Instability Analysis of Laminated Composite Beams Subjected to Parametric Axial Load

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Abstract—The integral form of equations of motion of composite beams subjected to varying time loads are discretized using a developed finite element model. The model consists of a straight five node twenty-two degrees of freedom beam element. The stability analysis of the beams is studied by solving the matrix form characteristic equations of the system. The principle of virtual work and the first order shear deformation theory are employed to analyze the beams with large deformation and small strains. The regions of dynamic instability of the beam are determined by solving the obtained Mathieu form of differential equations. The effects of non-conservative loads, shear stiffness, and damping parameters on stability and response of the beams are examined. Several numerical calculations are presented to compare the results with data reported by other researchers.

Keywords—Finite element beam model, Composite Beams, stability analysis

I. INTRODUCTION

COMPOSITE beams are increasingly being used in the design of high-performance load-carrying structures when high strength and stiffness to weight ratios are desired. The stability analysis of laminated composite structures is important to be investigated when such structures are subjected to varying time loads. Most of the studies in this field are associated to laminated composite plates analysis. In this study, the stability analysis of composite beams is investigated, which has not been studied as comprehensively as laminated composite plates.

The dynamic instability of structures occurs because of parametric resonance. The analytical dynamic instability analysis of the beams subjected to varying time loads has been studied extensively by Bolotin [1]. For prismatic composite laminated beam, consisting of individual rectangular component beams, which are rigidly connected together at their longitudinal edges to form of arbitrary cross section and the stress-strain system is time-dependent, using a numerical method such as finite element method must be considered.

The finite element method and numerical simulation have been widely used by researchers to study the dynamic analysis of laminated beams. Currently the demand for developing of beam elements and implementation of numerical tools to predict the response of such structures increases. Regular beam models can be used for moderately thick beams. But for slender beams that the length to thickness ratio is extremely high and geometrically nonlinear analysis is required, convergence may become very poor using such models. Also when the beam is shear deformable with small strains and large deformation, developing a model that can take in account the various coupling effects, such as stretching-bending coupling is important in dynamic analysis.

Earlier Kapania and Raciti [2] developed a twenty degrees of freedom beam element to analyze thick laminated beams. Yuan and Miller [3] developed a finite element model using a constant and quadratic shear strain in each lamina. Their model gives good results on the static analysis of short beams but has a high number of degrees of freedom, which is disadvantage. Manjunatha and Kant [4] developed a four nodes beam element with four, five, and six degrees of freedom per node for the analysis of laminated beams subjected to just static loading. Their model is based on the higher-order shear deformation theory. Bassiouni et al. [5] presented a two dimensional one lamina five node beam model with ten degrees of freedom to obtain the natural frequencies and mode shapes of the beams. Loja et al. [6] proposed a model based on a straight beam finite element with four nodes and fourteen degrees of freedom per node, considering bi-axial bending, stretching and twisting effects. Ramtekkar et al. [7] presented a six-node plane-stress mixed finite element model by using Hamilton's minimum energy principle, which can't include nonconservative loading.

Subramanian [8] developed a two-nodes beam element with eight degrees of freedom per node to investigate the vibration problems but coupling effects has not been considered in his model. In this study, a new beam model with optimum degrees of freedom is developed to descritize the equations of motion, which can acquire accurate results with faster convergence and less computation time. The model is a straight beam element with five nodes and twenty- two degrees of freedom with considering transverse bending, stretching and twisting coupling effects. The descritized equations of motion are solved using the symbolic computational algorithm to identify the boundary of instability in the load frequency dependence

plane. Several numerical results are presented to verify the performance of the developed model and formulation. Finally, the effects of nonconservative loads and damping on the boundaries of instability and dynamic response of the laminated composite beam are investigated.

