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Easy-Interactive Ordering of the Pareto Optimal Set with Imprecise Weights

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***Abstract*—**In the multi objective optimization, in the case when generated set of Pareto optimal solutions is large, occurs the problem to select of the best solution from this set. In this paper is suggested a method to order of Pareto set. Ordering the Pareto optimal set carried out in conformity with the introduced distance function between each solution and selected reference point, where the reference point may be adjusted to represent the preferences of a decision making agent. Preference information about objective weights from a decision maker may be expressed imprecisely. The developed elicitation procedure provides an opportunity to obtain surrogate numerical weights for the objectives, and thus, to manage impreciseness of preference. The proposed method is a scalable to many objectives and can be used independently or as complementary to the various visualization techniques in the multidimensional case.

***Keywords*—**Imprecise weights, Multiple objectives, Pareto optimality, Visualization.

I. Introduction

In multi-criteria/multi-objective decision making, the so-called Pareto set of non-dominated solutions can be huge and includes solutions with quite different properties even though they are all deemed as efficient. In order to make intelligent selections from a Pareto set, may become necessary to attract more information about the preferences of the entity making the selection. In such situations, there are essentially two ways to proceed when aiming to identify some ordering of a large Pareto optimal set and based upon that order aid a decision making agent in its selection of one or some alternatives for further review; 1) find a preferred subset based on some pre-defined aggregation rule [1]-[6]or 2), use some visualization technique in case the decision maker is human [7]-[10].

With respect to the first category, the concept "order of efficiency" is introduced in [1], allowing for setting up a preference ordering amongst Pareto optimal solutions. This concept of efficiency of order *k* provides a rigidly-defined means of branding some Pareto optimal solutions as being superior or more desirable than others, regardless of the specifics of the problem. In [2] is suggested the method to identifying preferred subsets of Pareto optimal solutions based on threshold values for each objective function. These preferred subsets can be obtained using by the some heuristics.

In [3] a method to obtaining smart Pareto sets for problems in the multidimensional case is developed, while in [4] a multi-attribute utility theory based approach for large solution sets is advocated for by introducing "almost-Pareto-sets" as a consequence of imprecisely specified trade-off coefficients and thereby creating an ordering by stepwise eliminating options from the set using an adapted dominance rule. In [5] was presented the algorithm for post-optimality analysis of Pareto optimal set based on maximizing a scalarizing function.

The method proposed in [6] presupposed the use of clustering technique for Pareto optimal set in order to visual presentation of clusters to a decision-maker instead numerical presentation a wide scattered Pareto set. Such representation of Pareto optimal set allows reducing the cognitive load on the decision-maker.

With respect to the second category, the visualization techniques are primarily developed in order to aid a human decision maker to understand the nature of the Pareto optimal set and identify more desirous solutions by means of subjective judgments, cf., e.g. [7]-[10]. To use of visualization technique is required decision maker's ability to understanding the given visualization technique. Advantages and disadvantages of various visualization techniques have been deliberated in [7]. In [8] is discussed visualizing techniques based on multivariate statistics visualization methods. In [9] is developed a user-friendly interactive approach to support non-sophisticated users. Their established approach is based on interactive Pareto frontier visualization taking into account preferences information and using an arbitration method. Since there are complications for effective visualization of four- or high- dimensional optimization problem, in [10] was proposed a method of visualization based on the choice of vertices of a regular tetrahedron as the basic points and mapping of the spatial coordinates of Pareto-optimal front through the space vector balance.

For two objectives, one technique is the use of a so-called scatterplot matrix showing the Pareto optimal solutions and the inherent trade-offs between two objectives at a time. In the case of three objectives, decision maps can be applied to represent trade-offs between three objectives simultaneously. In both of these cases, the number of relative comparison of objectives increases with increasing of the number of objectives. The case of more than three objectives can be explored by combining sub-problems by means of partly visualizations of two or three objectives.

Advantages and disadvantages of visualization techniques in the case of three objectives in a decision making situation

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have been discussed in [7]. However, the requirements for an effective utilization of visualization techniques, such as triangle factors, simplicity, persistence and completeness, are often not achievable in the case of more than three objectives [7].

It's not always straightforward for a decision maker to use his/her judgment on visually presented Pareto set. For instance, the trade-off surface can be visualized, but how to use the surface plot to select a final solution is not obvious to the user.

