

Heat Transfer in a Parallel-Plate Enclosure with Graded-Index Coatings on its Walls

Jiun-Wei Chen, Chih-Yang Wu, and Ming-Feng Hou

Abstract—A numerical study on the heat transfer in the thermal barrier coatings and the substrates of a parallel-plate enclosure is carried out. Some of the thermal barrier coatings, such as ceramics, are semitransparent and are of interest for high-temperature applications where radiation effects are significant. The radiative transfer equations and the energy equations are solved by using the discrete ordinates method and the finite difference method. Illustrative results are presented for temperature distributions in the coatings and the opaque walls under various heating conditions. The results show that the temperature distribution is more uniform in the interior portion of each coating away from its boundary for the case with a larger average of varying refractive index and a positive gradient of refractive index enhances radiative transfer to the substrates.

Keywords—Radiative transfer, parallel-plate enclosure, coatings, varying refractive index

I. INTRODUCTION

THE thermal barrier coatings (TBCs) are applied to reduce temperatures of metal walls in high temperature applications, where the hot surfaces that view each other can exchange thermal radiation. In most of radiative exchange calculations detailed in textbooks [1], the enclosure surfaces are opaque and each area of the enclosure has single radiating temperature. Only a few attempts [2]-[3] have so far been made at radiative exchange in an enclosure with semitransparent coatings on its walls. Those studies stem from the reasons that some of the TBCs are semitransparent and so the temperature is varying in a coating. The varying temperature is not only determined by surface radiative exchange, but also resulted from coupled conductive and radiative heat transfer in the coatings. The refractive indices of the coatings have a considerable effect on the temperature distribution and radiative heat flux. Surface reflections depend on the refractive indices on both sides of the interface; hence, the refractive indices influence both the transmission of the external energy into the medium and the amount of radiation reflected back into the interior. Within a medium, emission depends on its local refractive index squared. If the refractive index is varying in a

medium, it will cause portion of the radiation initially directing into a region with a smaller refractive index to be refracted backward. Siegel and Spuckler [4] were the first few researchers who analyzed the varying refractive index effects on radiative transfer in a semitransparent medium. Some researchers have subsequently investigated the effects of a varying refractive index on the radiative transfer [5]-[8] and the coupled radiative and conductive heat transfer within a graded index medium [9]-[11] in the last decade. However, no work has been reported so far on the effects of varying refractive indices on radiative exchange in an enclosure with semitransparent coatings on its walls.

In this work, temperature distributions in the semitransparent coatings and the substrates of a parallel-plate enclosure are calculated from radiative transfer equations and energy equations by using the discrete ordinates method and the finite difference method. The temperatures of the coatings and substrate surfaces are high enough to produce significant radiation effects. Each of the coating provides a thermal barrier to reduce temperature in its substrate. Illustrative results are presented for temperature distributions in the semitransparent coatings and the opaque walls. The influence of various heating conditions and varying refractive indices on the temperature distributions is investigated. The results are for gray emitting-absorbing coatings, but extensions to include spectral variations can be carried out as described in [2] and extensions to include scattering can be performed as described in [7].

II. MODELING AND COMPUTATIONAL METHOD

In this work, we consider steady state heat transfer in a parallel-plate enclosure on the inside with thermal barrier coatings. When the coatings are semitransparent, they exchange thermal radiation with one another. The physical model of the heat transfer is illustrated in Fig. 1, where the x-coordinate starts at the inside surface of each coating and extends outward. The energy equations for the semitransparent coatings and the opaque walls can be expressed as

$$k_{cj} \frac{d^2 T_{cj}(x_j)}{dx_j^2} - \frac{dq_j^r}{dx_j} = 0, \quad 0 \leq x_j \leq t_{cj}, \quad j = 1, 2, \quad (1)$$

$$k_{wj} \frac{d^2 T_{wj}(x_j)}{dx_j^2} = 0, \quad t_{cj} \leq x_j \leq t_{cj} + t_{wj}, \quad j = 1, 2, \quad (2)$$

where k denotes the thermal conductivity, T the temperature, the subscripts c and w the coatings and the opaque walls, respectively, the subscripts $j=1$ and $j=2$ the left-hand side

J.-W. Chen is now with the Department of Mechanical Engineering, National Cheng Kung University, Tainan, Taiwan 701, ROC (e-mail: n16941012@ccmail.ncku.edu.tw).

