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On $\lambda-$ Summable of Orlicz Space of Gai Sequences of Fuzzy Numbers

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Abstract—In this paper the concept of strongly $(\lambda_M)_p -$ Cesáro summability of a sequence of fuzzy numbers and strongly λ_M- statistically convergent sequences of fuzzy numbers is introduced.

Keywords—Fuzzy numbers, statistical convergence, Orlicz space, gai sequence.

I. Introduction

The concept of fuzzy sets and fuzzy set operations were first introduced Zadeh[18] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka[10] introduced bounded and convergent sequences of fuzzy numbers and studied their some properties. Matloka[10] also has shown that every convergent sequence of fuzzy numbers is bounded. Later on sequences of fuzzy numbers have been discussed by Nanda[12], Nuray [14], Kwon[9], Savas[15], Wu and Wang[17], Bilgin[3] Basarir and Mursaleen [2,11],Aytar[1], Fang and Huang[5], and many others. The notion of statistical convergence was introduced by Fast[6] and Schoenberg[16] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic thoery, number theory. Later on it was further investigated from the se quence space point of view and linked with summability theory by Fridy[7], Kwon[9], Nuray[14], Savas[15] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of Stone-$\breve{C}$ech compactification of the natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability. The notion depends on the density of subsets of the subset N of natural numbers. The natural density of a set A of positive integers is defined by

$$\delta(A) = lim_n \frac{1}{n} |\{k \leq n : k \in A\}|,$$

where $|\{k \leq n : k \in A\}|$ denotes the number of elements of $A \subseteq N$ not exceeding n [13]. It is clear that any finite subset of N have zero natural density and $\delta(A^c) = 1-\delta(A)$. If a property $P(k)$ holds for all $k \in A$ with $\delta(A) = 1$, we say that P holds for almost all k, we abbreviate this by "$a.a.k$". A sequence (x_k) is said to be statistically convergent to L if for every $\epsilon > 0, \delta(\{k \in N : |x_k - L| \geq \epsilon\}) = 0$. In this case we write $S - limx_k = L$. The existing literature on statistical convergence appears to have been restricted to real or complex sequences, but in [9] Kwon, Nuray [14] and Savas[15] extended the idea to apply to sequences of fuzzy numbers.

Let $C(R^n) = \{A \subset R^n : A\, compact\, and\, convex\}$. The space $C(R^n)$ has linear structure induced by the operations $A + B = \{a + b : a \in A, b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$ for $A, B \in C(R^n)$ and $\lambda \in R$. The Hausdorff distance between A and B of $C(R^n)$ is defined as

$$\delta_\infty(A, B) = max\{sup_{a\in A} inf_{b\in B} \|a - b\|, sup_{b\in B} inf_{a\in A} \|a - b\|\}$$

It is well known that $(C(R^n), \delta_\infty)$ is a complete metric space.

The fuzzy number is a function X from R^n to [0,1] which is normal, fuzzy convex, upper semi-continuous and the closure of $\{x \in R^n : X(x) > 0\}$ is compact. These properties imply that for each $0 < \alpha \leq 1$, the $\alpha-$level set $[X]^2 = \{x \in R^n : X(x) \geq \alpha\}$ is a nonempty compact convex subset of R^n, with support $X^0 = \{x \in R^n : X(x) > 0\}$. Let $L(R^n)$ denote the set of all fuzzy numbers. The linear structure of $L(R^n)$ induces the addition $X + Y$ and scalar multiplication $\lambda X, \lambda \in R$, in terms of $\alpha-$ level sets, by $|X + Y|^\alpha = |X|^\alpha + |Y|^\alpha, |\lambda X|^\alpha = \lambda |X|^\alpha$ for each $0 \leq \alpha \leq 1$. Define, for each $1 \leq q < \infty$,

$$d_q(X, Y) = \left(\int_0^1 \delta_\infty(X^\alpha, Y^\alpha)^q d\alpha\right)^{1/q}, and\, d_\infty = sup_{0\leq\alpha\leq1} \delta_\infty(X^\alpha, Y^\alpha),$$