II. FORMULATION

Consider a laminated prismatic composite four layer beam with uniform thickness and coordinate systems as shown in Fig. 1 are subjected to an axial harmonic varying time load.
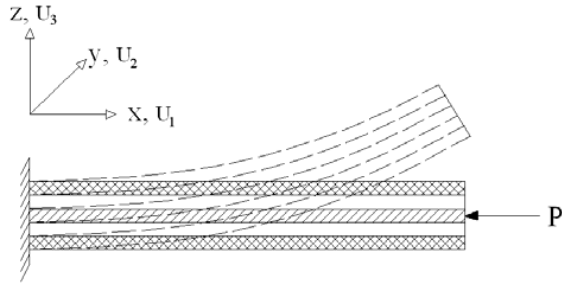

Fig. 1 The laminated composite beam with local coordinate system

The constitutive equations for the beam are

$$\mathbf{S} = \mathbf{E}\boldsymbol{\varepsilon} \tag{1}$$

where the resultant matrix $\mathbf{S}$, the strains matrix $\boldsymbol{\varepsilon}$, and the laminate stiffness matrix $\mathbf{E}$ are defined as;

$$\mathbf{S} = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \text{ and }$$

$$\mathbf{E} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & 0 & 0 \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & 0 & 0 \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & 0 & 0 \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & 0 & 0 \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & 0 & 0 \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{44} & A_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{45} & A_{55} \end{bmatrix} \tag{2}$$

The N_x, N_y, and N_{xy} are the membrane and shearing forces per unit length, M_x, M_y, and M_{xy} are the bending and twisting moments per unit length, and Q_x, Q_y are the shear forces through thickness per unit length.

The coefficients of the laminate stiffness matrix $\mathbf{E}$ are

described as;

$$A_{ij} = \int_{-h/2}^{h/2} Q_{ij} dz, \quad B_{ij} = \int_{-h/2}^{h/2} Q_{ij} z\, dz, \text{ and } D_{ij} = \int_{-h/2}^{h/2} Q_{ij} z^2 dz$$

for $i,j = 1,2,6$ and $A_{ij} = k_s \int_{-h/2}^{h/2} Q_{ij} dz$ for $i,j = 4,5$

whereas Q_{ij} are plane stiffness for $(i, j = 1, 2, 6)$ and Q_{ij} are the shear stiffness for $(i, j = 4, 5)$, and $k_s = 5/6$ is the shear correction factor.

To cover the effects of bending stretching, stretching-twisting and bending-twisting couplings on governing equations, membrane strains $\left(\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}, \gamma_{xz}, \gamma_{yz} \right)$ and the bending strains $\left(\kappa_{xx}, \kappa_{yy}, \kappa_{xy} \right)$ are defined in terms of the mid surface displacements and rotations;

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x} + y\dfrac{\partial \phi_3}{\partial x} + \dfrac{1}{2}[(\dfrac{\partial u_3}{\partial x})^2 + 2y\dfrac{\partial u_3}{\partial x}\cdot\dfrac{\partial \phi_1}{\partial x} + (y\dfrac{\partial \phi_1}{\partial x})^2] \\[2ex] \dfrac{1}{2}\phi_1^2 \\[2ex] \dfrac{\partial u_2}{\partial x} + \phi_3 + (\dfrac{\partial u_3}{\partial x} + y\dfrac{\partial \phi_1}{\partial x})\cdot\phi_1 \\[2ex] \dfrac{\partial \phi_1}{\partial x} \\[2ex] \dfrac{\partial \phi_2}{\partial y} \\[2ex] \dfrac{\partial \phi_1}{\partial y} + \dfrac{\partial \phi_2}{\partial x} \\[2ex] \dfrac{\partial u_3}{\partial x} + y\dfrac{\partial \phi_1}{\partial x} + \phi_2 \\[2ex] \dfrac{\partial u_3}{\partial y} + y\dfrac{\partial \phi_1}{\partial x} + \phi_1 \end{bmatrix} \tag{3}$$

The generalized displacement vector for each point of the beam is expressed as follow:

$$\mathbf{U}^{\mathrm{T}} = \{u_1, \quad u_2, \quad u_3, \quad \phi_1, \quad \phi_2, \quad \phi_3,\} \tag{4}$$