C.-Y. Wu is with the Department of Mechanical Engineering, National Cheng Kung University, Tainan, Taiwan 701, ROC (phone: 886-6-2757575-62151; fax: 886-6-2352973; e-mail: cywu@mail.ncku.edu.tw).

M.-F. Hou is with the Department of Mechanical Engineering, National Cheng Kung University, Tainan, Taiwan 701, ROC (e-mail: n1892116@ccmail.ncku.edu.tw).

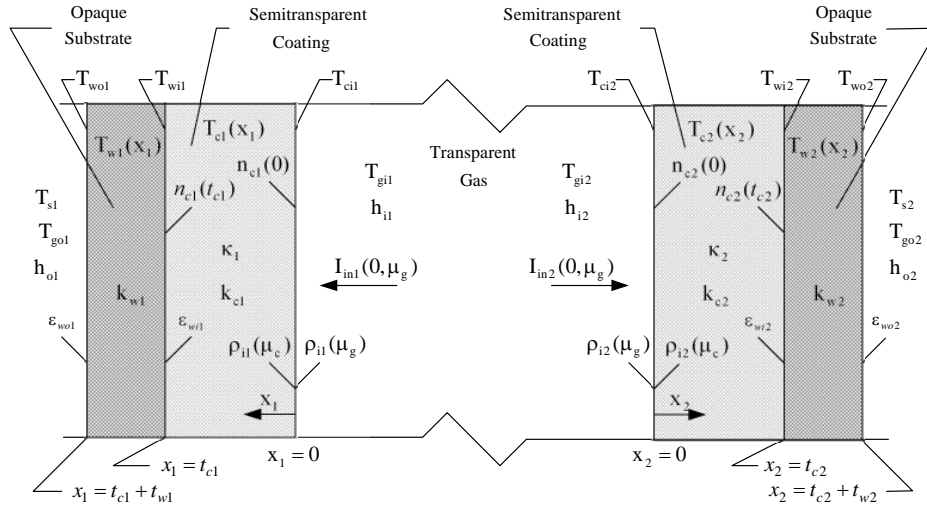


Fig. 1 Parallel-plate enclosure with nomenclature and coordinates

and the right-hand side of the enclosure, respectively, t the thickness of each of the coatings and the opaque walls, and q_j^r the radiative heat flux defined as

$$q_j^r(x_j) = 2\pi \int_{-1}^1 I_{cj}(x_j, \mu_c) \mu_c d\mu_c, \quad j = 1, 2, \quad (3)$$

with I_c denoting the radiative intensity and μ_c denoting the directional cosine of a ray propagating in the current coating. The exterior surfaces of the opaque walls of the enclosure are cooled by convection and radiation from surroundings, while the inside surfaces of the coatings are subject to convection from transparent gas within the enclosure. The boundary conditions of the above energy equations are

$$h_{ij}[T_{gij} - T_{cj}(x_j)] = -k_{cj} \frac{dT_{cj}(x_j)}{dx_j}, \quad x_j = 0, \quad j = 1, 2, \quad (4)$$

$$T_{cj}(x_j) = T_{wj}(x_j), \quad x_j = t_{cj}, \quad j = 1, 2, \quad (5)$$

$$-k_{cj} \frac{dT_{cj}(x_j)}{dx_j} + q_j^r(x_j) = -k_{wj} \frac{dT_{wj}(x_j)}{dx_j}, \quad x_j = t_{cj}, \quad j = 1, 2, \quad (6)$$

$$-k_{wj} \frac{dT_{wj}(x_j)}{dx_j} = \varepsilon_{woj} \sigma [T_{wj}^4(x_j) - T_{soj}^4] + h_{oj} [T_{wj}(x_j) - T_{soj}], \quad x_j = t_{cj} + t_{wj}, \quad j = 1, 2, \quad (7)$$

