

Analysis of Euler Angles in a Simple Two-Axis Gimbals Set

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Abstract—Any rotation of a 3-dimensional object can be performed by three consecutive rotations over Euler angles. Intrinsic rotations produce the same result as extrinsic rotations in transformation. Euler rotations are the movement obtained by changing one of the Euler angles while leaving the other two constant. These Euler rotations are applied in a simple two-axis gimbals set mounted on an automobiles. The values of Euler angles are $[\pi/4, \pi/4, \pi/4]$ radians inside the angles ranges for a given coordinate system and these actual orientations can be directly measured from these gimbals set of moving automobiles but it can occur the gimbals lock in application at $[\pi/2.24, 0, 0]$ radians. In order to avoid gimbals lock, the values of quaternion must be $[\pi/4.8, \pi/8.2, 0, \pi/4.8]$ radians. The four-gimbals set can eliminate gimbals lock.

Keywords—Intrinsic rotations, extrinsic rotations, Euler rotations, rotation matrices, quaternion

I. INTRODUCTION

THE Euler angles are three angles to describe the orientation of a rigid body [1]. To describe such an orientation in 3-dimensional space three parameters are required. Euler angles also represent three composed rotations that move a reference frame to a given frame. This is equivalent to saying that any orientation can be achieved by composing three elemental rotations and also equivalent to saying that any rotation matrix can be decomposed as a product of three elemental rotation matrices. The main aim is to apply in the simple two-axis gimbals set mounted on the automobiles. The objectives are:

- To denote three consecutive angles on a rigid body.
- To rotate mobile frame with intrinsic axes.
- To rotate mobile frame with respect to reference frame.
- To rotate the simple two-axis gimbals set with Euler rotations.

II. DEFINITION OF EULER ANGLES

Euler angles are a means of representing the orientation of any frame from a transformed frame. Fig. 1 illustrates the

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fixed system denoted in lower case (x,y,z) and the rotated system denoted in upper case letters (X,Y,Z). In Z - X' - Z'' transformation, if a reference frame and the one whose orientation are given, first the line of nodes (N) can be defined as the intersection of the xy and the XY coordinate systems. In other words, line of nodes is the line perpendicular to both z and Z axis. Then Euler angles are defined as:

- α (or ϕ) is the angle between the x -axis and the line of nodes.
- β (or θ) is the angle between the z -axis and the Z -axis.
- γ (or ψ) is the angle between the line of nodes and the X -axis.

Euler angles between two frames are defined only if both frames have the same handedness. Euler angles are just one of the several ways of specifying the relative orientation of two such coordinate systems. It can use the different sets of angles to describe these orientations leading to different transformations. Therefore Euler angles should always be preceded by sequence of angles [2].

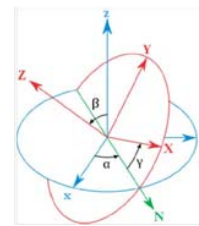


Fig. 1 Euler angles - first rotation over angle α , second rotation over angle β , and third rotation over angle γ with Z - X' - Z'' transformation

III. ANGLES SIGNS AND RANGES

Normally, angles are defined as positive when these angles rotate counter-clock-wise. The positive side will be the one of the positive axis of rotation. About the ranges:

- α and γ range are defined 2π radians. A valid range could be $[-\pi, \pi]$.
- β range covers π radians. For example could be $[0, \pi]$ or $[-\pi/2, \pi/2]$.

The angles α , β and γ are uniquely determined except for the singular case that the xy and the XY planes are identical, the z axis and the Z axis having the same or opposite directions. Indeed, if the z -axis and the Z -axis are the same ($\beta = 0$), only $(\alpha+\gamma)$ is uniquely defined and similarly, if the z -axis

and the Z-axis are opposite ($\beta = \pi$), only $(\alpha-\gamma)$ is uniquely defined. This phenomenon is known as gimbals lock of Mechanical Engineering [3].