where δ_∞ is the Hausdorff metric. Clearly $d_\infty(X, Y) = lim_{q\to\infty} d_q(X, Y)$ with $d_q \leq d_r$, if $q \leq r$ [4]. Throughout the paper, d will denote d_q with $1 \leq q \leq \infty$. Let w be set of all sequences of fuzzy numbers. The generalized de la Vallée-Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k\in I_n} x_k,$$

where $\lambda = (\lambda_n)$ is a nondecreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n + 1 = 1, \lambda_1 = 1, \lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$. A sequence $x(x_k)$ is said to be $(V, \lambda)-$ summable to a number L [8] if

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$t_n(x) \to L$ as $n \to \infty$. $(V,\lambda)-$ summability reduces to $(C,1)$ summability when $\lambda_n = n$ for all n. A complex sequence, whose k^{th} terms is x_k is denoted by $\{x_k\}$ or simply x. Let ϕ be the set of all finite sequences. Let ℓ_∞, c, c_0 be the sequence spaces of bounded, convergent and null sequences $x = (x_k)$ respectively. In respect of ℓ_∞, c, c_0 we have $\|x\| = \overset{sup}{k} |x_k|$, where $x = (x_k) \in c_0 \subset c \subset \ell_\infty$. A sequence $x = \{x_k\}$ is said to be analytic if $sup_k |x_k|^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by Λ. A sequence x is called gai sequence if $lim_{k\to\infty}(k!|x_k|)^{1/k} = 0$. The vector space of all gai sequences will be denoted by χ. Orlicz [26] used the idea of Orlicz function to construct the space (L^M). Lindenstrauss and Tzafriri [27] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (1 \le p < \infty)$. Subsequently different classes of sequence spaces defined by Parashar and Choudhary[28], Mursaleen et al.[29], Bektas and Altin[30], Tripathy et al.[31], Rao and subramanian[32] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in Ref[33].

Recall([26],[33]) an Orlicz function is a function $M : [0,\infty) \to [o,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \le M(x)+M(y)$ then this function is called modulus function, introduced by Nakano[34] and further discussed by Ruckle[35] and Maddox[36] and many others.

An Orlicz function M is said to satisfy Δ_2- condition for all values of u, if there exists a constant $K > 0$, such that $M(2u) \le KM(u)(u \ge 0)$. The Δ_2- condition is equivalent to $M(\ell u) \le K\ell M(u)$, for all values of u and for $\ell > 1$. Lindenstrauss and Tzafriri[27] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\, for some\, \rho > 0 \right\}. \tag{1}$$

The space ℓ_M with the norm

$$\|x\| = inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\} \tag{2}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p, 1 \le p < \infty$, the space ℓ_M coincide with the classical sequence space ℓ_p. Given a sequence $x = \{x_k\}$ its n^{th} section is the sequence $x^{(n)} = \{x_1, x_2, ..., x_n, 0, 0, ...\}$ $\delta^{(n)} = (0,0,...,1,0,0,...)$, 1 in the n^{th} place and zero's else where.

II. Definitions and Prelimiaries:

Let w denote the set of all fuzzy complex sequences $x = (x_k)_{k=1}^{\infty}$, and M be an Orlicz fucntion, or a modulus function. consider

$\chi_M = x \in w : lim_{k\to\infty}\left(M\left(\frac{(k!|x_k|)^{1/k}}{\rho}\right)\right) = 0\, for some\, \rho > 0$ and

$\Lambda_M = x \in w : sup_k \left(M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right) < \infty\, for some\, \rho > 0$

The space χ_M is a metric space with the metric

$$d(x,y) = inf \left\{ \rho > 0 : sup_k \left(M\left(\frac{(k!|x_k - y_k|)^{1/k}}{\rho} \right)\right) \le 1 \right\} \tag{3}$$

for all $x = \{x_k\}$ and $y = \{y_k\}$ in Γ_M.

