

N-Sun Decomposition of Complete, Complete Bipartite and Some Harary Graphs

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Abstract—Graph decompositions are vital in the study of combinatorial design theory. A decomposition of a graph G is a partition of its edge set. An n -sun graph is a cycle C_n with an edge terminating in a vertex of degree one attached to each vertex. In this paper, we define n -sun decomposition of some even order graphs with a perfect matching. We have proved that the complete graph K_{2n} , complete bipartite graph $K_{2n, 2n}$ and the Harary graph $H_{4, 2n}$ have n -sun decompositions. A labeling scheme is used to construct the n -suns.

Keywords—Decomposition, Hamilton cycle, n -sun graph, perfect matching, spanning tree.

I. INTRODUCTION

BY a graph $G = (V, E)$ we mean a simple undirected connected graph. A cycle of length n in G is denoted by C_n . An n -sun graph is a cycle C_n with an edge terminating from each vertex of C_n [1]. Thus every n -sun graph contains exactly one cycle of length n and n pendant vertices. A decomposition of a graph is a collection of edge-disjoint subgraphs G_1, G_2, \dots, G_n of G such that every edge of G belongs to exactly one G_i . Graph decompositions, known for its applications in combinatorial design theory, have been studied since the mid nineteenth century. Several decades after its introduction, Walecki had the credit of constructing Hamilton cycle decomposition of complete graphs [2]-[4]. In this paper we have decomposed complete graphs of even order, K_{2n} into n -suns. The decomposition is based on Walecki's construction of Hamilton cycles in complete graphs. A systematic approach to the decomposition with a labeling scheme is provided. By an orderly removal of edges from the cycles in n -suns, we have shown a spanning tree decomposition of K_{2n} . Every such spanning tree has the specialty of containing a perfect matching of K_{2n} . Complete bipartite graphs $K_{2n, 2n}$ are equally significant. Their n -sun decompositions are also given. The next type of graphs considered are the k -connected, $2n$ -vertex graphs having the smallest possible number of edges, called the Harary graphs. These graphs are widely used in interconnection network topology. For particular type of Harary graphs n -sun decomposition is studied.

II. PRELIMINARIES

A graph G in which any two distinct points are adjacent is called a *complete graph*, K_n . A complete bipartite graph $K_{m, n}$ is a graph whose vertices can be partitioned into two sets U and W such that every edge in $K_{m, n}$ has one end in U and the other end in W . Frank Harary constructed a class of graphs $H_{k, n}$, called the Harary graphs, beginning with an n -cycle graph whose vertices are consecutively numbered $0, 1, \dots, n-1$ clockwise around its perimeter. If k and n are even, form $H_{k, n}$ by joining each vertex to the nearest $k/2$ vertices in both directions around the circle. If k is odd and n is even, form $H_{k, n}$ by joining each vertex to the nearest $(k-1)/2$ vertices in each direction and to the diametrically opposite vertex. In both the cases $H_{k, n}$ is k -regular, k -connected n -vertex graph. We exclude the case of odd n in $H_{k, n}$ since n -sun is defined only for even order graphs.

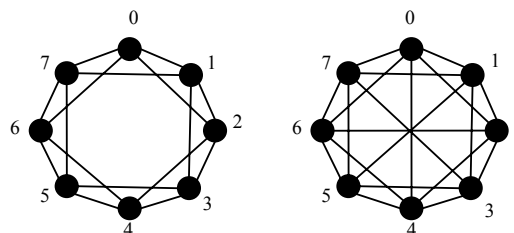


Fig. 1 The Harary graphs $H_{4,8}$ and $H_{5,8}$.

A spanning cycle in G is called a *Hamilton cycle* of G . In an even order graph G , a *perfect matching* or *1-factor*, denoted as I , is a set of mutually non-adjacent edges, which covers all vertices of G [5]. Thus a Hamilton cycle of a graph G of even order is the union of two perfect matching in G . Perfect matching exists in K_n and $H_{k, n}$ if and only if n is even; and for $K_{m, n}$, $m = n$.