IV. EULER ANGLES INTRINSIC ROTATIONS

Euler angles are given using the intrinsic rotations Z-X'-Z'', this means that these angles are equivalent to three concatenated intrinsic rotations around moving axes Z, X' and Z'' in that order. That order is non-commutative. It has to be applied in such a way that in the beginning one of the intrinsic axis moves together with the line of nodes. Starting with an initial set of mobile axes as illustrated in Fig. 2, say XYZ overlapping the reference axes xyz, three intrinsic rotations only about the mobile frame axes can be used to reach any target frame with an origin coincident with that of xyz from the reference frame. The values of the rotations are the Euler angles.

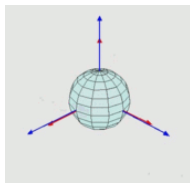


Fig. 2 Mobile frame rotations with respect to intrinsic axes

The position of the mobile axes can be reached using three rotations with angles α, β, γ in three ways as follows: The XYZ system rotates while the xyz is fixed. Starting with the XYZ system overlapping the reference frame xyz, the same rotations can be performed using only rotations around the mobile axes XYZ.

- Rotate the XYZ-system about the Z-axis by α . The X-axis now lies on the line of nodes.
- Rotate the XYZ-system again about the now rotated X-axis by β . The Z-axis is now in its final orientation, and the x-axis remains on the line of nodes.
- Rotate the XYZ-system a third time about the new Z-axis by γ .

V. EULER ANGLES EXTRINSIC ROTATIONS

This is the extrinsic rotations z-x-z, Euler angles are backward, meaning that the first angle is the intrinsic rotation and the last one is the precession. The extrinsic rotations about the reference frame axes as indicated in Fig. 3 can be used to reach any target frame. Let xyz system be fixed while the XYZ system rotates. Start with the rotating XYZ system coinciding with the fixed xyz system.

- Rotate the XYZ-system about the z-axis by γ . The X-axis is now at angle γ with respect to the x-axis.

- Rotate the XYZ-system again about the x-axis by β . The Z-axis is now at angle β with respect to the z-axis.
- Rotate the XYZ-system a third time about the z-axis by α . The first and third axes are identical.

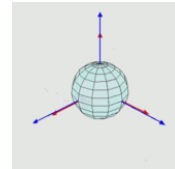


Fig. 3 Mobile frame rotations about the reference frame

The rotation is always around the origin. Off center rotations require first shifting the origin and then the object is rotated. Fig. 4 demonstrates mobile frame rotations with respect to intrinsic rotations and extrinsic rotations.

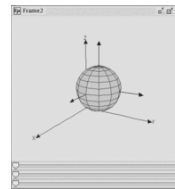


Fig. 4 Mobile frame rotations representing intrinsic rotations and extrinsic rotations

These extrinsic rotations can be shown to be equivalent to the intrinsic rotations: The successive frames deduced from the initial (e) reference frame may be defined as (e), (f), (g) as shown in the equation (1). The vectors u, v, w are the successive vectors obtained with that rotation. The column matrix representing a vector x in the frame (e) is $(x)_e$. If necessary a lower index will be also added to any matrix to operate in a specific frame. (Z_α) , (X_β) , (Z_γ) are called the successive rotations. Thus the intrinsic operations are described as :

$$(Z_\alpha)_e = (Z_\alpha) \quad (X_\beta)_f = (X_\beta) \quad (Z_\gamma)_g = (Z_\gamma) \quad (1)$$

When describing the intrinsic rotations in the (e) reference frame of course the matrices must be transformed to represent the rotations. Then by the rules of matrix algebra:

$$(t)_e = (Z_\gamma)_e (X_\beta)_e (Z_\alpha)_e (u)_e \quad (2)$$

$$(X_\beta)_e = (Z_\alpha)_e (X_\beta)_e (Z_\alpha)_e^t = (Z_\alpha) (X_\beta) (Z_\alpha)^t \quad (3)$$