The spac Λ_M is a metric space with the metric

$$d(x,y) = inf \left\{ \rho > 0 : sup_k \left(M\left(\frac{|x_k - y_k|^{1/k}}{\rho} \right)\right) \le 1 \right\} \tag{4}$$

for all $x = \{x_k\}$ and $y = \{y_k\}$ in Λ_M.

In the present paper we introduce and examine the concepts of λ_M- statistical convergence and strongly $(\lambda_M)_p-$ Cesáro convergence of sequences of fuzzy numbers. Firstly in section 2, we give the definition of λ_M- statistical convergence and strongly $(\lambda_M)_p-$ Cesáro convergence of sequence of fuzzy numbers. In section 3, we establish some inclusion relation between the sequences $s(\lambda_M)$ and $(\lambda_M)_p$. We now give the following new definitions which will be needed in the sequel.

A. Definition

Let $X = (X_k)$ be a sequence of fuzzy numbers. A sequence $X = (X_k)$ of fuzzy numbers is said to converge to fuzzy number X_0 if for every $\epsilon > 0$ there is a positive integer N_0 such that $\left(d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right), X_0\right)\right) < \epsilon$ for $k \ge N_0$. And $X = (X_k)$ is said to be Cauchy sequence if for every $\epsilon > 0$ there is a positive integer N_0 such that $\left(d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right), X_\ell\right)\right) < \epsilon$ for $k, \ell \ge N_0$.

B. Definition

A sequence $X = (X_k)$ of fuzzy numbers is said to be analytic if the set $\left\{M\left(\frac{|X_k|^{1/k}}{\rho}\right) : k \in N\right\}$ of fuzzy numbers is analytic.

C. Definition

A sequence $X = (X_k)$ of fuzzy numbers is said to be λ_M- statistically convergent to a fuzzy number X_0 if for every $\epsilon > 0$, we have

$\frac{1}{n}\left|\left\{k \in I_n : \left(d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right), X_0\right)\right) \ge \epsilon\right\}\right| \to 0\, as\, n \to \infty$

In this cas we shall write $S_{\lambda_M} - lim_{k\to\infty}\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right)\right) = X_0$

It can be shown that if a sequence $X = (X_k)$ of fuzzy numbers is convergent to a fuzzy number X_0. then it is statistically convergent to the fuzzy number X_0, but the converse does not hold. For example, we define $X = (X_k)$ such that

$$\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right)\right) = \begin{cases} A & \text{if } k = n^2, n = 1,2,3,\cdots \\ 0 & \text{otherwise} \end{cases}$$

Where A is a fixed fuzzy number. Then $X = (X_k)$ is statistically convergent but is not convergent.

D. Definition

A sequence $X = (X_k)$ of fuzzy numbers is said to be strongly λ_M-summable if there is a fuzzy number X_0 such that $\frac{1}{\lambda_n}\sum_{k\in I_n}\left(d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right),X_0\right)\right)\to 0\,as\,n\to\infty$

E. Definition

A sequence $X = (X_k)$ of fuzzy numbers is said to be strongly λ_M- Cesáro summable if there is a fuzzy number X_0 such that $\frac{1}{\lambda_n}\sum_{k\in I_n}\left(d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right),X_0\right)\right)^p\to 0\,as\,n\to\infty$ The set of all strongly $(\lambda_M)_p-$ Cesáro summable sequences of fuzzy numbers is denoted by $\lambda(M_p)$

F. Definition

A sequence $X = (X_k)$ of fuzzy numbers is said to be λ_M- statistically convergent or S_{λ_M} to a fuzzy number X_0 if for every $\epsilon > 0$, we have

$$\frac{1}{\lambda_n}\left|\left\{k\in I_n:\left(d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right),X_0\right)\right)\geq\epsilon\right\}\right|\to 0\,as\,n\to\infty$$

In this cas we shall write $S_{\lambda_M}-lim\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right)\right)=X_0$. In the special case $(\lambda_M)_n = n$ for all $n\in N$, then λ_M- statistically convergent is same as statistically convergent.