II. Multiobjective Optimization Problem

The multi-objective optimization problem has the following general form

$$\min_{x\in X}\{f_1(x),\dots,f_k(x)\} \quad (1)$$

subject to $x \in X \subset R^n$, where objective functions $f_i : X \to R$, the decision vector x belong to the feasible set X in the decision space R^n, a criterion vector $z = f(x)=(f_1(x), \dots , f_k(x))^T$ belongs to the criterion space R^k for all $x \in X$ and the feasible criterion set $Z = f(X) \subset R^k$.

A criterion vector $z^* = f(x)$, $x\in X$is said to dominate another vector $f(y)$, $y\in X$, if $f_i(x) \leq f_i(y)$ for all $i= 1,\dots,k$, and $f_i(x) < f_i(y)$ for at least one i. The criterion vector z^*called Pareto optimal or nondominated if there does not exit a criterion vector $z \in Z$ that dominates z^*. The set of nondominated solutions is called the Pareto optimal set and denoted by P. If $x\in X$, then $z=f(x) \in Z$ and the value of the ith objective of solution x is $z_i=f_i(x)$. We denote $z_j^+ = \max_{f(x)\in P} f_j(x)$ and $z_j^- = \min_{f(x)\in P} f_j(x)$, the maximum and minimum values of the jth objective for all $f(x) \in P$, and $j \in \{1,\dots,k\}$. Besides this information the elements of P are not arranged into any order.

Algorithms for solving multi-objective optimization problems usually divided into three categories: a priori, interactive and a posteriori/generating methods [11], [12], [6]. A priori and interactive algorithms imply involvement of a decision maker or group of decision makers in the searching of solutions before or during the optimization process, whereas a posteriori algorithms do not require the presence of additional information from a decision maker before or during the optimization process. A posteriori methods involve the generation of the Pareto optimal solutions and the presentation the sets of solutions to a decision maker for future analyzing. To presentation of Pareto optimal set is used visualizing techniques. We assume that we have a Pareto set that is generated with help of some a posteriori method. Commonly used a posteriori methods are described in [12], [11].Due to this, the method for ordering of found Pareto optimal set which presented in this paper allows the use each generating algorithm in an interactive manner.

III. The Ordering of the Pareto Optimal Set

In order to obtain a reasonable ordering for the elements of P set, we propose the use of a distance function between each solution and a selected reference point. As the distance function, we promote to use the Chebyshev distance such that for any two points belonging to a compact space$F \subset \mathbb{R}^n$, x, $y\in F$ with coordinates $x_i$and y_i, respectively, the Chebyshev distance is given by

$$d_\infty(x,y) = \max_i |x_i - y_i| \quad (2)$$

The Chebyshev distance is the maximum distance between the coordinates in any single dimension.

Thus, the Chebyshev distance indicates how two points are distinguished based upon the dimension in which they differ the most. The reason for using the Chebyshev distance is that the difference between the points is reflected more by differences in single dimensions rather than all dimensions considered together.

However, to allow for comparisons of objective functions between each other, the distance must be normalized.

We consider a normalized Chebyshev distance for two points x, $y\in F$ with coordinates $x_i$and y_i, respectively.

Definition 1 The normalized Chebyshev distance is

$$d_\infty^N(x,y) = \max_i \begin{cases} \frac{|x_i-y_i|}{u_i-l_i} \; if\ u_i \neq l_i \\ 0,\ if\ u_i = l_i \end{cases} \quad (3)$$

where $u_i = \max_{x\in F} x_i$ and $l_i = \min_{x\in F} x_i$.

Proposition 1.The normalized Chebyshev distance defined in definition 1 is a distance function on a given set F.

In conformity with the aforesaid, the rule for the ordering Pareto optimal set defined by sorting of the set in ascending order of the normalized Chebyshev distance between each solution and selected reference point. The reference point can be a middle point in the neutral preference case or the weighted middle point if preference information is available. Then, the reference point will be interpreted as a point that is neutral preference point or most desirable point.

A. Reference Point without Preference Information

In the case when preference information is lacking, we suggest to choose a middle point of a Pareto optimal set as a reference point. In this case, the choice of the middle point based on the reasoning that the finally selected alternative should be in the middle of the Pareto set [1], [13], [14].

We define the middle point of Pareto optimal set as $s = (s_1,\dots,s_k)$, where

$$s_i = \frac{1}{2}(z_i^+ + z_i^-) \text{ for all } i \in \{1,\dots,k\} \quad (4)$$

The middle point then symbolizes “the most central point” of the Pareto optimal set.