where h_{ij} and h_{oj} denote the convective heat transfer coefficient of the hot transparent gas inside the enclosure and that of the surroundings gas outside the enclosure, respectively, T_{gij} and T_{soj} the gas temperatures adjacent to boundaries inside enclosure and on outside of enclosure, respectively, T_{sj} the temperature of blackbody surroundings on outside of enclosure, the subscript g the gas inside or outside the enclosure, the subscript s the surroundings, ε_{woj} the emittance of the wall surface exposed to the surroundings and σ the Stefan-Boltzmann constant. The divergence of radiative heat flux,

dq_j^r/dx_j , can be expressed as

$$\frac{dq_j^r(x_j)}{dx_j} = \kappa_j [4n_{cj}^2(x_j) \sigma T_{cj}^4(x_j) - G_j(x_j)], \quad j = 1, 2, \quad (8)$$

where $n_{cj}(x_j) = n_{cj}(0) + x_j[n_{cj}(t_{cj}) - n_{cj}(0)]/t_{cj}$, κ_j denotes the absorption coefficient of each coating, and G_j is the incident radiation defined as

$$G_j(x_j) = 2\pi \int_{-1}^1 I_{cj}(x_j, \mu_c) d\mu_c, \quad j = 1, 2. \quad (9)$$

For simplicity, we use $n_{cj}(x_j) = n_{cj}(0) : n_{cj}(t_{cj})$ to denote the above linear refractive indices and \bar{n}_{cj} to denote their average.

For graded index coatings emitting and absorbing thermal radiation, the radiative transfer equations [6] are written as

$$\mu_c \frac{\partial I_{cj}(\tau_j, \mu_c)}{\partial \tau_j} + \gamma_j(\tau_j) \frac{\partial (1 - \mu_c^2) I_{cj}(\tau_j, \mu_c)}{\partial \mu_c} + I_{cj}(\tau_j, \mu_c) = \frac{n_{cj}^2(\tau_j) \sigma T_{cj}^4(\tau_j)}{\pi}, \quad 0 \leq \tau_j \leq \tau_{cj}, \quad j = 1, 2, \quad (10)$$

where

$$\gamma_j(\tau_j) = \frac{1}{n_{cj}(\tau_j)} \frac{dn_{cj}(\tau_j)}{d\tau_j}, \quad j = 1, 2, \quad (11)$$

with τ_j denoting the optical coordinate of a coating defined as $\tau_j = \kappa_j x_j$. The second term of (10) represents the angular redistribution of radiation caused by the spatially continuous variation of refractive index in each coating.

The boundary conditions of the radiative intensity are

$$I_{cj}^+(0, \mu_c) = n_{cj}^2(0) [1 - \rho_{ij}(\mu_g)] I_{inij}(0, \mu_g) + \rho_{ij}(\mu_c) I_{cj}^-(0, -\mu_c), \quad j = 1, 2, \quad (12)$$

$$I_{cj}^-(\tau_{cj}, -\mu_c) = n_{cj}^2(\tau_{cj}) \frac{\varepsilon_{wij} \sigma T_{wij}^4}{\pi} + 2(1 - \varepsilon_{wij}) \int_0^1 I_{cj}^+(\tau_{cj}, \mu_c) \mu_c d\mu_c, \quad j = 1, 2, \quad (13)$$

where $\rho_{ij}(\mu_g)$ denotes the interface reflectivity for radiation incident on the coating j from the coating on the other side of the enclosure, μ_g denotes the directional cosine of a ray propagating in the gas inside the enclosure, $\rho_{ij}(\mu_c)$ denotes the interface reflectivity for radiation propagating from the coating j toward the interface, τ_{cj} denotes the optical thickness of a coating defined as $\tau_j = \kappa_j t_{cj}$, ε_{wij} and T_{wij} are the emittance and temperature at the interface between the coating and the opaque wall denoted by j , respectively. The ρ_{ij} can be determined by the Fresnel relation [12]. The change of radiation propagation direction caused by refraction obeys Snell's law [12]. By summing all contributions of specular reflections on the interfaces between the coating and the gas inside the enclosure $I_{inj}(0, \mu_g)$ in (12) may be derived as