III. Main Results

A. Theorem

(i)If a sequence $X = (X_k)$ is strongly $(\lambda_M)_p-$ Cesáro summable to X_0, then it is λ_M- statistically convergent to X_0
(ii)If $X = (X_k)$ is a sequence λ_M- analytic and λ_M- statistically convergent to X_0, then it is strongly $(\lambda_M)_p-$ Cesáro summable to X_0, and hence X is strongly λ_M- Cesáro summable to X_0,
Proof: Let $\epsilon > 0$ and $X\in(\lambda_M)_p$. We have

$$\sum_{k\in I_n}\left(d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right),X_0\right)\right)$$
$$\geq\sum_{k\in I_n\,,\,d(X_k,X_0)\geq\epsilon}\left(d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right),X_0\right)\right)^p$$
$$\geq\left|\left\{k\in I_n:\left(d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right),X_0\right)\right)\geq\epsilon\right\}\right|\epsilon^p$$

Therefore λ_M is statistically convergent X_0.

(ii)Suppose that $X = (X_k)$ is analytic and λ_M- statistically convergent to X_0. Since $X\in\Lambda$, there exists a constant $M > 0$ such that $\left(d\left(M\left(\frac{|X_k|^{1/k}}{\rho}\right),X_0\right)\right)\leq M$ for all k. Let $\epsilon > 0$ be given and choose N_ϵ such that

$\frac{1}{\lambda_n}\left|\left\{k\in I_n:\left(d\left(M\left(\frac{|X_k|^{1/k}}{\rho}\right),X_0\right)\right)\geq\left(\frac{\epsilon}{2}\right)^{1/p}\right\}\right|\leq\frac{\epsilon}{2M^p}$ for all $n > N_\epsilon$, and

set $L_n=\left|\left\{k\in I_n:\left(d\left(M\left(\frac{|X_k|^{1/k}}{\rho}\right),X_0\right)\right)\geq\left(\frac{\epsilon}{2}\right)^{1/p}\right\}\right|$. Now for all $n > N_\epsilon$, we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{|X_k|^{1/k}}{\rho}\right),X_0\right)\right]^p=$$
$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{|X_k|^{1/k}}{\rho}\right),X_0\right)\right]^p+$$
$$\frac{1}{\lambda_n}\sum_{k\notin I_n}\left[d\left(M\left(\frac{|X_k|^{1/k}}{\rho}\right),X_0\right)\right]^p$$
$$\leq\frac{1}{\lambda_n}\left(\frac{\lambda_n\epsilon}{2M^p}\right)M^p+\frac{1}{\lambda_n}\left(\frac{\lambda_n\epsilon}{2}\right)=\epsilon$$

Hence $\left(M\left(\frac{|X_k|^{1/k}}{\rho}\right)\right)\to X_0\,(\lambda_M)_p$. Further we have,

$$\frac{1}{n}\sum_{k=1}^{n}\left[d\left(M\left(\frac{|X_k|^{1/k}}{\rho}\right),X_0\right)\right]=$$
$$\frac{1}{n}\sum_{k=1}^{n-\lambda_n}\left[d\left(M\left(\frac{|X_k|^{1/k}}{\rho}\right),X_0\right)\right]+$$
$$\frac{1}{n}\sum_{k\in I_n}\left[d\left(M\left(\frac{|X_k|^{1/k}}{\rho}\right),X_0\right)\right]$$
$$\leq\frac{1}{\lambda_n}\sum_{k=1}^{n-\lambda_n}\left[d\left(M\left(\frac{|X_k|^{1/k}}{\rho}\right),X_0\right)\right]+$$
$$\leq\frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{|X_k|^{1/k}}{\rho}\right),X_0\right)\right]$$
$$\leq\frac{2}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{|X_k|^{1/k}}{\rho}\right),X_0\right)\right]$$

Hence X is strongly Cesáro summable to X_0, since X is strongly λ_M- Cesáro summable to X_0. This completes the proof.