The middle point thus shows the same significance for all objective functions, and it represents a reasonable compromise between all the objectives since it is farthest from the all extreme points of the Pareto set [15].

The use of the middle point presupposes that preference information is unavailable, and the solutions that are nearest to the middle point conform more to a situation where we have equal importance in relation to all the objectives, i.e., that good performance on one objective is not preferred over any other objective. In accordance with this it is natural to search for the solution that is nearest to the middle point when suggesting a particular solution from the set *P*.

B. Reference Point with Preference Information

In the case when preference information is available, we suggest to choose a weighted middle point of a Pareto optimal set as a reference point. The relative importance of each objective is reflected in the weight of the objective function and, by convention, the sum of all weights equal one. In this case, the choice of the weighted middle point based on the reasoning that the weighted middle point relocated from the middle point in conformity with preference information. Denote by $w_i$is the weight of the *i*:th objective function $f_i(x)$, where $w_i \in [0,1] \forall i$ and $\sum_{i=1}^{k} w_i = 1$. Then we define a weighted middle point as $s_w = (s_{w1},\ldots,s_{wk})$, where

$$s_{wj} = \begin{cases} \frac{k}{2}\left(z_j^- - z_j^+\right) w_j + z_j^+, \\ \qquad if\ w_j \in \left[0, \frac{1}{k}\right] \\ \frac{k\left(z_j^+ - z_j^-\right)}{2(1-k)} w_j + \frac{(2-k)z_j^-}{2(1-k)} - \frac{k z_j^+}{2(1-k)}, \\ \qquad if\ \ w_j \in \left[\frac{1}{k}, 1\right] \end{cases} \tag{5}$$

Proposition 2.The weighted middle point coincides with the middle point in the case equal weights for all *j*.

Proof. Result follows directly by setting $w_j = \frac{1}{k}$ for all *j*.

However, when capturing user preference by using a weighted middle point, it is required that the decision makers can perform a precise determination of the weighting coefficients. In decision analysis this is considered as a cognitive demanding task [16] which is not desired for the application domain of concern for the approach suggested in this paper. Further, information about weighting coefficients is usually expressed in imprecise fashion since trade-offs is difficult to assess to any degree of precision.

C. Imprecise Information and Surrogate Weights

Several different techniques are developed, suggesting obtaining surrogate and conservative numerical weights given a decision maker's preferences when these are stipulated in imprecise fashion providing imprecise information.

A determination of weights enables to employ a weighted middle point representing a decision maker's preferences, and a few different techniques are developed suggesting obtaining surrogate and conservative numerical weights given a decision maker's preferences when these are stipulated in less-than-precise fashion. The widely used AHP method [17] takes advantage of pairwise comparisons between objectives such that the decision maker is asked for ratio statements (selected from a pre-defined scale) between pairs of objectives (given in a matrix) in terms of their relative importance. Surrogate weights are then derived from the matrix eigenvector such that the weights add up to one.

As for ranking statements, it is assumed that a decision maker can at least assess that one objective is more important than another, indicating that for any two objectives statements such as $w_i > w_j$ can be elicited from the decision maker. In the case of neutral preference, weights are equal such that $\frac{1}{k}$. A determination of weights enables to vary weighted middle point according with a decision maker's preference.

The Rank Order Centroid (ROC) procedure is a purely rank based elicitation procedure for weights where the ranking of objectives is determined by a decision maker [18]. This method has been shown to provide efficacious surrogate weights with limited cognitive demands on the decision maker [19]. The method transforms a ranking of objectives into surrogate weights given as the centroid (centre of mass) of the polytope spanned by the constraints $w_i > w_j$and $w_i \geq 0$ for all *i*and $\sum w_i = 1$. This procedure can be applied directly to obtain exact weights in case if the importance ranking of objectives is known.

In the considered case, it is possible to extend the ROC procedure. Preferential uncertainties can be represented by desirable values and ordinal information on importance and/or cardinal relation information. Information about the ranking of objectives may be combined with information regarding the value of the objective function that is desired by a decision maker. The suggested procedure consists of followings steps: a choice of desirable values for one or more objective functions, a determination of the sequence priorities of objective functions, calculation of weights relating to these values according to the formula (6), and finally calculation all weights according to the formula (7).

