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2 – Block 3 - Point Modified Numerov Block Methods for Solving Ordinary Differential Equations

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Abstract**—**In this paper, linear multistep technique using power series as the basis function is used to develop the block methods which are suitable for generating direct solution of the special second order ordinary differential equations of the form $y'' = f(x, y)$, $a \leq x \leq b$ with associated initial or boundary conditions. The continuous hybrid formulations enable us to differentiate and evaluate at some grids and off – grid points to obtain two different three discrete schemes, each of order $(4,4,4)^T$, which were used in block form for parallel or sequential solutions of the problems. The computational burden and computer time wastage involved in the usual reduction of second order problem into system of first order equations are avoided by this approach. Furthermore, a stability analysis and efficiency of the block method are tested on linear and non-linear ordinary differential equations whose solutions are oscillatory or nearly periodic in nature, and the results obtained compared favourably with the exact solution.

Keywords**—**Block Method, Hybrid, Linear Multistep Method, Self – starting, Special Second Order.

I. Introduction

LET us consider the numerical solution of the special second order ordinary differential equation of the form

$$y'' = f(x, y), \quad a \leq x \leq b \tag{1}$$

with associated initial or boundary conditions. The mathematical models of most physical phenomena especially in mechanical systems without dissipation leads to special second order initial value problem of type (1). Solutions to initial value problem of type (1) according to Fatunla [1], [2] are often highly oscillatory in nature and thus, severely restrict the mesh size of the conventional linear multistep method. Such system often occurs in mechanical systems without dissipation, satellite tracking, and celestial mechanics.

Lambert [3] and several authors, have written on conventional linear multistep method

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j}, k \geq 2 \tag{2}$$

or compactly in the form

$$\rho(E) y_n = h^2 \delta(E) f_n \tag{3}$$

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where E is the shift operator specified by $E^j y_n = y_{n+j}$ while ρ and δ are characteristics polynomials and are given as:

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j, \ \delta(\xi) = \sum_{j=0}^{k} \beta_j \xi^j \tag{4}$$

y_n is the numerical approximation to the theoretical solution $y(x)$ and $f_n = f(x_n, y_n)$.

In the present consideration, our motivations for the study of this approach is a further advancement in efficiency, i.e obtaining the most accuracy per unit of computational effort, that can be secured with the group of methods proposed in this paper over Awoyemi [4], and Yahaya and Mohammed [5]. However, some authors have proposed solutions to special second order initial value problems of ordinary differential equations using different approaches. In particular, Awoyemi [4], Yahaya and Adegboye [6], Fudziah *et al.* [7], and Onumanyi *et al* [8] developed linear multistep methods for special second order ordinary differential equations.

Hairer *et al.* [9], Numerov's method is a numerical method developed by Boris Vasil'evich Numerov. This method is used to solve ordinary differential equations of second order in which the first-order term does not appear. It is a fourth-order linear multistep method. The three term recurrence formula

$$y_{n+1} = 2y_n - y_{n-1} + \frac{1}{12} h^2 (f_{n+1} + 10 f_n + f_{n-1}) \tag{5}$$

is called the Numerov method for efficient solution of type (1) on a discrete mesh point with variable step-size h of the form $h = x_n - x_{n-1}$ $n = 0,1,\ldots,N$. The formula (5) is accurate of order four with an error term whose coefficient $c_6 = -1/240$.

In a block method, a set of new values that are obtained by each application of the scheme is referred to as "block". In r-point block method, each application of the schemes generates a block of r new equally spaced solution values simultaneously. The computation which proceeds in blocks is based on the computed values at earlier blocks. The computed values at the previous k-block are used to compute the current block containing r-points and the method is called r-point k-block method, Fudziah *et al.* [7].