$$(Z_\gamma)_e = (Z_\alpha)_e (X_\beta)_f (Z_\gamma)_g (X_\beta)_f^t (Z_\alpha)_e^t = (Z_\alpha) (X_\beta) (Z_\gamma) (X_\beta)^t (Z_\alpha)^t \quad (4)$$

$$(t)_e = [(Z_\alpha) (X_\beta) (Z_\gamma) (X_\beta)^t (Z_\alpha)^t] [(Z_\alpha) (X_\beta) (Z_\alpha)^t] (u)_e = (Z_\alpha) (X_\beta) (Z_\gamma) (u)_e \quad (5)$$

$$(R)_e = (Z_\gamma)_e (X_\beta)_e (Z_\alpha)_e = (Z_\alpha) (X_\beta) (Z_\gamma) \quad (6)$$

The equation (5) can then of course be interpreted in extrinsic manner as a succession of rotations around the (e) axes.

VI. EULER ANGLES EULER ROTATIONS

Euler rotations are defined as the movement obtained by changing one of the Euler angles while leaving the other two constant. Euler rotations are never expressed in terms of the external frame, or in terms of the co-moving rotated body frame, but in a mixture. These Euler rotations constitute a rotation system of mixed axes, where the first angle rotates the line of nodes around the external axis z, the second rotates around the line of nodes and the third one is an intrinsic rotation around an axis fixed in the body that moves.

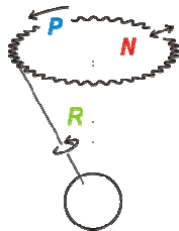


Fig. 5 Euler rotations of the Earth. Intrinsic (green), precession (blue) and nutation (red)

These rotations are called precession, nutation, and intrinsic rotation as shown in Fig. 5. When these rotations are applied over individual frames, only precession is valid as a rotation operator, and only precession can be expressed in general as a matrix in the basis of the space.

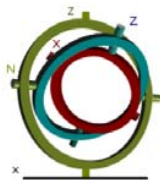


Fig. 6 Three axes z-x-z-gimbals showing Euler angles

If the simple two-axis gimbals set are assumed, it is able to move each with respect to the former according to just one angle, there will be one initial, one final and two in the middle, which are called intermediate frames. The two in the middle work as two gimbal rings that allow the last frame to reach any orientation in space. Fig. 6 indicates three axes z-x-z-gimbals showing Euler angles. External frame and external axis 'x' are not shown.

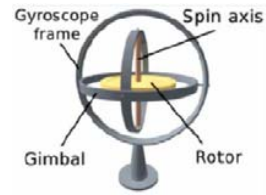


Fig. 7 Illustration of the simple two-axis gimbals set

The simple two-axis gimbals set are illustrated as given in Fig. 7. In this Figure, the outer gimbal or ring, which is the gyroscope frame, is mounted so as to pivot about an axis in its own plane determined by the support. This outer gimbal possesses one degree of rotational freedom and its axis possesses none. The next inner gimbal is mounted in the gyroscope frame so as to pivot about an axis in its own plane that is always perpendicular to the pivotal axis of the outer gimbal. This inner gimbal has two degrees of rotational freedom and its axis possesses one. Likewise, next innermost gimbal is attached to the inner gimbal, which has three degrees of rotational freedom and its axis possesses two.



(a) Gimbals set operation (b) Gyroscope frame precession

Fig. 8 The simple two-axis gimbals set operation in the automobiles and precession of the gyroscope frame

In Fig. 8, as this gimbals set keeps the rotation axes constant, angles measured in the gyro frame are equivalent to angles measured in the lab frame together with precession of the gyroscope frame when the second and third Euler angles of the frame are constant.

These gimbals set is a device for measuring or maintaining orientation based on the principles of conservation of angular momentum [4]. In essence, a mechanical gyroscope is a spinning wheel or disk whose axle is free to take any orientation. This orientation changes very little in response to a given external torque without the large angular momentum associated with the gyroscope's high rate of spin. Since external torque is minimized by mounting gimbals, its orientation remains nearly fixed, regardless of any motion of the platform on which it is mounted.