Step 1:
Identify desirable values of *i*:th objective function s_{wi} such as $s_{wi} \in [z_i^{min}, z_i^{max}]$, $i \in \boldsymbol{I} \subset \{1, \ldots, k\}$

Step 2:
Determine the sequence priorities of objective functions:

$$f_{i_j}(x) \succcurlyeq f_{i_{j+1}}(x)$$

Step 3:
Calculate $w_i$as

$$w_i = \begin{cases} \frac{s_{wi} - z_i^+}{\frac{k}{2}\left(z_i^- - z_i^+\right)}, \\ \qquad if\ s_{wi} \in [\frac{1}{2}(z_i^- + z_i^+), z_i^+] \\ \frac{2(1-k)s_{wi} - (2-k)z_i^- + k z_i^+}{k\left(z_i^+ - z_i^-\right)}, \\ \qquad if\ s_{wi} \in [z_i^-, \frac{1}{2}(z_i^- + z_i^+)] \end{cases} \tag{6}$$

Step 4:
If $\sum_{i \in I} w_i$>=1 go to step 1 and select other desirable values otherwise calculate remaining weights according formula

$$w_j = \left(1 - \sum_{i \in I} w_i\right) \left(\frac{1}{l} \sum_{t=j}^{l} \frac{1}{t}\right) \tag{7}$$

where l=dim($\{1,\ldots,k\}\backslash \boldsymbol{I}$).

Proposition 3 The sum of weights defined by formulas (6)-(7) is equal to 1, i.e. $\sum_{i=1}^{k} w_i = 1$.

Proof. This follows directly from the formulas (5) and (7).

With this procedure we can calculate weights based on the preference statements from decision maker: a ranking of priorities of objective functions and determination of desirable values. Note that the considering of origin as reference point may be acceptable in some case. Then the ordering of the set occurs in accordance increasing distance from the origin to some maximum remote objective. However, it is difficult to speak about compromise between all objectives, because it is not clear how are located other objectives in relation this maximal remote objective, how significant is the difference between other objectives.

IV. Example

The suggested method has been demonstrated for three objectives in the case with predetermined weights in [20]. In this work a mixed integer multi-objective optimization method for intelligent matching of goods with freight transports in intermodal logistics was developed, so that a buyer of transport services is delivered a set of feasible and existing transport alternatives and where this buyer is interested in minimizing cost, time, and emissions.

In this case, the preferences of the decision maker might not be known and must be roughly obtained rapidly. In other words, any interaction with the method should be easy to adopt with a low effort by the user of the method, and, preferably, the interaction should be able to be bypassed if interaction is not desired by the user.

In order to demonstrate the suggested approach we consider a problem with four objectives f_1, f_2, f_3, f_4 and a Pareto optimal set consisting of eight solutions. A scatter plot matrix enables us to display the relocation of weighted middle point in relation to the middle point of Pareto optimal set, and thus, it makes sense use it in this case.

TABLE I
Values of Alternatives A-H

	f_1	f_2	f_3	f_4
A	11147	170,5	4000	0,0011
B	11193	224,5	3000	0,00095
C	11238	218,5	3200	0,00099
D	11243	172,5	3800	0,0012
E	11282	164,5	4000	0,00093
F	11338	218,5	2800	0,00089
G	11378	166,5	3800	0,00091
H	12018	164,5	4200	0,00096

Location of alternatives A-H displayed on scatterplot matrix Fig. 1. The middle point calculated according to formula (5) is s= (11582,45; 194,5; 3500; 0,001045).In this case, it is not obvious which alternative is nearest to the middle point. The ordering in conformity with the normalized Chebyshev distance between each solution and the middle point s should be following.

TABLE II
The Ordering for Alternatives A-H in the Case of Neutral Preference

Alternatives	The normalized Chebyshev distance between each solution and the middle point
C	0,4
G	0,47
A	0,5
B	0,5
D	0,5
E	0,5
F	0,5
H	0,5

This ordering envisages the neutral preference of a decision maker and, accordingly, the alternative C can be considered as the most preferred.

With the intention of demonstration how imprecise information can be handled, information on a desirable value of an objective function and the sequence priorities of objective functions should be elicited. A decision maker can express for one or more desirable values and ordinal information on importance for each objective function. Then, the above described procedure can be applied.

Step 1:
Let the value of objective function f_2 is desired 170,5.

Step 2:
Let the sequence of priorities be $f_2 \succcurlyeq f_1 \succcurlyeq f_3 \succcurlyeq f_4$.