A *Hamilton decomposition* is a partitioning of the edge set of G into Hamilton cycles if G is $2d$ -regular or into Hamilton cycles and a perfect matching if G is $(2d+1)$ -regular [6]. The complete graph K_n has Hamilton decomposition for all $n > 2$. Any complete graph K_n can be decomposed into $(n-1)/2$ Hamilton cycles if n is odd and $(n-2)/2$ Hamilton cycles plus a perfect matching if n is even. For convenience in labeling, we denote the even order complete graph as K_{2n} . In the decomposition of K_{2n} into n -suns we choose C_n to be the Hamilton cycles of its subgraph, K_n . The complete bipartite graph $K_{n, n}$ can be decomposed into $n/2$ Hamilton cycles when

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n is even and $(n-1)/2$ Hamilton cycles and a perfect matching when n is odd.

III. MAIN DEFINITIONS AND RESULTS

We define a new kind of decomposition for even order graphs with a perfect matching. An n -sun decomposition of an even order graph G with a perfect matching is partitioning the edge set into n -suns and m (>0) copies of K_2 which forms either a perfect matching or a Hamilton cycle or both.

A graph G is said to have *total n -sun decomposition* if every edge belongs to exactly one n -sun of the decomposition; i.e. $m = 0$ in the n -sun decomposition. An example graph is shown in Fig. 2.

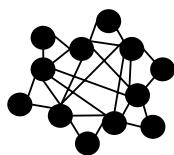


Fig. 2 A graph with total n -sun decomposition

A graph with $2n$ vertices may have an n -sun as its subgraph but need not have n -sun decomposition which can be observed from Fig. 3.

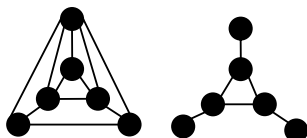


Fig. 3 A graph with six vertices and one its 3-suns

Regularity of graphs does not play a role in total n -sun decomposition. Not all regular graphs have n -sun decomposition. An example shown in Fig. 4 is the Petersen graph P , whose edge set can be partitioned into a 5-sun (bold line) and a cycle C_5 (dotted line).

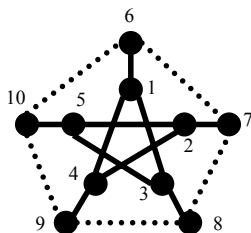


Fig. 4 Decomposition of Petersen's graph into 5-sun and C_5

It is interesting to note that $P + 5K_2$ can be decomposed into two n -suns by a choice of $5K_2 = \{(1, 7), (2, 8), (3, 9), (4, 10), (5, 6)\}$.

Lemma 3.1: A necessary condition for the n -sun decomposition of $G(V_{2n}, E)$ into t isomorphic copies of n -suns and mK_2 is that $|E|$ is divisible by $2tn + m$.

Proof: Since there are $2n$ edges in G , $2nt$ edges are required

to construct t isomorphic copies of n -suns and m edges in K_2 .

We show below that K_{2n} can be decomposed into $n-1$ n -suns and a perfect matching when n is odd. For n even, K_{2n} can be decomposed into $n-2$ n -suns, a Hamilton cycle and a perfect matching. A proper labeling of G with k vertices is a bijection $f: V(G) \rightarrow \{0, 1, 2, \dots, k-1\}$. The method of building n -suns uses a labeling scheme and Walecki's construction for the Hamilton decomposition of complete graphs.

Theorem 3.2: The complete graph K_{2n} has n -sun decomposition for all odd $n \geq 3$.