$$I_{in1}(0, \mu_g) = \frac{1}{1 - \rho_{i1}(\mu_g)\rho_{i2}(\mu_g)} \left\{ \frac{1}{n_{c2}^2(0)} [1 - \rho_{i2}(\mu_c)] I_{c2}^-(0, -\mu_c) + \frac{\rho_{i2}(\mu_g)}{n_{c1}^2(0)} [1 - \rho_{i1}(\mu_c)] I_{c1}^-(0, -\mu_c) \right\}, \quad (14)$$

$$I_{in2}(0, \mu_g) = \frac{1}{1 - \rho_{i1}(\mu_g)\rho_{i2}(\mu_g)} \times \left\{ \frac{\rho_{i1}(\mu_g)}{n_{c2}^2(0)} [1 - \rho_{i2}(\mu_c)] I_{c2}^-(0, -\mu_c) + \frac{1}{n_{c1}^2(0)} [1 - \rho_{i1}(\mu_c)] I_{c1}^-(0, -\mu_c) \right\}. \quad (15)$$

The system of equations to be solved is (1), (2) and (10) for the six unknowns, I_{cj} , T_{cj} and T_{wj} . To simplify the analysis, we assume that the properties are constant, except the refractive indices of the coatings. An iterative procedure is adopted to estimate the temperature distributions and then solve the radiative intensities. Employing the discrete ordinates method, we can solve the radiative part, (10)-(15). The numerical solution procedure is based on a discrete representation for the angular variations of the radiative intensities and a finite difference approximation for the spatial derivatives of the intensities. Since the derivation of the discrete ordinates equations has been presented [6]-[7], only a brief description of the derivation is given here. The radiative intensities I_{cj} is calculated for a set of M different directional cosines μ_m with an equally spaced distribution of μ over $[-1, 1]$ and the constant quadrature weights $w_m = 2/M$, $m = 1, 2, \dots, M$. Each of the coatings is divided into a number of slices of equal width. Using the numerical technique presented by Lemonnier and Le Dez [6], we get the finite difference form of angular redistribution term. The solutions of the discrete ordinates equations are performed iteratively by sweeping through each direction. The direction of integration is in accord with the direction of propagation of the radiation ray, with iterative recalculation of the boundary conditions and G_j , which depend

on the intensity distribution. The iterative procedure is continued until $|(G^{current} - G^{previous}) / \sigma T_r^4| \leq 10^{-7}$ at each grid.

Here, we set $T_r = T_{gi2}$. The divergences of radiative heat fluxes are then evaluated and the energy equations are solved to obtain improved temperature distributions with which to continue the iteration. The temperature and radiation distributions are iterated until $|(T^{current} - T^{previous}) / T_r| \leq 10^{-7}$ at each grid.

III. RESULTS AND DISCUSSION

The heat transfer behavior in the thermal barrier coatings and the substrates of a parallel-plate enclosure is determined by many parameters, such as optical thicknesses, convection coefficients, thermal conductivities and varying refractive indices. Since the influence of varying refractive indices on the heat transfer has not been investigated, we would like to lay some emphasis on the effect of varying refractive indices on the heat transfer in the coatings and the substrates of the enclosure. Temperature distributions of the coatings and the substrates are evaluated for various heating and cooling conditions; the temperatures are high enough and conduction-radiation parameters, $N_{cj} = k_{cj} / (t_{cj} \sigma T_r^3)$ and $N_{wj} = k_{wj} / (t_{wj} \sigma T_r^3)$, are selected to provide significant radiation effects. The ratio of thermal conductivity and the geometrical thickness in the substrate is 10 times larger than in the coating that is more of an insulating material.