A. Definition : Consistent, Lambert [3]

The linear multistep method (2) is said to be consistent if it has order $p \geq 1$, that is if

$$\sum_{j=0}^{k} \alpha_j = 0 \text{ and } \sum_{j=0}^{k} j \alpha_j - \sum_{j=0}^{k} \beta_j = 0 \tag{6}$$

Introducing the first and second characteristics polynomials (4), we have from (5) LMM type (2) is consistent if $\rho(1) = 0,\ \rho^1(1) = \delta(1)$

B. Definition: Zero Stability, Lambert [3]

A linear multistep method type (2) is zero stable provided the roots $\xi_j, j = 0(1)k$ of first characteristics polynomial $\rho(\xi)$ specified as $\rho(\xi) = \det\left|\sum_{j=0}^{k} A(i)\xi^{(k-i)}\right| = 0$ satisfies $|\xi_j| \leq 1$ and for those roots with $|\xi_j| = 1$ the multiciplicity must not exceed two. The principal root of $\rho(\xi)$ is denoted by $\xi_1 = \xi_2 = 1$.

C. Definition: Convergence, Lambert [3]

The necessary and sufficient conditions for the linear multistep method type (2) is said to be convergent if it is consistent and zero stable.

D. Definition: Order and Error Constant, Lambert [3]

The linear multistep method type (2) is said to be of order p if $c_0 = c_1 = \cdots c_{P+1} = 0$ but $c_{p+2} \neq 0$ and c_{p+2} is called the error constant, where

$$c_0 = \sum_{j=0}^{k} \propto_j = \propto_0 + \propto_1 + \propto_2 + \ \ldots\ + \propto_k$$

$$c_1 = \sum_{j=0}^{k} j \propto_j = (\propto_1 + \ 2 \propto_2 + 3 \propto_3 + \ \ldots + k \propto_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k)$$

$$c_2 = \sum_{j=0}^{k} \frac{1}{2!} j^2 \propto_j - \sum_{j=0}^{k} \beta_j = \left\{\begin{array}{l} \frac{1}{2!}(\propto_1 + \ 2^2 \propto_2 + 3^3 \propto_3 + \ \ldots + k^2 \propto_k) \\ -(\beta_1 + 2\beta_2 + 3\beta_3 + \ \ldots + k\beta_k) \end{array}\right\}$$

$$c_q = \sum_{j=1}^{k}\left\{\frac{1}{q!} j^q \propto_j - \frac{1}{(q-2)!} j^{q-2}\beta_j\right\} = \left\{\begin{array}{c} \frac{1}{q!}(\propto_1 + \ 2^q \propto_2 + 3^q \propto_3 + \ \ldots + k^q \propto_k) \\ -\frac{1}{(q-1)!}\left(\beta_1 + 2^{(q-1)}\beta_2 + 3^{(q-1)}\beta_3 + \cdots + k^{(q-1)}\beta_k\right) \end{array}\right\} \qquad (7)$$

E. Theorem: Lambert, [3]

Let $f(x, y)$ be defined and continuous for all points (x, y) in the region D defined by $\{(x, y) : a \leq x \leq b, -\infty < y < \infty\}$ where *a and b* finite, and let there exist a constant L such that for every x, y, y^* such that (x, y) *and* (x, y^*) are both in D:

$$|f(x, y) - f(x, y^*)| \leq L|y - y^*| \qquad (8)$$

Then if η is any given number, there exist a unique solution $y(x)$ of the initial value problem (1), where $y(x)$ is continuous and differentiable for all (x, y) *in D*. The inequality (7) is known as a Lipschitz condition and the constant L as a Lipschitz constant.