III. EQUATIONS OF MOTION

The governing equations of motion corresponding to the constitutive Eq.(1) are derived using the dynamic version of principle of virtual work;

$$\int_0^T (\delta W_I - \delta W_E - \delta K)\,dt = 0 \tag{5}$$

The δW_E, δW_I, δK are the virtual work done by external forces, the virtual work done by internal forces, and the virtual kinetic energy respectively and defined as follow;

$$\delta W_I = \int_0^l \int_{-b/2}^{b/2} \left(\delta\boldsymbol{\varepsilon}^{\mathrm{T}} \mathbf{S} \right) dy\, dx \tag{6}$$

$$\delta W_E = \int_0^l \int_{-b/2}^{b/2} f_i . \delta U_j . dydx \qquad (7)$$

$$\int_0^T \delta K dt = \int_0^T \int_0^l \delta \mathbf{U}^{\mathrm{T}} \mathbf{M} \ddot{\mathbf{U}} dxdt \qquad (8)$$

where f_i are surface forces per unit area acting on the beam and δU_i are virtual displacements. The double dot over the variables denote the derivative with respect to time and $\mathbf{M}$ represents the mass matrix. It is perceptible that all terms of the integral form of the equations of motion Eq.(5) are displacement dependents and can be discretized through a well established beam finite element model.

IV. FINITE ELEMENT MODEL

A five-node twenty-two degrees of freedom beam element based on the first order shear deformation theory is developed to discretize the integral form of equations of motion. The effects of bending-stretching, shear-stretching, bending-twisting couplings, transverse shear deformation, and continuity have been considered to define optimum degrees of freedom for each node to acquire fast and accurate results.
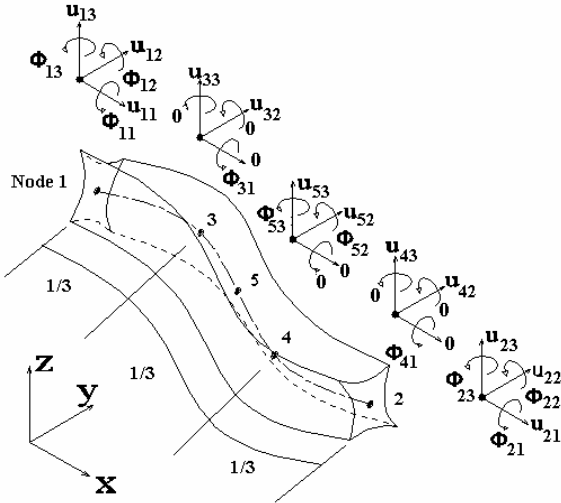

Fig. 2 The developed beam model

The nodal displacement $\mathbf{U}^e$ can be expressed as the generalized global displacement $\mathbf{U}$ and shape functions $\mathbf{N}$ as defined with the following equation;

$$\mathbf{U} = \mathbf{N}\mathbf{U}^e \qquad (9)$$

The shape functions matrix $\mathbf{N}$ comply with Lagrangian cubic and quadratic interpolation polynomials. All the nodal displacements and rotations are measured at the mid-surface and expressed as;

$$\mathbf{U}^e = \mathbf{U}^e_{1,2} + \mathbf{U}^e_{3,4,5} \qquad (10)$$

for end nodes 1 and 2:

$$\mathbf{U}^e_{1,2} = \{u_{11}, u_{12}, u_{13}, \phi_{11}, \phi_{12}, \phi_{13}, u_{21}, u_{22}, u_{23}, \phi_{21}, \phi_{22}, \phi_{23}\}$$

and for nodes 3,4, and 5:

$$\mathbf{U}^e_{3,4,5} = \{u_{32}, u_{33}, \phi_{31}, u_{42}, u_{43}, \phi_{41}, u_{52}, u_{53}, \phi_{52}, \phi_{53}\}$$

Nodal displacements are a combination of the axial displacements (u_{11}, u_{21}), the lateral displacements $(u_{12}, u_{22}, u_{32}, u_{42}, u_{52})$, the transverse displacements $(u_{13}, u_{23}, u_{33}, u_{43}, u_{53})$, and the rotations $(\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}, \phi_{13}, \phi_{23}, \phi_{31}, \phi_{41}, \phi_{52}, \phi_{53})$.