B. Theorem

Let (X_k) and (Y_k) be sequence of fuzzy numbers.
(i)If $S_{\lambda_M}-lim\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right)\right)=X_0$ and $c\in R$, then $S_{\lambda_M}-lim\left(cM\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right)\right)=cX_0$
(ii)If $S_{\lambda_M}-lim\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right)\right)=X_0$ and $S_{\lambda_M}-lim\left(M\left(\frac{(k!|Y_k|)^{1/k}}{\rho}\right)\right)=Y_0$, then $S_{\lambda_M}-lim\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right)+\left(M\left(\frac{(k!|Y_k|)^{1/k}}{\rho}\right)\right)\right)=X_0+Y_0$

Proof: Let $\alpha\in[0,1]$ and $c\in R$. Let $\left(M\left(\frac{(k!|X_k^\alpha|)^{1/k}}{\rho}\right)\right),\left(M\left(\frac{(k!|Y_k^\alpha|)^{1/k}}{\rho}\right)\right),X_0^\alpha$ and Y_0^α be $\alpha-$ level sets of $\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right)\right),\left(M\left(\frac{(k!|Y_k|)^{1/k}}{\rho}\right)\right)X_0$ and Y_0 respectively. Since $\delta_\infty\left(cM\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right),X_0^\alpha\right)=|c|\,\delta_\infty\left(M\left(\frac{(k!|X_k^\alpha|)^{1/k}}{\rho}\right),X_0^\alpha\right)$, we have $d\left(cM\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right),cX_0\right)=|c|\,d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right),X_0\right)$. For given $\epsilon > 0$ we have $\frac{1}{\lambda_n}\left|\left\{k\in I_n:\left(d\left(cM\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right),X_0\right)\right)\geq\epsilon\right\}\right|\leq\frac{1}{\lambda_n}\left|\left\{k\in I_n:\left(d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right),X_0\right)\right)\geq\frac{\epsilon}{|c|}\right\}\right|$. Hence $S_{\lambda_M}-lim\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right)\right)=cX_0$.
(ii) Suppose that $S_{\lambda_M}-lim\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right)\right)=X_0$ and $S_{\lambda_M}-lim\left(M\left(\frac{(k!|Y_k|)^{1/k}}{\rho}\right)\right)=Y_0$. Firstly we have,

$$\delta_\infty\left(M\left(\frac{(k!|X_k^\alpha|)^{1/k}+(k!|Y_k^\alpha|)^{1/k}}{\rho}\right),X_0^\alpha+Y_0^\alpha\right)\leq$$
$$\delta_\infty\left(M\left(\frac{(k!|X_k^\alpha|)^{1/k}}{\rho}+\frac{(k!|Y_k^\alpha|)^{1/k}}{\rho}\right),\left(M\left(\frac{(k!|Y_k^\alpha|)^{1/k}}{\rho}\right)\right)+X_0^\alpha\right)+$$
$$\delta_\infty\left(M\left(\frac{(k!|Y_k^\alpha|)^{1/k}}{\rho}\right)+X_0^\alpha,X_0^\alpha+Y_0^\alpha\right)=$$
$$\delta_\infty\left(M\left(\frac{(k!|X_k^\alpha|)^{1/k}}{\rho}\right),X_0^\alpha\right)+\delta_\infty\left(M\left(\frac{(k!|Y_k^\alpha|)^{1/k}}{\rho}\right),Y_0^\alpha\right)$$

By Minskowki's inequality we get

$$d\left(M\left(\frac{(k!|X_k^\alpha|)^{1/k}+(k!|Y_k|)^{1/k}}{\rho}\right),X_0+Y_0\right)\leq$$
$$d\left(M\left(\frac{(k!|X_k^\alpha|)^{1/k}}{\rho}\right),X_0\right)+d\left(M\left(\frac{(k!|Y_k^\alpha|)^{1/k}}{\rho}\right),Y_0\right)$$