Proof: Consider the complete graph K_{2n} where n is odd. Split the vertex set $V = \{v_0, v_1, \dots, v_{n-1}, v_n, v_{n+1}, \dots, v_{2n-1}\}$ of K_{2n} into two such that $V_1 = \{v_0, v_1, \dots, v_{n-1}\}$ and $V_2 = \{v_n, v_{n+1}, \dots, v_{2n-1}\}$. Let X and Y be the induced subgraph of K_{2n} with vertex subsets V_1 and V_2 respectively. Then X and Y are complete graphs of odd order and have $n(n-1)/2$ edges each. The remaining n^2 edges of K_{2n} form an edge cut (whose removal disconnects K_{2n}). To construct n -suns for K_{2n} , we first find Hamilton cycles of X and Y using Walecki's construction. Then to each Hamilton cycle, adjoin n edges from the edge cut. A similar functional notation of [2] is adopted in finding the Hamilton cycles of X and Y .

Since n is odd, X can be decomposed into $(n-1)/2$ Hamilton cycles and a perfect matching. Hence the maximum number of edge-disjoint n -suns possible using the Hamilton cycles in X is $(n-1)/2$. Let C be the Hamilton cycle,

$v_0 v_1 v_2 v_{n-1} v_3 v_{n-2} v_4 v_{n-3} \dots v_{\frac{n-1}{2}+3} v_{\frac{n-1}{2}} v_{\frac{n-1}{2}+2} v_{\frac{n-1}{2}+1} v_0$ and α

be the permutation $(v_0)(v_1 v_2 \dots v_{n-1})$. Then $C, \alpha(C), \alpha^2(C), \dots, \alpha^{\frac{n-3}{2}}(C)$ is a Hamilton cycle decomposition of X . For simplicity, let $\Phi_k = \alpha^{k-1}(C)$ denote the k^{th} Hamilton cycle and $\Phi_k(v_i)$ denote vertex v_i in that Hamilton cycle,

where $k = 1, 2, \dots, \frac{n-1}{2}$.

Append the edges to these Hamilton cycles as $\Phi_k(v_i) = \begin{cases} n+k+i-1 & \text{if } k+i-1 < n \\ [(k+i-1) \bmod n] + n & \text{if } k+i-1 \geq n \end{cases}, i = 0, 1, \dots, n-1.$

Similarly in Y let C' be the Hamilton cycle $v_n v_{n+1} v_{n+2} v_{2n-1} v_{n+3} v_{2n-2} v_{n+4} v_{2n-3} \dots v_{\frac{3n-1}{2}+3} v_{\frac{3n-1}{2}} v_{\frac{3n-1}{2}+2}$

$v_{\frac{3n-1}{2}+1} v_n$ and β be the permutation $(v_n)(v_{n+1} v_{n+2} \dots v_{2n-1})$.

Then $C', \beta(C'), \beta^2(C'), \dots, \beta^{\frac{n-3}{2}}(C')$ is a Hamilton cycle decomposition of Y . Let $\Phi'_k(v_i)$ denote vertex v_i in the k^{th}

Hamilton cycle $\Phi'_k = \beta^{k-1}(C')$ in Y , where $k = 1, 2, \dots, \frac{n-1}{2}$.

Append the edges to these Hamilton cycles as $\Phi'_k(v_{n+i}) = \begin{cases} k+i & \text{if } k+i < n \\ (k+i) \bmod n & \text{if } k+i \geq n \end{cases}, i = 0, 1, \dots, n-1.$

Finally, the perfect matching given by (v_i, v_j) where $i = 0,$

$$1, \dots, n-1 \text{ and } j = \begin{cases} \frac{3n-1}{2} + i, \text{ if } i < \frac{n+1}{2} \\ \left[\left(\frac{3n-1}{2} + i \right) \bmod n \right] + n, \text{ if } i \geq \frac{n+1}{2} \end{cases} \text{ and the } n-$$

suns decompose K_{2n} .

When n is odd, the total of $n(2n-1)$ edges in K_{2n} are divided as follows. There are $2n$ edges in an n -sun and hence $2nt$ edges are in t isomorphic copies of n -suns and n edges in the perfect matching. Also $n(2n-1) = 2nt + n$ implies $t = n-1$. Thus when n is odd, the total number of n -suns in the decomposition of K_{2n} is $n-1$ which is exactly the same number as in the decomposition of K_{2n} into Hamilton cycles. An illustration of the n -sun decomposition of K_{14} into six 7-suns and a perfect matching is shown in Fig. 7 of Appendix.