The discretization of equation (5) over the domain leads to the element tangent stiffness matrix, mass matrix, and force vectors. Substituting equations (6)-(10) into equation (4), the element dynamic equations of motion in matrix form are obtained as follows;

$$\mathbf{M}^e \ddot{\mathbf{U}}^e_t + \left(\mathbf{K}^e_E + \mathbf{K}^e_G + \mathbf{K}^e_L \right) \mathbf{U}^e_t = \mathbf{F}^e \qquad (11)$$

where the element mass matrix $\mathbf{M}^e$, and the elastic stiffness matrix $\mathbf{K}^e_E$, the geometric stiffness matrix $\mathbf{K}^e_G$, and the loading stiffness matrix $\mathbf{K}^e_L$ are given as;

$$\mathbf{M}^e = \rho \iiint_V \mathbf{N}^{\mathrm{T}} \mathbf{N} dV \qquad (12)$$

$$\mathbf{K}^e_E = \int_0^{l_e} \int_{-b/2}^{b/2} \mathbf{B}^{\mathrm{T}} \mathbf{D} \, \mathbf{B} dydx \qquad (13)$$

$$\mathbf{K}^e_G = \int_0^{l_e} \int_{-b/2}^{b/2} \left\{ \frac{\partial \boldsymbol{\varepsilon}^{\mathrm{T}}}{\partial \mathbf{U}^e} \mathbf{S} \right\} dydx \qquad (14)$$

$$\mathbf{K}^e_L = \frac{\partial \mathbf{F}^e}{\partial \mathbf{U}^e} \qquad (15)$$

The total external nodal loads applied on the surface of the beam is defined as, $\mathbf{F}^e = \int_0^{l_e} \int_{-b/2}^{b/2} \mathbf{N}^{\mathrm{T}} f_i dydx$. In this case, the nodal loads $\mathbf{F}^e$ are applied on the last element, therefore the nodal and the global loads are same $\mathbf{F}^e = \mathbf{F}$.

By assembling of all elements, the global finite element dynamic equation of motion is obtained as;

$$\mathbf{M}\ddot{\mathbf{U}}_t + \left(\mathbf{K}_E + \mathbf{K}_G + \mathbf{K}_L \right) \mathbf{U}_t = \mathbf{F} \qquad (16)$$

If the compression varying time periodic load applied on the beam is introduced in the form of $F = F_0 + F_t \cos\theta t$, the loading stiffness matrix can be separated to static and dynamic stiffness matrices;

$$\mathbf{K}_L = \mathbf{K}^0_L + \mathbf{K}^t_L \cos\theta t \qquad (17)$$

Therefore, dynamic equation of motion becomes;

$$\mathbf{M}\ddot{\mathbf{U}}_t + \left(\mathbf{K}_E + \mathbf{K}_G - \mathbf{K}^0_L - \mathbf{K}^t_L \cos\theta t \right) \mathbf{U}_t = \mathbf{F} \qquad (18)$$

and free vibration analysis of the beam about the equilibrium state when $\mathbf{F} = 0$ yields to:

$$\mathbf{M}\ddot{\mathbf{U}}_t + \left(\mathbf{K}_E + \mathbf{K}_G - \mathbf{K}^0_L - \mathbf{K}^t_L \cos\theta t \right) \mathbf{U}_t = 0 \qquad (19)$$

V. STABILITY ANALYSIS

Equation (19) is a set of coupled Mathieu equations which govern the motion of the beam with periodic solutions. The periodic motion is usually the boundary case of vibrations with unboundedly increasing amplitudes. Therefore, a dynamic instability analysis is essentially about the determination of the boundaries of the dynamic instability regions. The trivial solution of the equation (19) with period of $2T$ where $T = \frac{2\pi}{\theta}$ in Fourier series form are expressed as the following:

$$\mathbf{U}_t = \sum_{m=1,3,5,\dots}^{\infty} (\mathbf{A}_m \sin \frac{m\theta t}{2} + \mathbf{B}_m \cos \frac{m\theta t}{2}) \qquad (20)$$