Therefore given $\epsilon > 0$ we have
$\frac{1}{\lambda_n}\left|\left\{k \in I_n : d\left(M\left(\frac{(k!|X_k|)^{1/k}+(k!|Y_k|)^{1/k}}{\rho}\right), X_0+Y_0\right) \geq \epsilon\right\}\right| \leq$
$\frac{1}{\lambda_n}\left|\left\{k \in I_n : d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right), X_0\right) \geq \frac{\epsilon}{2}\right\}\right| +$
$\frac{1}{\lambda_n}\left|\left\{k \in I_n : d\left(M\left(\frac{(k!|Y_k|)^{1/k}}{\rho}\right), Y_0\right) \geq \frac{\epsilon}{2}\right\}\right|$
Hence $S_{\lambda_M} - lim\left(M\left(\frac{(k!|X_k|)^{1/k}+(k!|Y_k|)^{1/k}}{\rho}\right)\right) = X_0+Y_0$. This completes the proof.

C. Theorem

If a sequence $X = (X_k)$ is statistically convergent to X_0 and $lim\,inf_{(n)}\left(\frac{(\lambda_M)_n}{n}\right) > 0$, then it is λ_M- statistically convergent to X_0

Proof: For given $\epsilon > 0$, we have $\left|\left\{k \in n : d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right), X_0\right) \geq \epsilon\right\}\right| \supset \left|\left\{k \in I_n : d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right), X_0\right) \geq \epsilon\right\}\right|$. Therefore
$\frac{1}{n}\left|\left\{k \leq n : d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right), X_0\right) \geq \epsilon\right\}\right| \geq$
$\frac{1}{n}\left|\left\{k \in I_n : d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right), X_0\right) \geq \epsilon\right\}\right| \geq$
$\frac{(\lambda_M)_n}{n}\frac{1}{(\lambda_M)_n}\left|\left\{k \in I_n : d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right), X_0\right) \geq \epsilon\right\}\right|$.
Taking lim as $n \to \infty$ and using $lim\,inf_{(n)}\left(\frac{(\lambda_M)_n}{n}\right) > 0$, we get $X = (X_k)$ is λ_M- statistically convergent to X_0. This completes the proof.

D. Definition

Let $p = (p_k)$ be any sequence of positive real numbers. Then we define $(\lambda_M)_p = X = (X_k)$:
$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right), X_0\right)\right]^{p_k} \to 0\, as\, n \to \infty$.
Suppose that p_k is a constant for all k, then $(\lambda_M)_p = \lambda_M$.

E. Theorem

Let $0 \leq p_k \leq q_k$ and let $\left\{\frac{q_k}{p_k}\right\}$ be bounded. Then $(\lambda_M)_p \subset (\lambda_M)_q$

Proof: Let
$$X \in (\lambda_M)_q \tag{5}$$
$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right), X_0\right)\right]^{q_k} \to 0\, as\, n \to \infty$.
Let $t_k = \frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right), X_0\right)\right]^{p_k}$ and $\lambda_k = \frac{p_k}{q_k}$ Since $p_k \leq q_k$, we have $0 \leq \lambda_k \leq 1$.
Take $0 < \lambda < \lambda_k$. Define $u_k = t_k\,(t_k \geq 1)$;
$u_k = 0\,(t_k < 1)$ and $v_k = 0\,(t_k \geq 1)$;
$v_k = t_k\,(t_k < 1)\,.\, t_k = u_k + v_k$. (i.e)$t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$.
Now it follows that
$$u_k^{\lambda_k} \leq u_k \leq t_k \quad and \quad v_k^{\lambda_k} \leq v_k^{\lambda}. \tag{6}$$
Since $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$, then $t_k^{\lambda_k} \leq t_k + v_k^{\lambda}$