Corollary 3.3: $K_{2n} + C_n$ can be decomposed into n n -suns when n is odd.

From the previous theorem there are $n-1$ n -suns of K_{2n} . Add n multi edges $(v_0, v_1), (v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_0)$ to form an n -cycle of K_{2n} . The perfect matching of K_{2n} in the previous theorem with the multi edges forms another n -sun.

Spanning trees are well known in the literature as minimally connected subgraphs of a graph. They find immense applications in networks whenever there is a necessity of unique paths between vertices. K_{2n} can be decomposed into n spanning trees [6]. By using the labeling scheme of Theorem 3.2, and properly removing one edge from each cycle of n -sun and adding them with the perfect matching, we get spanning tree decomposition for K_{2n} .

Corollary 3.4: When n is odd, K_{2n} can be decomposed into n spanning trees each containing a perfect matching.

In Theorem 3.3, delete edges (v_k, v_{k+1}) from $\alpha^{k-1}(C)$ and edges (v_{n+k}, v_{n+k+1}) from $\beta^{k-1}(C)$, $k = 1, 2, \dots, \frac{n-1}{2}$. The edge

deleted n -suns form $n-1$ spanning trees. The edges deleted from the n -suns when added with the perfect matching give another spanning tree.

Theorem 3.5: The complete graph K_{2n} has n -sun decomposition for all even $n \geq 4$.

Proof: The procedure for the decomposition is the same as that for odd n except for a slight change in the labels. Let the notations V_1, V_2, X and Y be as in Theorem 3.2 except that n is odd in this case. In X , let C be the Hamilton cycle $V_0 V_1 V_2 \dots V_{n-1} V_n V_{n+1} \dots V_{2n-1} V_0$ and α be the permutation $(v_0)(v_1 v_2 \dots v_{n-1})$ as in the odd case. Then $C, \alpha(C), \alpha^2(C), \dots, \alpha^{\frac{n-4}{2}}(C)$ is a Hamilton cycle decomposition of X .

Let $\Phi_k = \alpha^{k-1}(C)$ and $\Phi_k(v_i)$ denote vertex v_i in the k^{th} Hamilton cycle of X , where $k = 1, 2, \dots, \frac{n-2}{2}$. Append the edges to the Hamilton cycles of X as

$$\Phi_k(v_i) = \begin{cases} n+k+i-1 & \text{if } k+i-1 < n \\ [(k+i-1) \bmod n] + n & \text{if } k+i-1 \geq n \end{cases}, k = 1, 2, \dots, \frac{n-2}{2} \text{ and } i = 0, 1, \dots, n-1.$$

Similarly in Y , $C', \beta(C'), \beta^2(C'), \dots, \beta^{\frac{n-4}{2}}(C')$ is a Hamilton cycle decomposition where C' is the cycle $V_n V_{n+1} V_{n+2} \dots V_{2n-1} V_{n+3} V_{n+4} \dots V_{3n-2} V_{3n-1} V_{3n} V_{3n+1} V_{3n+2} V_{3n+3} \dots V_{4n-1} V_{4n} V_{4n+1} V_{4n+2} \dots V_{5n-1} V_{5n} V_{5n+1} V_{5n+2} \dots V_{6n-1} V_{6n} V_{6n+1} V_{6n+2} \dots V_{7n-1} V_{7n} V_{7n+1} V_{7n+2} \dots V_{8n-1} V_{8n} V_{8n+1} V_{8n+2} \dots V_{9n-1} V_{9n} V_{9n+1} V_{9n+2} \dots V_{10n-1} V_{10n} V_{10n+1} V_{10n+2} \dots V_{11n-1} V_{11n} V_{11n+1} V_{11n+2} \dots V_{12n-1} V_{12n} V_{12n+1} V_{12n+2} \dots V_{13n-1} V_{13n} V_{13n+1} V_{13n+2} \dots V_{14n-1} V_{14n} V_{14n+1} V_{14n+2} \dots V_{15n-1} V_{15n} V_{15n+1} V_{15n+2} \dots V_{16n-1} V_{16n} V_{16n+1} V_{16n+2} \dots V_{17n-1} V_{17n} V_{17n+1} V_{17n+2} \dots V_{18n-1} V_{18n} V_{18n+1} V_{18n+2} \dots V_{19n-1} V_{19n} V_{19n+1} V_{19n+2} \dots V_{20n-1} V_{20n} V_{20n+1} V_{20n+2} \dots V_{21n-1} V_{21n} 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required decomposition.