One problem with gimbals systems is gimbals lock. Gimbals lock occurs when two axes in the three gimbals system align. When that happens, the object's movement is limited. An entire range of motion becomes impossible. There are two ways to avoid gimbals lock. One is to adjust the gimbals, either by maneuvering the surface so that the gimbals swing another way or by resetting the gimbals. If gimbals lock does occur, the gimbals must be reset to work again. Another

solution is to add more gimbals to the system. Adding a fourth gimbal helps to eliminate gimbals lock.

VII. ROTATION MATRICES FOR EULER ANGLES

Using the sequence of Euler angles, it is possible to change to and from matrix convention. Fixed axes and column vectors, with intrinsic rotations and the right-handed rule for the positive sign of the angles are assumed. This means that a transformation (ZXZ) is the result of performing first an intrinsic Z rotation, followed by an X and a Z rotation, in the moving axes. Its matrix is the product of Rot(Z,θ₁) Rot(X,θ₂) Rot(Z,θ₃) like this:

$$\begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 \\ \sin \theta_1 & \cos \theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_2 & -\sin \theta_2 \\ 0 & \sin \theta_2 & \cos \theta_2 \end{bmatrix} \begin{bmatrix} \cos \theta_3 & -\sin \theta_3 & 0 \\ \sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sub-indexes refer to the order in which the angles are applied. Trigonometric notation has been simplified. For example, c₁ means cosθ₁ and s₂ means sinθ₂ and θ₁ is the external angle between fixed axis **x** and line of nodes and θ₃ the internal angle from the line of nodes to rotated axis **X**. TABLE I can be used both ways, to obtain an orientation matrix from Euler angles and to obtain Euler angles from the matrix. The possible combinations of rotations equivalent to Euler angles are shown in TABLE I.

TABLE I
ROTATION MATRICES FOR INTRINSIC ROTATIONS

	C ₂	-S ₂ C ₃	S ₂ S ₃
ZXZ	C ₁ S ₂	C ₁ C ₂ C ₃ - S ₁ S ₃	-C ₁ C ₂ S ₃ - S ₁ C ₃
	S ₁ S ₂	C ₁ S ₃ + S ₁ C ₂ C ₃	C ₁ C ₃ - S ₁ C ₂ S ₃
	C ₂	S ₂ S ₃	S ₂ C ₃
YXX	S ₁ S ₂	C ₁ C ₃ - S ₁ C ₂ S ₃	-C ₁ S ₃ - S ₁ C ₂ C ₃
	-C ₁ S ₂	C ₁ C ₂ S ₃ + S ₁ C ₃	C ₁ C ₂ C ₃ - S ₁ S ₃
	C ₁ C ₃ - S ₁ C ₂ S ₃	S ₁ S ₂	C ₁ S ₃ + S ₁ C ₂ C ₃
YXY	S ₂ S ₃	C ₂	-S ₂ C ₃
	-C ₁ C ₂ S ₃ - S ₁ C ₃	C ₁ S ₂	C ₁ C ₂ C ₃ - S ₁ S ₃
	C ₁ C ₂ C ₃ - S ₁ S ₃	-C ₁ S ₂	C ₁ C ₂ S ₃ + S ₁ C ₃
YZY	S ₂ C ₃	C ₂	S ₂ S ₃
	-C ₁ S ₃ - S ₁ C ₂ C ₃	S ₁ S ₂	C ₁ C ₃ - S ₁ C ₂ S ₃
	C ₁ C ₂ C ₃ - S ₁ S ₃	-C ₁ C ₂ S ₃ - S ₁ C ₃	C ₁ S ₂
ZYZ	C ₁ S ₃ + S ₁ C ₂ C ₃	C ₁ C ₃ - S ₁ C ₂ S ₃	S ₁ S ₂
	-S ₂ C ₃	S ₂ S ₃	C ₂
	C ₁ C ₃ - S ₁ C ₂ S ₃	-C ₁ S ₃ - S ₁ C ₂ C ₃	S ₁ S ₂
ZXZ	C ₁ C ₂ S ₃ + S ₁ C ₃	C ₁ C ₂ C ₃ - S ₁ S ₃	-C ₁ S ₂
	S ₂ S ₃	S ₂ C ₃	C ₂

The possible combinations of rotations for gimbals lock at (θ₂ = 0, π) equivalent to Euler angles are given in TABLE II and TABLE III.