Step 3:
According to formula (6), the second weight is w_2 =0,85.

Step 4:
The other weights calculated on the formula (7) are w_1=0,092, w_3=0,042, w_4 =0,016.

The weighted middle point relating to predetermined sequence of priorities and the desirable value 170,5 calculated on the formula (5) is s_w= (11858,24, 170,5, 4083,33, 0,00119). The weighted middle point s_w is shown in the 4-objectives scatter plot matrix in Fig. 1.

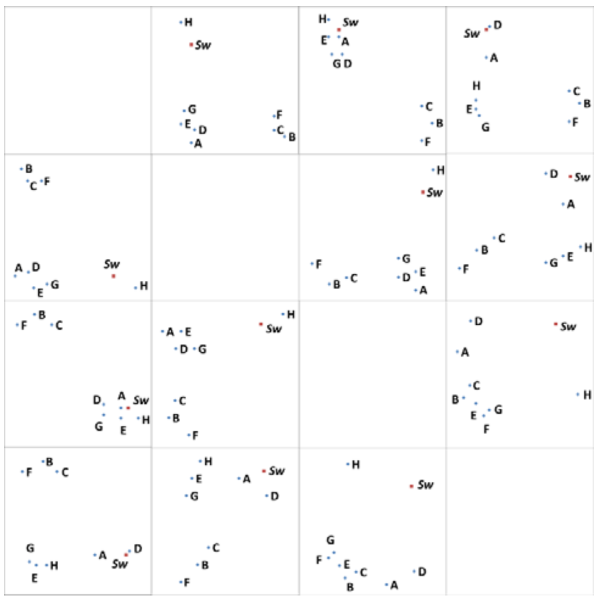

Fig. 1 Scatterplot matrix for four objectives f_1, f_2, f_3, f_4

The weighted middle point s_w relocated from the middle point s in relation with information on the desirable value 170.5 and predetermined sequence of priorities. In this case, pairwise trade-offs between two objectives in relation of the weighted middle point is shown on scatterplot matrix, and for final choice is required to consideration of twelve scatterplot. With ordering of alternatives according the defined normalized Chebyshev distance between the weighted middle point and each alternative is easily visible location of each alternative relative the weighted middle point in the four-dimensional case.

The ordering according to the preferences of the decision maker will be according to Table III.

TABLE III
THE ORDERING FOR ALTERNATIVES A-H IN THE CASE OF INDICATED PREFERENCE

Alternatives	The normalized Chebyshev distance between each solution and the weighted middle point
D	0,706436
H	0,74086
C	0,8
A	0,816667
E	0,837634
B	0,9
G	0,902151
F	0,966667

Thus, in the case if the desirable value of second objective function equals 170.5 and the sequence of priorities is determined as $f_2 \succcurlyeq f_1 \succcurlyeq f_3 \succcurlyeq f_4$, the alternative D (11243;172,5;3800;0,0012) is most promising.

In this example, it can see that it's straightforward method, which includes the expression preference about the desirable value of some objective value, putting the sequence of priorities or determining that other objective functions have neutral priorities, after calculating the weights and making final decision. A decision maker does not have to posit exact weight of each objective function.

Imprecise information from a decision maker in the form of desirable value and sequence of priorities can be taken into account; this additional information can be converted through proposed procedure into surrogate numerical weights, and the obtained result will be based on the decision maker's preference.

V. CONCLUSION

In this article, we suggest an approach that enables to combine described above two ways to identifying of final choice of one or some alternatives but where the interaction can be omitted or kept a low level of complexity from a decision maker's perspective as the method provide reasonable results given numerically imprecise decision maker statements.

This paper has presented the method to identify the ordering of Pareto optimal set based on the preference information from a decision maker, where the relative importance of objective functions can be expressed precise and/or imprecise. The ordering of Pareto optimal set is based on employing the introduced distance function between each point of the Pareto set and a selected reference point. The imprecise expression implies using preference statements in the form of ranking of priorities of objective functions and determination of desirable values. The method is considered as an easy-interactive method since; 1) a reference point may be suggested without interaction, and 2) the cognitive demand put on the decision maker in a desired interaction may be kept at a low level through the use of the rank based approach. The method is also scalable to decision problems with many objectives. In conclusion, the suggested method can be used independently or complementary to the different visualization techniques in the multi-dimensional case.

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