In the case of odd order complete graphs K_{2n+1} , n -sun decomposition is not possible, since the maximum matching is $2n$.

Corollary 3.7: $K_{2n+1} - K_{1, 2n}$ has an n -sun decomposition.

Proof: In K_{2n+1} , removal of a star subgraph $K_{1, 2n}$, results in a complete graph of even order which has n -sun decomposition.

In the next section we discuss about the n -sun decomposition of complete bipartite graphs $K_{n, n}$, for n even. Since the minimum cycle length in bipartite graphs is four and to append pendants, $n \geq 4$. $K_{n, n}$ is split into two complete bipartite subgraphs $K_{n/2, n/2}$ and $K'_{n/2, n/2}$; the remaining edges forming an edge cut of $K_{n, n}$. Any r -partite graph can be decomposed into edge-disjoint Hamilton cycles [7]. We find Hamilton cycles in each of the subgraphs $K_{n/2, n/2}$ and $K'_{n/2, n/2}$ and append edges from the edge cut for the pendants of the n -sun.

We brief now the procedure to find edge-disjoint Hamilton cycles in $K_{m, m}$ which uses consecutive perfect matchings (1-factors). Let the vertex bipartition of $K_{m, m}$ be $\{u_0, u_1, \dots, u_{m-1}\}$ and $\{v_0, v_1, \dots, v_{m-1}\}$. Let the set of perfect matching be $\{F_j = \bigcup_{i=0}^{m-1} u_i v_{i+j-1}, j = 1, 2, \dots, m\}$, the suffices of v are taken modulo m . When m is even, $\{F_{2k-1} \cup F_{2k}, k = 1, 2, \dots, m/2\}$ gives the set of $m/2$ edge disjoint Hamilton cycles of $K_{m, m}$. For odd m , the decomposition is $(m-1)/2$ Hamilton cycles plus the perfect matching $\{F_m\}$.

The next two theorems give the labeling scheme for the n -sun decomposition for $K_{n, n}$, $n \geq 4$.

Theorem 3.8: The complete bipartite graph $K_{n, n}$ has n -sun decomposition for all $n/2$ even.

Proof: To simplify the labeling scheme, let the vertices of $K_{n, n}$ be partitioned into P, Q, R and S where

$$P = \{u_0, u_1, \dots, u_{(n-2)/2}\}, Q = \{u'_0, u'_1, \dots, u'_{(n-2)/2}\},$$

$$R = \{v_0, v_1, \dots, v_{(n-2)/2}\} \text{ and } S = \{v'_0, v'_1, \dots, v'_{(n-2)/2}\}.$$

Let $K_{n/2, n/2}$ and $K'_{n/2, n/2}$ be the induced subgraphs formed by $P \cup R$ and $Q \cup S$ respectively as shown in Fig. 6.

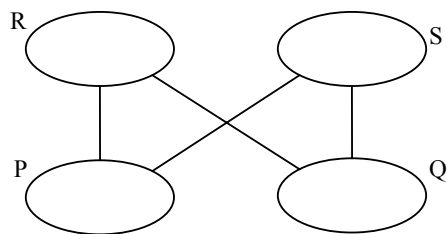


Fig. 6 Diagram representing the vertex partitioning of $K_{n, n}$.