TABLE II
ROTATION MATRICES FOR GIMBALS LOCK AT Θ₂ = 0

	1	0	0
ZXZ	0	cos (θ ₁ + θ ₃)	- sin (θ ₁ + θ ₃)
	0	sin (θ ₁ + θ ₃)	cos (θ ₁ + θ ₃)
	1	0	0
YXX	0	cos (θ ₁ + θ ₃)	- sin (θ ₁ + θ ₃)
	0	sin (θ ₁ + θ ₃)	cos (θ ₁ + θ ₃)
	cos (θ ₁ + θ ₃)	0	sin (θ ₁ + θ ₃)
YXY	0	1	0
	- sin (θ ₁ + θ ₃)	0	cos (θ ₁ + θ ₃)
	cos (θ ₁ + θ ₃)	0	sin (θ ₁ + θ ₃)
YZY	0	1	0
	- sin (θ ₁ + θ ₃)	0	cos (θ ₁ + θ ₃)
	cos (θ ₁ + θ ₃)	- sin (θ ₁ + θ ₃)	0
ZYZ	sin (θ ₁ + θ ₃)	cos (θ ₁ + θ ₃)	0
	0	0	1
	cos (θ ₁ + θ ₃)	- sin (θ ₁ + θ ₃)	0
ZXZ	sin (θ ₁ + θ ₃)	cos (θ ₁ + θ ₃)	0
	0	0	1

TABLE III
ROTATION MATRICES FOR GIMBALS LOCK AT Θ₂ = π

	-1	0	0
ZXZ	0	cos (θ ₁ - θ ₃)	sin (θ ₁ - θ ₃)
	0	sin (θ ₁ - θ ₃)	- cos (θ ₁ - θ ₃)
	-1	0	0
YXX	0	cos (θ ₁ - θ ₃)	sin (θ ₁ - θ ₃)
	0	sin (θ ₁ - θ ₃)	- cos (θ ₁ - θ ₃)
	- cos (θ ₁ - θ ₃)	0	sin (θ ₁ - θ ₃)
YXY	0	-1	0
	sin (θ ₁ - θ ₃)	0	cos (θ ₁ - θ ₃)
	- cos (θ ₁ - θ ₃)	0	sin (θ ₁ - θ ₃)
YZY	0	-1	0
	sin (θ ₁ - θ ₃)	0	cos (θ ₁ - θ ₃)
	cos (θ ₁ - θ ₃)	sin (θ ₁ - θ ₃)	0
ZYZ	sin (θ ₁ - θ ₃)	- cos (θ ₁ - θ ₃)	0
	0	0	-1
	cos (θ ₁ - θ ₃)	sin (θ ₁ - θ ₃)	0
ZXZ	sin (θ ₁ - θ ₃)	- cos (θ ₁ - θ ₃)	0
	0	0	-1

VIII. QUATERNION

Unit quaternion [5], also known as Euler parameters provide another mechanism for representing 3D rotations. This is equivalent to the special unitary group description. Expressing rotations in 3D as unit quaternion instead of matrices has some advantages:

- Concatenating rotations is computationally faster and numerically more stable.
- Extracting the angle and axis of rotation is simpler.
- Interpolation is more straightforward.

IX. RELATIONSHIP BETWEEN EULER ANGLES AND QUATERNION

The rotations in three dimensions can be parameterized using both Euler angles and unit quaternion. The unit quaternion in this application is referred as "Euler parameters".