II. Derivation of the Proposed Method

We proposed an approximate solution to (1) in the form:

$$y(x) = \sum_{j=0}^{t+m-1} a_j x^i = y_{n+j},\ i = 0(1)m + t - 1 \qquad (9)$$

$$y''(x) = \sum_{j=0}^{t+m-1} i(i-1)a_j x^{i-2} = f_{n+j} \qquad (10)$$

i = 2,3,…,m+t-1 with m = 4, t = 2 and p = m+t-1 where the $a_j, j = 0,1,(m + t - 1)$are the parameters to be determined, t and m are points of interpolation and collocation respectively. Where P, is the degree of the polynomial interpolant of our choice. Specifically, we collocate equation (10) at $x = x_{n+j}$, $j = 0(1)k - 1$ and interpolate equation (9) at $x = x_{n+j}$, $j = 0(1)k$ using the method described above. Putting in the matrix equation form and then solved to obtain the values of parameters α_j's, j = 0, 1, … which is substituted in (9) yields, after some algebraic manipulation, the new continuous form for the solution.

$$y(x) = \sum_{j=0}^{k-1} \propto_j(x) y_{n+j} + \sum_{j=0}^{k} \beta_j(x) f_{n+j} \qquad (11)$$

A. Derivation of First Block Method

Let us consider the numerical solution of the second order differential system of type (1). Put equations (9) and (10) in matrix equation form, which when solved either by matrix inversion techniques or Gaussian elimination method to obtain the values of the parameters $\propto_j, j = 0,1, m + t - 1$ and then substituting them into equation (9) to give the continuous form:

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + h^2[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_{4/3}(x)f_{n+4/3} + \beta_2(x)f_{n+2}] \qquad (12)$$

We set $\eta = (x - x_{n+1})$

$$\begin{aligned} y(x) &= \left\{-\left(\frac{\eta}{h}\right)\right\} y_n + \left\{\left(\frac{h+\eta}{h}\right)\right\} y_{n+1} \\ &+ \left\{\frac{-9(\eta)^5 + 20h(\eta)^4 - 10h^2(\eta)^3 + 39h^4(\eta)}{480h^3}\right\} f_n \\ &+ \left\{\frac{9(\eta)^5 - 5h(\eta)^4 - 30h^2(\eta)^3 + 30h^3(\eta)^2 + 46h^4(\eta)}{60h^3}\right\} f_{n+1} \\ &+ \left\{\frac{-27(\eta)^5 + 90h^2(\eta)^3 - 63h^4(\eta)}{160h^3}\right\} f_{n+\frac{4}{3}} \\ &+ \left\{\frac{9(\eta)^5 + 10h(\eta)^4 - 10h^2(\eta)^3 + 11h^4(\eta)}{240h^3}\right\} f_{n+2} \end{aligned} \qquad (13)$$

Evaluating the continuous scheme of equation (13) at some selected point and its first derivative w.r.t x at x = x_0 yield the following schemes:

(a) $y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{12}\{f_n + 10f_{n+1} + f_{n+2}\}$

(b) $y_{n+\frac{4}{3}} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{h^2}{486}\left\{\begin{array}{c} 13f_n + 142f_{n+1} \\ -54f_{n+\frac{4}{3}} \\ +7f_{n+2} \end{array}\right\}$

(c) $hz_0 - y_{n+1} + y_n = \frac{h^2}{120}\left\{\begin{array}{c} -29f_n - 78f_{n+1} \\ +54f_{n+4/3} - 7f_{n+2} \end{array}\right\}$ (14)

Interestingly, the already well-known Numerov's scheme of order four with an error constant $c_6 = -{}^{1}/_{240}$ was recovered in equation (a).

B. Derivation of Second Block Method

Using the same procedure as in first block method, but with one off grid point at interpolation, we obtain another

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_{4/3}(x)y_{n+4/3} + h^2[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2}] \qquad (15)$$