The maximum number of n -suns possible in the decomposition of $K_{n, n}$ is $n/2$ since the maximum number of

Hamilton cycles in $K_{n/2, n/2}$ and $K'_{n/2, n/2}$ put together is $n/2$.

Let $\{F_j = \bigcup_{i=0}^{(n-2)/2} u_i v_{i+j-1}, j = 1, 2, \dots, n/2\}$, be the set of edge disjoint collection of perfect matchings of $K_{n/2, n/2}$, the suffix of v being taken modulo $n/2$. Let $H_k = F_{2k-1} \cup F_{2k}, k = 1, 2, \dots, n/4$ be a Hamilton cycle decomposition of $K_{n/2, n/2}$. For each cycle H_k , append edges for the pendants from the edge cut as follows: $H_k(u_i) = v'_p, H_k(v_i) = u'_p, p = k+i-1$ is taken modulo $n/2, i = 0, 1, 2, \dots, (n-2)/2$ and $k = 1, 2, \dots, n/4$.

Similarly let $\{F'_j = \bigcup_{i=0}^{(n-2)/2} u'_i v'_{i+j-1}, j = 1, 2, \dots, n/2\}$, be the perfect matchings of $K'_{n/2, n/2}$, the suffix of v being taken modulo $n/2$. Let $H'_k = F'_{2k-1} \cup F'_{2k}, k = 1, 2, \dots, n/4$ be a Hamilton cycle decomposition of $K'_{n/2, n/2}$. Append edges for the pendants of n -sun as $H'_k(u'_i) = v_q$ and $H'_k(v'_i) = u_q$ where $i = 0, 1, \dots, (n-2)/2, k = 1, 2, \dots, n/4$ and $q = k+i-1+(n/4)$ is taken modulo $n/2$. The Hamilton cycles with the appended edges form n -suns. Since every edge is in exactly one n -sun, $K_{n, n}$ has a total n -sun decomposition.

Theorem 3.9: The complete bipartite graph $K_{n, n}$ has n -sun decomposition for all $n/2$ odd.

Proof: Let the notations be as in Theorem 3.8 where $n/2$ is odd and $k = 1, 2, \dots, (n-2)/2$. The proof is similar to that of even $n/2$ but the maximum number of n -suns possible in this case is $(n-2)/2$. By the construction, two sets of perfect matching of $K_{n, n}$ are left out after the n -suns are constructed. They are $F_{n/2} \cup F'_{n/2}$ and $\{u_i v'_r\} \cup \{u'_i v_r\}, i = 0, 1, \dots, (n-2)/2$ and $r = i+(n-2)/4$ is taken modulo $n/2$. These two matching forms a Hamilton cycle.

Examples of n -sun decompositions of $K_{4, 4}$ and $K_{6, 6}$ are shown in Fig. 9 and Fig. 10 respectively of Appendix.

The following theorem is on the n -sun decomposition of certain type of even order Harary graphs $H_{k, 2n}, k < 2n$.

Theorem 3.10: If the Harary graph $H_{k, 2n}$ has total n -sun decomposition, then $H_{k+1, 2n}$ has n -sun decomposition for all n .

Proof: If $H_{k, 2n}$ has total n -sun decomposition, then every edge is in some n -sun of the decomposition. By construction of the Harary graphs, $H_{k+1, 2n}$ is $H_{k, 2n}$ plus a perfect matching formed by the edges with diametrically opposite vertices as end vertices. The n -suns of $H_{k, 2n}$ and the perfect matching decompose $H_{k+1, 2n}$.

Theorem 3.11: The Harary graph $H_{4, 2n}$ has total n -sun decomposition for all n .