The unit quaternion can be described as:

$$q = [q_0 \ q_1 \ q_2 \ q_3]^T$$

$$|q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

The orthogonal matrix corresponding to a counterclockwise / right-handed rotation by the unit quaternion $q = q_0 + iq_1 + jq_2 + kq_3$ are given by the homogeneous expression:

$$\begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_0q_3 + 2q_1q_2 & 2q_0q_2 - 2q_1q_3 \\ 2q_0q_3 - 2q_1q_2 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_0q_1 + 2q_2q_3 \\ 2q_0q_2 + 2q_1q_3 & 2q_0q_1 - 2q_2q_3 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

The orthogonal matrix corresponding to a counterclockwise / right-handed rotation with Euler angles $[\theta_1, \theta_2, \theta_3]$ with ZXZ transformation is given by:

$$\begin{bmatrix} C_1 C_3 - S_1 C_2 S_3 & -C_1 S_3 - S_1 C_2 C_3 & S_1 S_3 \\ C_1 C_2 S_3 + S_1 C_3 & C_1 C_2 C_3 - S_1 S_3 & -C_1 S_2 \\ S_2 S_3 & S_2 C_3 & C_2 \end{bmatrix}$$

By quaternion from Euler angles:

$$q = R_z(\theta_1) R_x(\theta_2) R_z(\theta_3)$$

$$= [\cos \theta_1/2 + k \sin \theta_1/2] [\cos \theta_2/2 + i \sin \theta_2/2] [\cos \theta_3/2 + k \sin \theta_3/2]$$

$$q = \begin{bmatrix} C_{\theta_1/2} C_{\theta_2/2} C_{\theta_3/2} - S_{\theta_1/2} C_{\theta_2/2} S_{\theta_3/2} \\ C_{\theta_1/2} S_{\theta_2/2} C_{\theta_3/2} + S_{\theta_1/2} S_{\theta_2/2} S_{\theta_3/2} \\ C_{\theta_1/2} S_{\theta_2/2} S_{\theta_3/2} - S_{\theta_1/2} S_{\theta_2/2} C_{\theta_3/2} \\ C_{\theta_1/2} C_{\theta_2/2} S_{\theta_3/2} + S_{\theta_1/2} C_{\theta_2/2} C_{\theta_3/2} \end{bmatrix}$$

Euler angles from quaternion:

$$\begin{bmatrix} \varphi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} -\text{atan2}(2q_0q_2 - 2q_1q_3, 2q_0q_1 + 2q_2q_3) \\ \text{acos}(q_0^2 - q_1^2 - q_2^2 + q_3^2) \\ \text{atan2}(2q_0q_2 + 2q_1q_3, 2q_0q_1 - 2q_2q_3) \end{bmatrix}$$

TABLE IV and TABLE V can be used to obtain quaternion from Euler angles and Euler angles from quaternion. The possible solutions are shown in TABLE IV and V.