The results shown in Figs. 2-5 are the dimensionless temperatures, $\Theta_{wj} = T_{wj} / T_r$ and $\Theta_{cj} = T_{cj} / T_r$, of the substrates and their coatings that each is being heated at its exposed surface by convection of a hot transparent gas in the enclosure. Each coating provides a thermal barrier to reduce temperatures in the opaque substrate, which is cooled on the outside by convection and radiation. For comparison purpose, we first apply the codes developed for the general cases with varying refractive indices to the special cases with constant refractive indices and various heating and cooling boundary conditions. The dimensionless temperatures obtained by the present codes are shown in Fig. 2 and are in good agreement with those obtained by a combined analytical two-flux and numerical iteration method [3]. The conduction, radiation and convection parameters for Fig. 2 are given in the figure captions. Besides, the dash lines show the temperatures for the cases with opaque coatings and the temperatures for the cases with semitransparent coatings are shown as solid lines.

Temperature distributions for the cases with a fixed value of $n_{cj}(t_{cj}) - n_{cj}(0)$ and various values of \bar{n}_{cj} are shown in Fig. 3. A general trend can be found from Fig. 3 is that for an increased \bar{n}_{cj} the temperature distributions are more uniform in the interior portion of each coating away from its boundary. This is probably the result of increased reflections at the exposed surfaces. The curves shown in Fig. 3(a) are for the cases with symmetric heating and cooling boundary conditions. The 'S-shaped' curves of the temperatures in semitransparent coatings

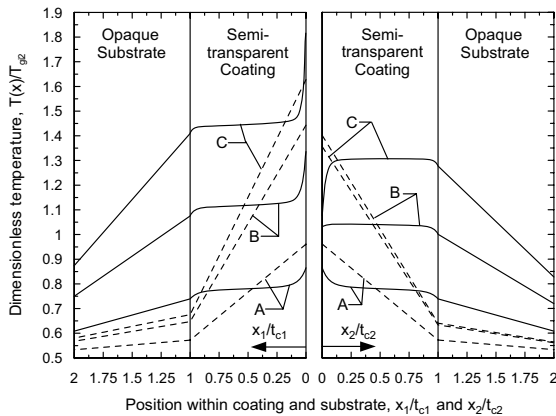
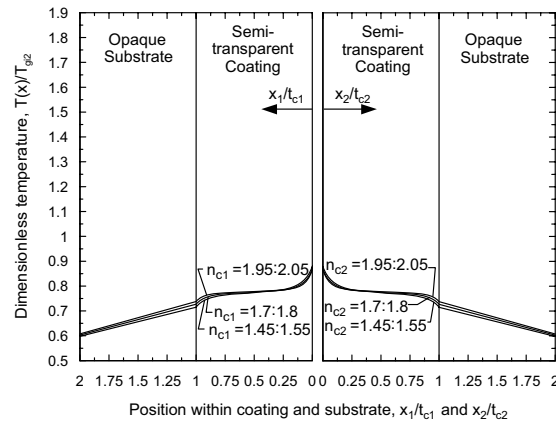


Fig. 2 Temperature distributions for the cases with $\tau_{cj} = 1$, $\Theta_{gi2} = 1$, $\Theta_{goj} = 0.5$, $H_{oj} = 1$, $N_{cj} = 0.1$, $N_{wj} = 1.0$, $\varepsilon_{wij} = \varepsilon_{woj} = 0.3$ and $n_{cj}(x_j) = 2$; case A: $\Theta_{gi1} = 1$, $H_{ij} = 1$, case B: $\Theta_{gi1} = 2$, $H_{ij} = 1$, case C: $\Theta_{gi1} = 2$, $H_{ij} = 10$