We set $\eta = (x - x_{n+1})$

$$\begin{aligned} y(x) &= \left\{\frac{81(\eta)^5 - 270h^2(\eta)^3 + 29h^4(\eta)}{160h^5}\right\} y_n \\ &+ \left\{\frac{-81(\eta)^5 + 270h^2(\eta)^3 - 149h^4(\eta) + 40h^2}{40h^5}\right\} y_{n+1} \\ &+ \left\{\frac{243(\eta)^5 - 810h^2(\eta)^3 + 567h^4(\eta)}{160h^5}\right\} y_{n+\frac{4}{3}} \\ &+ \left\{\frac{-57(\eta)^5 + 40h(\eta)^4 + 110h^2(\eta)^3 - 13h^4(\eta)}{960h^3}\right\} f_n \\ &+ \left\{\frac{-141(\eta)^5 - 40h(\eta)^4 + 470h^2(\eta)^3 + 240h^3(\eta)^2 - 129h^4(x - x_{n+1})}{480h^3}\right\} f_{n+1} \\ &+ \left\{\frac{3(\eta)^5 + 8h(\eta)^4 + 6h^2(\eta)^3 - h^4(\eta)}{192h^3}\right\} f_{n+2} \end{aligned} \qquad (16)$$

Evaluating (16) at a certain point and taking the first and its second derivatives w.r.t x at some selected points yield the following schemes:

$$\begin{aligned} &\text{(a) } y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{12}\{f_n + 10f_{n+1} + f_{n+2}\} \\ &\text{(b) } 9y_{n+\frac{4}{3}} - 12y_{n+1} + 3y_n = \frac{h^2}{54}\left\{13f_n + 142f_{n+1} - 54f_{n+\frac{4}{3}} + 7f_{n+2}\right\} \\ &\text{(c) } hz_0 + \frac{81}{20}y_{n+4/3} - \frac{32}{5}y_{n+1} + \frac{47}{20}y_n = \frac{h^2}{15}\{-2f_n + 8f_{n+1}\} \end{aligned} \qquad (17)$$

Interestingly, the scheme obtained in (a) above turn out to be the popular discrete Numerov's scheme of order four with an error constant $c_6 = -{}^{1}/_{240}$

Consider the schemes obtained in equation (14) and (17). Taylor series expansion about y(x) was used, which constitute the members of zero stable block integrators each of order $(4,4,4,)^T$ with $c_6 = \left(-\frac{1}{240}, -\frac{313}{262440}, \frac{23}{4320}\right)^T$ and $c_6 = \left(-\frac{1}{240}, -\frac{313}{29160}, \frac{1}{2025}\right)^T$ respectively.

The application of the block integrators with n = 0, gives the accurate values of $y_1, y_2, \ldots, y_k$ as shown in Table I – VI.

To start the IVP integration on the sub interval $\{x_0, x_2\}$, we consider equation (14) to give the 1 – block – 3 point method. Thus, produces simultaneously values for $y_1, y_2, \ldots$ without recourse to any Predictor like Aladeselu [10] and Awoyemi [4] to provide y_1 and y_2 in the main method. Hence, this is an improved over these reported works. Though this does not becloud the contribution of these authors.

III. Stability Analysis

Recall, that, it is a desirable property for a numerical integrator to produce solution that behave similar to the theoretical solution to a problem at all times. Thus, several definitions, which call for the method to posses some "adequate" region of absolute stability can be found in several literatures. See Lambert [3], Fatunla [1], [2] e.t.c

The Fatunla's approach states that the block method is presented as a single block r-point multi-step method of the form:

$$A^{(0)}Y_m = \sum_{i=1}^{k} A^i Y_{m-i} + h^2 \sum_{i=0}^{k} B^{(i)} F_{m-i} \qquad (18)$$

where, h is a fixed mesh size within a block, A^i, B^i, i = 0 (1)k are r x r matrix coefficients, and A^0 is r by r identity matrix, Y_m, Y_{m-i}, F_m and F_{m-i} are vectors of numerical estimates. Following Fatunla [3], [4]; the three integrator proposed in this report in equation (14) are put in the matrix equation form and for easy analysis the result was normalized to obtain;

(i) Convergence Analysis of the first Block Method with one Off-grid Point at Collocation:

The method is expressed in the form of type (18) to give

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+\frac{4}{3}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} y_{n-1} \\ y_{n-\frac{2}{3}} \\ y_n \end{bmatrix} + h^2 \left\{ \begin{bmatrix} -\frac{1}{9} & \frac{7}{486} & -\frac{14}{81} \\ 0 & \frac{1}{12} & -1 \\ \frac{9}{20} & -\frac{7}{120} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+\frac{4}{3}} \\ f_{n+2} \end{bmatrix} + \begin{bmatrix} 0 & \frac{13}{486} & -\frac{13}{243} \\ 0 & \frac{1}{12} & -\frac{1}{6} \\ 0 & -\frac{29}{120} & \frac{29}{60} \end{bmatrix} \begin{bmatrix} f_{n-1} \\ f_{n-\frac{2}{3}} \\ f_n \end{bmatrix} \right\} \qquad (19)$$

The first characteristics polynomial of the block hybrid method (14) is given by

$$\rho(\lambda) = \det[\lambda I - A_1^{(1)}] \qquad (20)$$

Substituting the values of λI and $A_1^{(1)}$ in equation (19), gives $\Rightarrow \lambda_1 = \lambda_2 = 0$ or $\lambda_3 = 1$

From equations (18) and (20), the block hybrid method (14) is zero stable and is also consistent as its order $(4,4,4)^T > 1$, thus, it is convergent following Henrici [11] and Fatunla [2].

(ii) Convergence Analysis of the Block Method with one Off-Grid Point at Interpolation:

The method expressed in the form of (18) to give:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+\frac{4}{3}}\\y_{n+2}\end{bmatrix}=\begin{bmatrix}-1&-\frac{9}{10}&2\\-3&-\frac{27}{10}&6\\-\frac{47}{20}&-\frac{423}{200}&\frac{47}{10}\end{bmatrix}\begin{bmatrix}y_{n-1}\\y_{n-\frac{2}{3}}\\y_n\end{bmatrix}+h^2\left\{\begin{bmatrix}\frac{1}{12}&\frac{9}{20}&-1\\\frac{7}{54}&\frac{53}{90}&-\frac{14}{9}\\0&\frac{6}{25}&-\frac{8}{15}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+\frac{4}{3}}\\f_{n+2}\end{bmatrix}+\begin{bmatrix}\frac{1}{12}&\frac{3}{40}&-\frac{1}{6}\\\frac{13}{54}&\frac{13}{60}&-\frac{13}{27}\\-\frac{2}{15}&-\frac{3}{25}&\frac{4}{15}\end{bmatrix}\begin{bmatrix}f_{n-1}\\f_{n-\frac{2}{3}}\\f_n\end{bmatrix}\right\} \tag{21}$$

The first characteristics polynomial of the block hybrid method (17) is given as in (20)

Substituting the values of λI and $A_1^{(1)}$ in equation (4.4), gives $\rho(\lambda)=\lambda^2(\lambda-1)$, which implies, $\lambda_1=\lambda_2=0$ or $\lambda_3=1$

From equations (18) and (21), the block hybrid method (17) is zero stable and is also consistent as its order $(4,4,4)^T > 1$, thus, it is convergent as in [11] and [2].

IV. Implementation of the Methods

This section deal with numerical experiments by considering the derived discrete schemes in block form for solution of both stiff and non-stiff differential equations of second order initial value problems. The idea is to enable us see how the proposed methods performs when compared with exact solutions. The results are summarized in Table I to VI.