TABLE IV
THE POSSIBLE SOLUTIONS OF QUATERNION FROM EULER ANGLES

XZX	$C_{1/2}C_{2/2}C_{3/2} - S_{1/2}C_{2/2}S_{3/2}$
	$C_{1/2}C_{2/2}S_{3/2} + S_{1/2}C_{2/2}C_{3/2}$
	$-C_{1/2}S_{2/2}S_{3/2} + S_{1/2}S_{2/2}C_{3/2}$
	$C_{1/2}S_{2/2}C_{3/2} + S_{1/2}S_{2/2}S_{3/2}$
XYX	$C_{1/2}C_{2/2}C_{3/2} - S_{1/2}C_{2/2}S_{3/2}$
	$C_{1/2}C_{2/2}S_{3/2} + S_{1/2}C_{2/2}C_{3/2}$
	$C_{1/2}S_{2/2}C_{3/2} + S_{1/2}S_{2/2}S_{3/2}$
	$C_{1/2}S_{2/2}S_{3/2} - S_{1/2}S_{2/2}C_{3/2}$
YXY	$C_{1/2}C_{2/2}C_{3/2} - S_{1/2}C_{2/2}S_{3/2}$
	$C_{1/2}S_{2/2}C_{3/2} + S_{1/2}S_{2/2}S_{3/2}$
	$C_{1/2}C_{2/2}S_{3/2} + S_{1/2}C_{2/2}C_{3/2}$
	$-C_{1/2}S_{2/2}S_{3/2} + S_{1/2}S_{2/2}C_{3/2}$
YZY	$C_{1/2}C_{2/2}C_{3/2} - S_{1/2}C_{2/2}S_{3/2}$
	$C_{1/2}S_{2/2}S_{3/2} - S_{1/2}S_{2/2}C_{3/2}$
	$C_{1/2}C_{2/2}S_{3/2} + S_{1/2}C_{2/2}C_{3/2}$
	$C_{1/2}S_{2/2}C_{3/2} + S_{1/2}S_{2/2}S_{3/2}$
ZYZ	$C_{1/2}C_{2/2}C_{3/2} - S_{1/2}C_{2/2}S_{3/2}$
	$-C_{1/2}S_{2/2}S_{3/2} + S_{1/2}S_{2/2}C_{3/2}$
	$C_{1/2}S_{2/2}C_{3/2} + S_{1/2}S_{2/2}S_{3/2}$
	$C_{1/2}C_{2/2}S_{3/2} + S_{1/2}C_{2/2}C_{3/2}$
ZXZ	$C_{1/2}C_{2/2}C_{3/2} - S_{1/2}C_{2/2}S_{3/2}$
	$C_{1/2}S_{2/2}C_{3/2} + S_{1/2}S_{2/2}S_{3/2}$
	$C_{1/2}S_{2/2}S_{3/2} - S_{1/2}S_{2/2}C_{3/2}$
	$C_{1/2}C_{2/2}S_{3/2} + S_{1/2}C_{2/2}C_{3/2}$

TABLE V
THE POSSIBLE SOLUTIONS OF EULER ANGLES FROM QUATERNION

XZX	φ	$\text{atan2}(2q_0q_2 + 2q_1q_3, 2q_0q_3 - 2q_1q_2)$
	θ	$\text{acos}(q_0^2 + q_1^2 - q_2^2 - q_3^2)$
	ψ	$-\text{atan2}(2q_0q_2 - 2q_1q_3, 2q_0q_3 + 2q_1q_2)$
XYX	φ	$-\text{atan2}(2q_0q_3 - 2q_1q_2, 2q_0q_2 + 2q_1q_3)$
	θ	$\text{acos}(q_0^2 + q_1^2 - q_2^2 - q_3^2)$
	ψ	$\text{atan2}(2q_0q_3 + 2q_1q_2, 2q_0q_2 - 2q_1q_3)$
YXY	φ	$\text{atan2}(2q_0q_3 + 2q_1q_2, 2q_0q_1 - 2q_2q_3)$
	θ	$\text{acos}(q_0^2 - q_1^2 + q_2^2 - q_3^2)$
	ψ	$-\text{atan2}(2q_0q_3 - 2q_1q_2, 2q_0q_1 + 2q_2q_3)$
YZY	φ	$-\text{atan2}(2q_0q_1 - 2q_2q_3, 2q_0q_3 + 2q_1q_2)$
	θ	$\text{acos}(q_0^2 - q_1^2 + q_2^2 - q_3^2)$
	ψ	$\text{atan2}(2q_0q_1 + 2q_2q_3, 2q_0q_3 - 2q_1q_2)$
ZYZ	φ	$\text{atan2}(2q_0q_1 + 2q_2q_3, 2q_0q_2 - 2q_1q_3)$
	θ	$\text{acos}(q_0^2 - q_1^2 - q_2^2 + q_3^2)$
	ψ	$-\text{atan2}(2q_0q_1 - 2q_2q_3, 2q_0q_2 + 2q_1q_3)$
ZXZ	φ	$-\text{atan2}(2q_0q_2 - 2q_1q_3, 2q_0q_1 + 2q_2q_3)$
	θ	$\text{acos}(q_0^2 - q_1^2 - q_2^2 + q_3^2)$
	ψ	$\text{atan2}(2q_0q_2 + 2q_1q_3, 2q_0q_1 - 2q_2q_3)$