Proof: By construction of Harary graphs, every vertex v_i is adjacent to $v_{i-1}, v_{i-2}, v_{i+1}$ and v_{i+2} , the suffices being taken modulo $2n$. Thus there exists cycles $C_1: v_1 v_3 v_5 \dots v_{2n-1} v_1$ and $C_2: v_0 v_2 v_4 \dots v_{2n-2} v_0$ and perfect matchings $M_1: \{v_0 v_1, v_2 v_3, v_4 v_5, \dots, v_{2n-2} v_{2n-1}\}$ and $M_2: \{v_1 v_2, v_3 v_4, v_5 v_6, \dots, v_{2n-1} v_0\}$ mutually disjoint to each other. The subgraphs $C_1 \cup M_1$ and $C_2 \cup M_2$ form n -suns of $H_{4, 2n}$.

Corollary 3.12: The Harary graph $H_{5, 2n}$ has an n -sun decomposition.

It can be easily seen that the Harary graphs $H_{k, k+2}$ (k even) is nothing but the complete graphs K_{k+2} with a perfect matching removed. Hence it has a total $(k+2)$ -sun decomposition when $(k+2)/2$ is odd and a $(k+2)$ -sun decomposition when $(k+2)/2$ is even. The n -sun decomposition of other Harary graphs need attention.

An example of 5-sun decomposition of $H_{5, 10}$ is shown in Fig. 11 of Appendix.

IV. CONCLUSION AND FUTURE STUDIES

The aim of this communication has been to present a new kind of decomposition of K_{2n} , $K_{n, n}$ and $H_{4, n}$. It is hoped that this decomposition may stimulate further studies on n -sun decompositions. The n -suns and the Hamilton cycles of an even order graph have close association since both are spanning subgraphs containing perfect matching (one in n -sun and two in Hamilton cycle) and exactly one cycle. Also the deletion of any one edge of the cycle in the n -sun or Hamilton cycle results in a spanning tree where the tree contains a perfect matching. Since the maximum degree of a vertex in the n -sun is three, the spanning trees obtained from n -suns also have the same maximum degree. The important feature of the spanning trees using n -sun is that the diameter is $(n/2)+1$.

The definition of n -sun has wide scope in finding such decomposition for graphs in general. The labeling procedure used for complete graphs and complete bipartite graphs can be used to decompose product graphs into n -suns. In the case of Harary graphs, we have found a labeling scheme for four and five regular graphs. Any possible extension or a generalized labeling scheme for n -sun decomposition of Harary graphs can be found out. Finding a necessary and sufficient condition for the existence of n -sun and total n -sun decomposition will be well appreciated. Tree decomposition for $K_{n, n}$ and Harary graphs using n -suns can be studied. By suitably choosing the labels one may try to obtain a graceful labeling for the n -sun graphs discussed in the paper.