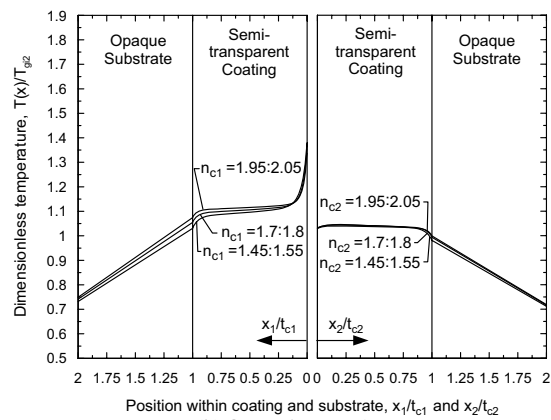
are typical of a planar medium with heat transfer in one direction by combined conduction and radiation, while the steady-state heat conduction in an opaque wall generates linear temperature distribution [3]. For the curves shown in Fig. 3(b), the temperature of gas convectively heating coating 1 is increased to twice that of gas close to coating 2. The temperature distribution in coating 1 still is 'S-shaped' as energy is flowing from the surface at $x_1/t_{c1} = 0$, through the coating and substrate to the cooled exterior boundary at $x_1/t_{c1} = 2$. The radiative heat transfers from coating 1 to coating 2 and is absorbed by coating 2. The interior temperature of coating 2 is raised above the gas temperature at $x_2/t_{c2} = 0$. Thus, heat now transfers from coating 2 to the gas by convection and the maximum temperature appears inside the coating. For the curves shown in Fig. 3(c), the convective heat transfer coefficients at $x_j/t_{cj} = 0$ are increased by a factor of 10 while the gas temperatures on both inside surfaces are kept the same as those assigned for the cases shown in Fig. 3(b). Now, the temperature distributions shown in Fig. 3(c) are similar to those shown in Fig. 3(b), but the temperatures rises higher and the gradients of the temperatures around the interfaces increases.

From the Fresnel relation, one may find both $\rho_{ij}(\mu_g)$ and $\rho_{ij}(\mu_c)$ increase with the increase of $n_{cj}(0)$; however, the increase of the former is much less than that of the latter. For the cases shown in Figs. 3(a)-3(c), the overall emissive power of coating 1 increases with the increase of \bar{n}_{c1} , and so the radiating power to the cooler coating 2 also increases. However, this effect is offset by the interface reflection of radiation propagating in the coating toward the interface that also increases with the increase of \bar{n}_{c1} corresponding to the increase of $n_{c1}(0)$. Besides, all reflections between the interfaces of the coating and the gas inside the enclosure also have contributions to the radiative heat transfer between the coatings. Therefore,

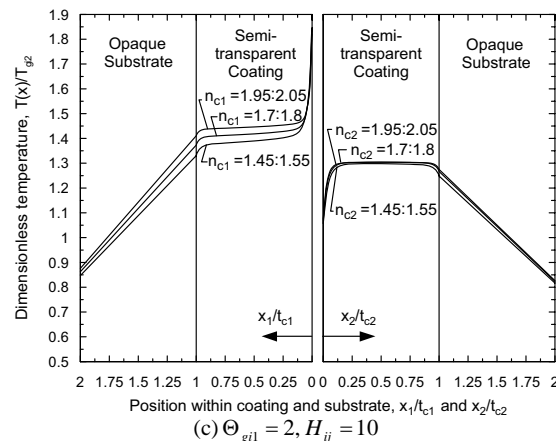
the overall temperature distribution in coating 2 only increases a little bit with the increase of \bar{n}_{c1} .



(a) $\Theta_{gi1} = 1$, $H_{ij} = 1$



(b) $\Theta_{gi1} = 2$, $H_{ij} = 1$



(c) $\Theta_{gi1} = 2$, $H_{ij} = 10$

Fig. 3 Effects of \bar{n}_{cj} on temperature distributions for the cases with

$$\Theta_{gi2} = 1, \Theta_{goj} = 0.5, H_{oj} = 1, N_{cj} = 0.1, N_{wj} = 1.0, \varepsilon_{wij} = \varepsilon_{woj} = 0.3, \tau_{cj} = 1 \text{ and } n_{cj}(t_{cj}) - n_{cj}(0) = 0.1$$