X. HIGHER DIMENSIONS

It is possible to define parameter analogous to the Euler angles in dimensions higher than three. The number of degrees of freedom of a rotation matrix is always less than the dimension of the matrix squared. That is, the elements of a rotation matrix are not all completely independent. The rotation matrix in dimension 2 has only one degree of freedom, since all four of its elements depend on a single angle of rotation. A rotation matrix in dimension 3 which has

nine elements has three degrees of freedom, corresponding to each independent rotation, by its three Euler angles. Any set of 6 parameters defines that the rotation matrix could be considered an extension of Euler angles to dimension 4.

In general, the number of Euler angles in dimension D is quadratic in D; since any one rotation consists of choosing two dimensions to rotate between, the total number of rotations available in dimension D is:

$$N_{rot} = \binom{D}{2} = D(D - 1) / 2$$

which for D = 2, 3, 4 yields $N_{rot} = 1, 3, 6$.

XI. RESULTS OF EULER ANGLES

The values of Euler angles, then the problem of gimbals lock, and then the values of quaternion are shown in TABLE VI.

TABLE VI
THE VALUES OF EULER ANGLES, THE PROBLEM OF GIMBALS LOCK AND THE VALUES OF QUATERNION

Method	The Values of Euler Angles							
	$\alpha_1(\varphi_1)$	$\beta_1(\theta_1)$	$\gamma_1(\psi_1)$	$\alpha_2(\varphi_2)$	$\beta_2(\theta_2)$	$\gamma_2(\psi_2)$		
Rotation matrices	$\pi / 4$	$\pi / 4$	$\pi / 4$	$3\pi / 4$	$3\pi / 4$	$3\pi / 4$		
Method	The Problems of Gimbals Lock							
Rotation matrices	$\pi / 2.24$	0	0	$\pi / 1.81$	π	0		
Method	The Values of Quaternion							
	$\alpha_1(\varphi_1)$	$\beta_1(\theta_1)$	$\gamma_1(\psi_1)$	$\delta_1(\omega_1)$	$\alpha_2(\varphi_2)$	$\beta_2(\theta_2)$	$\gamma_2(\psi_2)$	$\delta_2(\omega_2)$
Quaternion	$\pi / 4.8$	$\pi / 8.2$	0	$\pi / 4.8$	$\pi / 1.3$	$\pi / 1.14$	π	$\pi / 1.3$

XII. DISCUSSION

From the given rotation matrix, there are two possible solutions when Euler angles are defined as the sequence of

rotations, but only one inside the angles ranges is valid. This is because the sequence of rotations to reach the target frame is not unique if the ranges are not defined. In rotation matrices, Euler angles are computed by using Pseudo-code from the given matrix. Quaternion can be employed to obtain the values of quaternion from Euler angles. There has to be more than one gimbal in the automotives because the intrinsic rotation angle cannot be read from a single gimbal. Normally there are at least three for redundancy. Therefore the simple two-axis gimbals set are used to know the actual orientations of moving automotives and then Euler angles are directly measurable.

XIII. CONCLUSION

The values of Euler angles are $[\pi/4, \pi/4, \pi/4]$ radians inside the angles ranges for the simple two-axis gimbals set mounted in the automotives but it can occur the gimbals lock at Euler angles $[\pi/2.24, 0, 0]$ radians. Euler parameters can eliminate the problem of gimbals lock at Mechanical Engineering. If the values of quaternion are at $[\pi/4.8, \pi/8.2, 0, \pi/4.8]$ radians inside the angles ranges from Euler angles, the gimbals lock can be avoided. As the simple two-axis gimbals set keeps the rotation axes constant, angles measurements from the lab frame are the same as angles measurements from the gyro frame. Therefore the main advantage over orientations description is that Euler angles or Euler parameters can control the orientations of these gimbals set from moving automotives.

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