APPENDIX

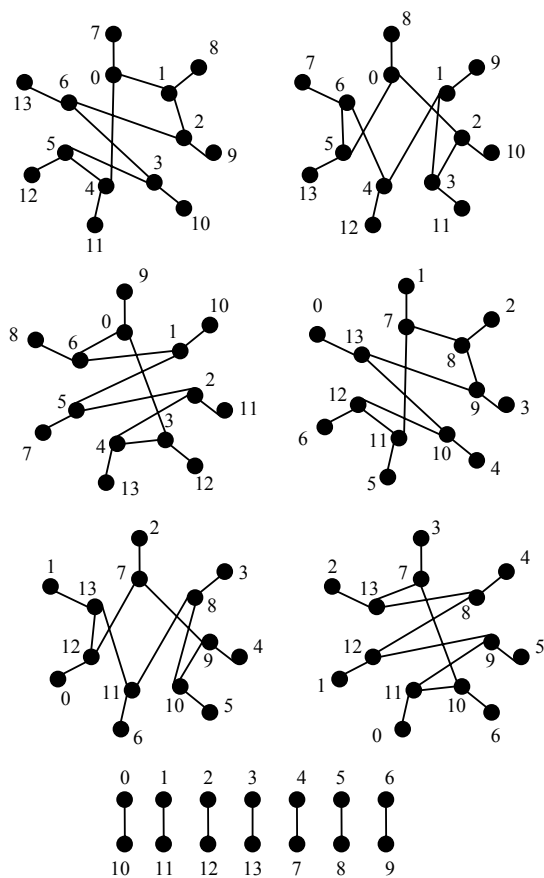


Fig. 7 A 7-sun decomposition of K_{14}

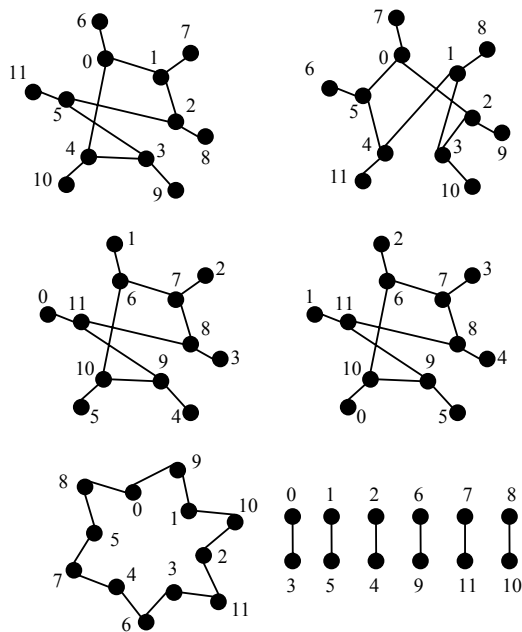


Fig. 8 A 6-sun decomposition of K_{12}

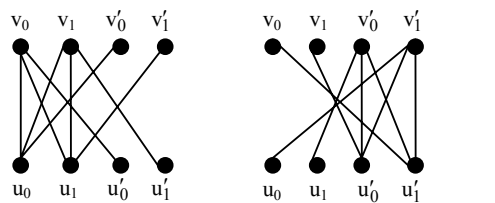


Fig. 9 A 4-sun decomposition of $K_{4,4}$

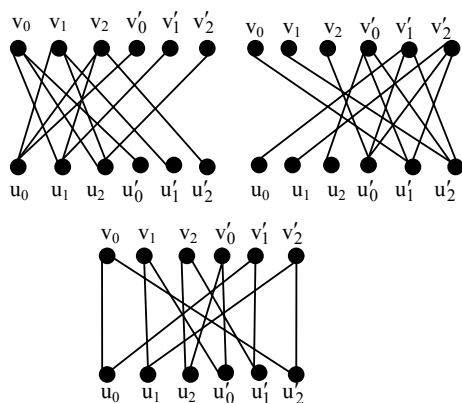


Fig. 10 A 6-sun decomposition of $K_{6,6}$

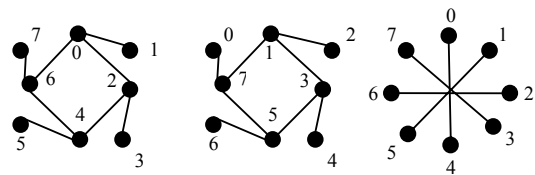


Fig. 11 A 4-sun decomposition of $H_{5,8}$

REFERENCES

- [1] W. D. Wallis, "Magic graphs," Birkhauser, 2000.
- [2] B. Alspach, J. C. Bermond and Sotteau, "Decompositions into cycles I: Hamilton decompositions, cycles and rays," Kluwer Academic Press 1990, pp. 9–18.
- [3] B. Alspach, "The wonderful Walecki construction," 2006, April [Lecture notes].
- [4] B. Alspach and H. Gavlas, "Cycle decompositions of K_n and $K_n - I$," *J. Combin. Theory Ser. B*, Vol. 81, pp. 77–99, 2001.
- [5] D. B. West, "Introduction to graph theory," Pearson Education Pte.Ltd., 2002.
- [6] J. L. Gross and J. Yellen, "Handbook of Graph theory," CRC Press, 2004.
- [7] R. Laskar and B. Auerbach, "On decomposition of r -partite graphs into edge-disjoint Hamilton circuits", *Discrete Math.* Vol. 14, pp. 265-268, 1976.