$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right), X_0\right)^{q_k}\right]^{\lambda_k}$
$\leq \frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right), X_0\right)\right]^{q_k}$
$\Rightarrow \frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right), X_0\right)^{q_k}\right]^{p_k/q_k}$
$\leq \frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right), X_0\right)\right]^{q_k}$
$\Rightarrow \frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right), X_0\right)\right]^{p_k}$
$\leq \frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right), X_0\right)\right]^{q_k}$.
But $\frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right), X_0\right)\right]^{q_k} \to 0\, as\, n \to \infty$.
Hence $\frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right), X_0\right)\right]^{p_k} \to 0\, as\, n \to \infty$. Hence
$$X \in (\lambda_M)_p \tag{7}$$
From (5) and (7) we get $(\lambda_M)_q \subset (\lambda_M)_p$. This completes the proof.

F. Theorem

(a)Let $0 < inf p_k \leq p_k \leq 1$. Then $(\lambda_M)_p \subset \lambda_M$ (b) Let $1 \leq p_k \leq sup\, p_k < \infty$. Then $\lambda_M \subset (\lambda_M)_p$.

Proof:(a)Let
$$X \in (\lambda_M)_p \tag{8}$$
$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!\,|X_k|)^{1/k}}{\rho}\right), X_0\right)\right]^{p_k} \to 0\, as\, n \to \infty.$$
Since $0 < inf\, p_k \leq p_k \leq 1$
$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!\,|X_k|)^{1/k}}{\rho}\right), X_0\right)\right] \leq \frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!\,|X_k|)^{1/k}}{\rho}\right), X_0\right)\right]^{p_k}$$
$$X \in \lambda_M \tag{9}$$
Thus
$$(\lambda_M)_p \subset \lambda_M. \tag{10}$$
This completes the proof.

Proof:(b) Let $p_k \geq 1$ for each k and $sup\, p_k < \infty$. Let $X \in \lambda_M$
$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!\,|X_k|)^{1/k}}{\rho}\right), X_0\right)\right] \to 0\, as\, n \to \infty \tag{11}$$
Since $1 \leq p_k \leq sup\, p_k < \infty$ we have
$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!\,|X_k|)^{1/k}}{\rho}\right), X_0\right)\right]^{p_k} \leq \frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!\,|X_k|)^{1/k}}{\rho}\right), X_0\right)\right]$$
$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right), X_0\right)\right]^{p_k} \to 0\, as\, n \to \infty$ (by using 11)

Therefore $X \in (\lambda_M)_p$. This completes the proof.

G. Theorem

Let $0 < p_k \leq q_k < \infty$ for each k. Then $(\lambda_M)_p \subseteq (\lambda_M)_q$

Proof: Let

$$X \in (\lambda_M)_p \tag{12}$$

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!\,|X_k|)^{1/k}}{\rho}\right),X_0\right)\right]^{p_k} \to 0\, as\, n \to \infty \tag{13}$$

This implies that $\frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right),X_0\right)\right] \leq 1$, for sufficiently large n. Since M is non-decreasing, we get

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!\,|X_k|)^{1/k}}{\rho}\right),X_0\right)\right]^{q_k}$$
$$\leq \frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!\,|X_k|)^{1/k}}{\rho}\right),X_0\right)\right]^{p_k}$$

$\Rightarrow \frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(M\left(\frac{(k!|X_k|)^{1/k}}{\rho}\right),X_0\right)\right]^{q_k} \to 0\, as\, n \to \infty$ (by using 12). Hence

$$X \in (\lambda_M)_q \tag{14}$$

From (12) and (14) we get $(\lambda_M)_p \subseteq (\lambda_M)_q$. This completes the proof.

IV. Conclusion

The above results are constructed with the concept of strongly $(\lambda_M)_p-$ Ces*á*ro summability of a gai sequence of fuzzy numbers and strongly λ_M- statistically convergent sequences of fuzzy numbers.

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