A. *Numerical Experiment*

1. From Yahaya and Mohammed [5];

Consider the IVP $y''=-100y$; $y(0)=1$, $y'(0)=10$. The exact solution is $y(x)=\sin 10x+\cos 10x$

TABLE I
Results for the Proposed Method Presented in Equation (14) with One Off - Grid Point at Collocation

H	X	Exact Value	Approximate Value	Yahaya and Mohammed [5]	Error of Proposed Method (14)
0	0	1	1	0	0
0.001	0.001	1.009949834	1.009949835	3.00000E-09	1.00000E-09
	0.002	1.019798673	1.019798676	3.00000E-09	3.00000E-09
	0.0013	1.012915135	1.013244054		3.28919E-04
0.0025	0.0025	1.024684912	1.024684913	3.00000E-09	1.00000E-09
	0.005	1.048729430	1.048729433	1.00000E-09	3.00000E-09
	0.0013	1.032449560	1.032771658		3.22098E-04
0.005	0.005	1.048729430	1.048729430	1.00000E-09	0.00000E+00
	0.01	1.094837582	1.094837584	3.00000E-09	2.00000E-09
	0.0067	1.064706224	1.064395897		3.10327E-04

TABLE II
Results for the Proposed Method Presented in Equation (17) with One Off - Grid Point at Interpolation

H	X	Exact Value	Approximate Value	Yahaya and Mohammed [5]	Error of Proposed Method (17)
0	0	1	1	0	0
0.001	0.001	1.009949834	1.009949837	3.00000E-09	3.00000E-09
	0.002	1.019798673	1.019798679	3.00000E-09	6.00000E-09
	0.0013	1.012915135	1.013244055		3.28920E-04
0.0025	0.0025	1.024684912	1.024684914	3.00000E-09	2.00000E-09
	0.005	1.048729430	1.048729433	1.00000E-09	3.00000E-09
	0.0013	1.032449560	1.032771659		3.22099E-04
0.005	0.005	1.048729430	1.048729432	1.00000E-09	2.00000E-09
	0.01	1.094837582	1.094837587	3.00000E-09	5.00000E-09
	0.0067	1.064706224	1.064395899		3.10325E-04

B. *Numerical Experiment*

2. From Awoyemi [4]; Consider a Non-Linear IVP; $y''=2y^3; y(1)=1, y'(1)=-1$, whose exact solution is $y(x)=1/x$

TABLE III
RESULTS FOR THE PROPOSED METHOD PRESENTED IN EQUATION (14) WITH ONE OFF - GRID POINT AT COLLOCATION

N	x	Exact Value	Approximate Value	Awoyemi [4]	Error of Proposed Method (14)
0	1	1	1	0	0
1	1.1	0.909090109	0.9090930164	2.8483722E-03	2.90740E-06
2	1.2	0.833333333	0.8333392266	2.26883436E-01	5.89360E-06
3	1.3	0.769230769	0.7692374245	7.3968630E+00	6.65550E-06
4	1.4	0.714285714	0.7142938547	2.1168783E-01	8.14070E-06
5	1.5	0.666666667	0.6666751611	3.3156524E-01	8.49410E-06
6	1.6	0.625	0.6250092535	4.3968593E-01	9.25350E-06
7	1.7	0.588235294	0.5882447490	5.3903097E-01	9.45500E-06
8	1.8	0.555555556	0.5555654825	6.3121827E-01	9.92650E-06
9	1.9	0.526315789	0.5263258495	7.1723621E-01	1.00605E-06
10	2.0	0.5	0.5000103946	7.9776590E-01	1.03946E-05

TABLE IV
RESULTS FOR THE PROPOSED METHOD PRESENTED IN EQUATION (17) WITH ONE OFF - GRID POINT AT INTERPOLATION

N	x	Exact Value	Approximate Value	Awoyemi [4]	Error of Proposed Method (17)
0	1	1	1	0	0
1	1.1	0.909090109	0.9090930164	2.8483722E-03	2.90740E-06
2	1.2	0.833333333	0.8333392265	2.26883436E-01	5.89350E-06
3	1.3	0.769230769	0.7692374244	7.3968630E+00	6.65540E-06
4	1.4	0.714285714	0.7142938546	2.1168783E-01	8.14060E-06
5	1.5	0.666666667	0.6666751610	3.3156524E-01	8.4940E-06
6	1.6	0.625	0.6250092534	4.3968593E-01	9.25340E-06
7	1.7	0.588235294	0.5882447489	5.3903097E-01	9.45490E-06
8	1.8	0.555555556	0.5555654824	6.3121827E-01	9.92640E-06
9	1.9	0.526315789	0.5263258494	7.1723621E-01	1.00604E-06
10	2.0	0.5	0.5000103945	7.9776590E-01	1.03945E-05