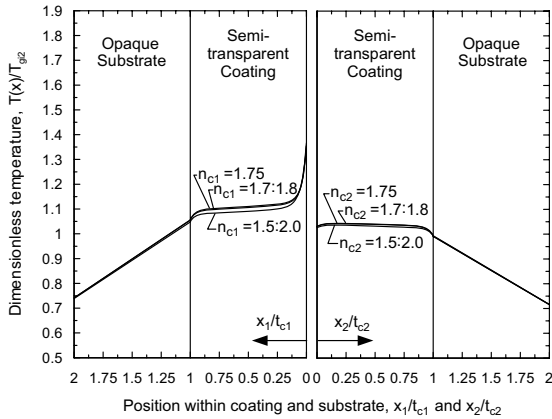


Fig. 4 Effects of refractive index gradients on temperature distributions for the cases with $\Theta_{goj} = 0.5$, $\Theta_{gil} = 2$, $\Theta_{gi2} = 1$, $H_{oj} = H_{ij} = 1$, $N_{cj} = 0.1$, $N_{wj} = 1.0$, $\varepsilon_{wij} = \varepsilon_{woj} = 0.3$, $\tau_{cj} = 1$ and $\bar{n}_{cj} = 1.75$

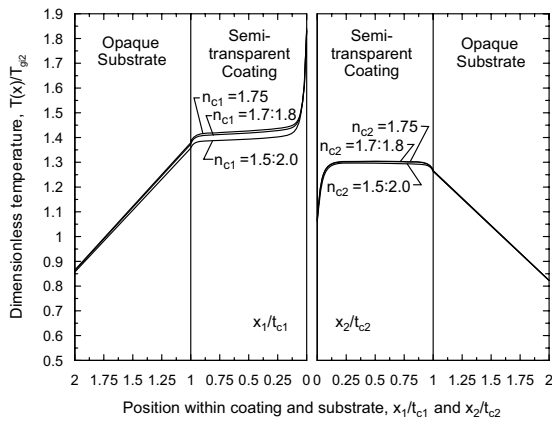


Fig. 5 Effects of refractive index gradients on temperature distributions for the cases with $\Theta_{goj} = 0.5$, $\Theta_{gil} = 2$, $\Theta_{gi2} = 1$, $H_{ij} = 10$, $H_{oj} = 1$, $N_{cj} = 0.1$, $N_{wj} = 1.0$, $\varepsilon_{wij} = \varepsilon_{woj} = 0.3$, $\tau_{cj} = 1$ and $\bar{n}_{cj} = 1.75$

The effects of refractive index gradients (n'_{cj}) on temperature distributions for the cases with a fixed value of \bar{n}_{cj} are shown in Figs. 4 and 5. The internal reflection due to refraction in the present cases with a positive n'_{cj} makes a part of backward radiation turn to forward direction, and a positive n'_{cj} makes the direction of a forward radiation stream tends to positive x_j direction and shortens the path length and reduces the attenuation of the intensity in a forward direction. These effects enhance forward radiative transfer, and increase with the increase of n'_{cj} . For the cases shown in Figs. 4 and 5, the portion of coating 1 around the surface at $x_1/t_{c1} = 0$ heated by the gas inside the enclosure has highest temperature and is radiating to the cooler portion of coating 1 and coating 2. A positive n'_{cj} enhances radiative transfer to the substrate, and so decreases the

temperature in the portion of coating 1 away from the exposed surface. Meanwhile, the convective heat transfer by the gas causes the steep temperature distributions in the portion of coatings around the exposed surfaces. Thus, for a case with a smaller n'_{cj} , the temperature distribution is more uniform in the interior portion of each coating away from its boundary. Besides, the comparison of Figs. 4 and 5 shows that the effects of refractive index gradients on temperature distributions are more evident in the cases with higher temperatures.

IV. CONCLUDING REMARKS

In this work, we investigate the influence of varying refractive indices on the heat transfer in the TBCs and the substrates of a parallel-plate enclosure under various heating conditions. The following may be observed from the temperature distributions in the semitransparent coatings and the substrates calculated from radiative transfer equations and energy equations. (i) For the case with larger \bar{n}_{cj} the temperature distribution is more uniform in the interior portion of each coating away from its boundary. (ii) A positive n'_{cj} enhances radiative transfer to the substrate, and so decreases the temperature in the portion of coating 1 away from the exposed surface. (iii) The effects of refractive index gradients on temperature distributions are more evident in the cases with higher temperatures.

ACKNOWLEDGMENT

This work is supported by the National Science Council of the Republic of China on Taiwan through Grant NSC 99-2221-E-006-080-MY2.

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