangent vector ξ at each point of the transformation; hence, the arbitrariness in the coordinates of ξ is removed. The characteristic tangent vector obtained by replacing the coordinates for ξ from (12) with the coordinate values defined by the two differential equations (15), is called the vortex vector (section IVC). This vector gives the direction (the vortex direction) of the system phase flow, with the vortex lines (integral curves of the differential equations passing through points of a closed curve) called the vortex tube.

B. Solutions

The differential equations are now examined with the focus of obtaining a positive economic growth rate with respect to the time-rate of change of capital and labor. Focusing on the differential equation $\frac{dK_i}{dt} = - \left(\frac{\partial Y}{\partial L^i} \right)$, note that a decrease in the rate of capital $\frac{dK_i}{dt}$ implies a positive economic growth with respect to labor; hence, a decrease in capital can be offset by increased labor. Focusing on the differential equation $\frac{dL^i}{dt} = + \left(\frac{\partial Y}{\partial K_i} \right)$, it is noted that an increase in the rate of labor $\frac{dL^i}{dt}$ implies a positive economic growth rate with respect to capital.

Consider the solutions to these characteristic differential equations. The equation $\frac{dK_i}{dt} = - \left(\frac{\partial Y}{\partial L^i} \right)$, has the solution

$$K_i = - \left(\frac{\partial Y}{\partial L^i} \right) + \text{constant of integration} \tag{16}$$

By plotting K_i vs t , a straight line is predicted with a slope $- \left(\frac{\partial Y}{\partial L^i} \right)$; thus, the rate of change of economic growth with respect to labor can be computed from (K_i, t) data. Following the same procedure for the equation $\frac{dL^i}{dt} = + \left(\frac{\partial Y}{\partial K_i} \right)$, leads to the solution

$$L^i = -\left(\frac{\partial Y}{\partial K_i}\right) + \text{constant of integration} \quad (17)$$

In this case the straight line predicts a slope of $\left(\frac{\partial Y}{\partial K_i}\right)$; hence, the rate of change of economic growth with respect to capital can be computed from (L^i, t) data. These solutions therefore provide a quantitative measure of economic growth based on observations of capital and labor as functions of time.

C. Vortex vector, Lagrangian

By substituting the coordinate values from $\frac{dK_i}{dt} = -\left(\frac{\partial Y}{\partial L^i}\right)$

and $\frac{dL^i}{dt} = +\left(\frac{\partial Y}{\partial K_i}\right)$ into (12), the vortex vector is obtained as

$$\mathbf{R} = -\left(\frac{\partial Y}{\partial L^i}\right)\frac{\partial}{\partial K^i} + \left(\frac{\partial Y}{\partial K_i}\right)\frac{\partial}{\partial L^i} + \frac{\partial}{\partial t} \quad (18)$$

The Lagrangian of the system is obtained [8] by contracting the characteristic differential one-form \mathbf{dS} with the vortex vector \mathbf{R} , giving for the Lagrangian,

$$\mathbf{dS}(\mathbf{R}) = K_i \left(\frac{\partial Y}{\partial K_i}\right) - Y \quad (19)$$

D. Integral Invariant of economics

Let γ_1 and γ_2 be two closed in a $(2n + 1)$ -dimensional manifold M^{2n+1} . The vortex lines passing through points of γ_1 and γ_2 form a vortex tube for the extended phase space (K_i, L^i, t) with $\gamma_1 - \gamma_2 = \partial\sigma$, where σ is a section of the vortex tube and $\partial\sigma$ is the boundary of σ . The vortex lines of $\omega (\equiv \mathbf{dS})$ on the extended phase space give a one-to-one projection onto the t -axis. By Stokes' formula,

$$\oint_{\gamma_1} \omega - \oint_{\gamma_2} \omega = \int_{\partial\sigma} \omega = \int_{\sigma} \mathbf{d}\omega \quad (20)$$

However, in a previous discussion it was shown that the equations

$$\frac{dL^i}{dt} = +\left(\frac{\partial Y}{\partial K_i}\right) \quad \text{and} \quad \frac{dK_i}{dt} = -\left(\frac{\partial Y}{\partial L^i}\right) \quad (21)$$

arrive only when $\mathbf{d}\omega(\xi, \eta) = 0$. Hence, the integral of $\mathbf{d}\omega$ is zero, implying

$$\oint_{\gamma_1} \omega = \oint_{\gamma_2} \omega \quad (22)$$

IV. CONCLUSION

The principle applied in this paper is identical to the one applied in other areas of Hamiltonian geometry (optics, thermodynamics, Black holes, classical electromagnetism, classical string theory, and Navier-Stokes dynamics). By applying exterior calculus to economic growth dynamics, a set of differential equations and a characteristic tangent vector for economic growth are constructed. Since a critical and quantitative means of measuring economic growth as a function of capital and labor is an extremely useful societal tool, it is expected that the results presented here will focus more attention to this area of mathematical economics and to other applications of this differential geometric model of dynamics.

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REFERENCES

- [1] Carl P. Simon and Lawrence E. Blume, Mathematics for Economists, Norton and Company, New York, 1994.
- [2] R. J. Barro and X. Sala-i-Martin, Economic Growth, MIT Press, Boston, 2004.
- [3] Angel de la Fuente, Mathematical Methods and Models for Economists, Cambridge University Press, Cambridge, 2000.
- [4] J. Mimkes, "A Thermodynamic Formulation of Economics" in Econophysics and Sociophysics, B. K. Chakrabarti, A. Chatterjee, (Eds.), Wiley, Hoboken, 2006.
- [5] R. N. Mantegna and H. E. Stanley, An Introduction to Econophysics: Correlations and Complexity in Finance, Cambridge University Press, Cambridge, 2007.
- [6] V.I. Arnold, Mathematical Methods of Classical Mechanics, Springer, New York, 2010.
- [7] Troy L. Story, Thermodynamics on one-forms, J. Chem. Phys., 88(1988), 1192-1197.
- [8] Troy L. Story, Dynamics on differential one-forms, J. Math. Chem., 29(2001), 85-96.
- [9] Troy L. Story, Navier-Stokes dynamics on a differential one-form, RIMS Kokyuroku Bessatsu, B1(2007), 365-382.