C. *Numerical Experiment*

3. From Adeboye [12]; Consider the BVP, $y'' - y = 4x - 5; y(0) = y(1) = 0$, with h = 0.10, whose exact solution is

$$y = \frac{1}{e^{-1} - e}\left[(5e - 1)e^{-x} - (5e^{-1} - 1)e^{x}\right] - 4x + 5$$

TABLE V
RESULTS FOR THE PROPOSED METHOD PRESENTED IN EQUATION (14) WITH ONE OFF - GRID POINT AT COLLOCATION

x	Exact Solution	Approximate Value	Adeboye [12]	Error of Proposed Method (14)
0.00	0.00000000000	0.0000000000	3.379500000E-06	0.000000000E+00
0.10	0.14735784232	0.1473578176	6.598600000E-06	2.470000000E-08
0.20	0.25015214537	0.2501520771	9.454000000E-06	6.820000000E-08
0.30	0.31341504348	0.3134149552	1.156300000E-05	8.820000000E-08
0.40	0.34178302747	0.3417829037	1.204180000E-05	1.237000000E-07
0.50	0.33954334810	0.3395432096	8.902600000E-06	1.385000000E-07
0.60	0.31067692433	0.3106767591	1.922800000E-06	1.652000000E-07
0.70	0.25889818576	0.2588980094	2.803580000E-05	1.763000000E-07
0.80	0.18769224781	0.1876920512	8.259870000E-05	1.966000000E-07
0.90	0.10034979197	0.1003495874	1.870490000E-04	2.045000000E-07
1.00	0.00000000000	-0.0000002194	3.379500000E-06	2.194000000E-07

TABLE VI
RESULTS FOR THE PROPOSED METHOD PRESENTED IN EQUATION (17) WITH ONE OFF - GRID AT INTERPOLATION

x	Exact Solution	Approximate Value	Adeboye [1]	Error of Proposed Method (17)
0.00	0.00000000000	0.0000000000	3.379500000E-06	0.000000000E+00
0.10	0.14735784232	0.1473578182	6.598600000E-06	2.410000000E-08
0.20	0.25015214537	0.2501520782	9.454000000E-06	6.710000000E-08
0.30	0.31341504348	0.3134149558	1.156300000E-05	8.760000000E-08
0.40	0.34178302747	0.3417829049	1.204180000E-05	1.225000000E-07
0.50	0.33954334810	0.3395432108	8.902600000E-06	1.373000000E-07
0.60	0.31067692433	0.3106767591	1.922800000E-06	1.626000000E-07
0.70	0.25889818576	0.2588980102	2.803580000E-05	1.755000000E-07
0.80	0.18769224781	0.1876920529	8.259870000E-05	1.949000000E-07
0.90	0.10034979197	0.1003495889	1.870490000E-04	2.030000000E-07
1.00	0.00000000000	-0.0000002164	3.379500000E-06	2.164000000E-07

V. CONCLUSION

As can be observed from the tables of results, we can conclude that the present methods have many attractive features among which are: at $x = x_{n+2}$, the very popular Numerov's method was recovered, it allows the block formulation, as such it is self – starting and eliminate the use of predictor – corrector method. However, for appropriate choice of k, overlap of solutions model is eliminated.

The order and error constants of the discrete schemes for k = 2 at both collocation and interpolation constitute the members of zero stable block integrator of order $(4,4,4)^T$ respectively. All the new proposed block methods are more accurate when compared with exact solution as well as compared with the solutions obtained by other methods, especially for k = 2 with one off grid point at interpolation. The method is also an efficient technique for finding approximate solutions to both stiff and non – stiff initial value problem.

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