$\mathbf{A}_m$ and $\mathbf{B}_m$ are vectors independent of time (t). Substituting Eq. (20) into Eq. (19) and equating the sum of the coefficients of identical $\left(\sin \frac{m\theta t}{2}\right)$ and $\left(\cos \frac{m\theta t}{2}\right)$ to zero leads a set of the first order determinant to zero:

$$
\begin{vmatrix}
\mathbf{K}_E + \mathbf{K}_G - \mathbf{K}_L^0 \pm \frac{\mathbf{K}_L^t}{2} - \frac{\mathbf{M}\theta^2}{4} & -\frac{\mathbf{K}_L^t}{2} & 0 & \cdot \\
-\frac{\mathbf{K}_L^t}{2} & \mathbf{K}_E + \mathbf{K}_G - \mathbf{K}_L^0 - \frac{9\mathbf{M}\theta^2}{4} & -\frac{\mathbf{K}_L^t}{2} & \cdot \\
0 & -\frac{\mathbf{K}_L^t}{2} & \mathbf{K}_E + \mathbf{K}_G - \mathbf{K}_L^0 - \frac{25\mathbf{M}\theta^2}{4} & \cdot \\
\cdot & \cdot & \cdot &
\end{vmatrix} = \mathbf{0}
$$

$$(21)$$

Equations (21) are characteristic equations of the system, which determines boundaries of dynamic instability regions. It is important to note that these equations have symmetric coefficients and are useful for systems without damping in account.

VI. STABILITY ANALYSIS WITH DAMPING IN ACCOUNT

Damping of composite laminated beams plays a vital role in the dynamic behaviour analysis of structures by controlling the resonant vibrations and thus reducing of the bounded instability regions. This damping depends on lamina material properties as well as layer orientations, and stacking sequence. A damping analysis procedure of laminated composites has been developed initially by Adams and Bacon [9]. In this procedure the energy dissipation can be described as separable energy dissipations associated to the individual stress components. This analysis was refined in a later paper of Ni and Adams [10]. In their study, the damping of orthotropic beams is considered as a function of material orientation such as cross-ply laminates, angle-ply laminates, and more general types of symmetric laminates. Herein, damping of composite materials based on dissipation energy associated by strain energy is considered. The equilibrium state equation of motion for the beam with damping in account is defined as;

$$\mathbf{M}\ddot{\mathbf{U}}_t + \mathbf{C}\dot{\mathbf{U}}_t + \left\{(\mathbf{K}_E + \mathbf{K}_G) - \mathbf{K}_L^0 - \mathbf{K}_L^t \cos\theta t\right\}\mathbf{U}_t = 0 \quad (22)$$

Solving Eq. (22) with same approach as described previously, leads to the eigenvalue equations in matrix form as:

$$
\begin{vmatrix}
\mathbf{K}_E + \mathbf{K}_G - \mathbf{K}_L^0 + \frac{\mathbf{K}_L^t}{2} - \frac{\mathbf{M}\theta^2}{4} & -\frac{1}{2}\mathbf{C}\theta \\
\frac{1}{2}\mathbf{C}\theta & \mathbf{K}_E + \mathbf{K}_G - \mathbf{K}_L^0 - \frac{\mathbf{K}_L^t}{2} - \frac{\mathbf{M}\theta^2}{4}
\end{vmatrix} = 0 \quad (23)
$$

Expansion of the above determinant in second order form, yields the equations of the boundary of principal instability regions of the system.

VII. NUMERICAL RESULTS

To study the performance of the present model and eigenvalue formulation, isotropic and orthotropic straight cantilever and simply supported beams were examined in different cases.

Case-1: Isotropic clamped-free beam

In this case, the natural frequencies of a cantilever isotropic straight beam using the developed model were determined and the results compared to exact solution of an Euler-Bernouli beam. The material and geometric properties are $l = 0.1935$ m , $E = 20.7$ GPa , $I = 3.47$ cm^4 , $bh = 0.000645$ m^2 , $v = 0.3$, $\rho = 1968$ kg/m^3 . Table-1 compares the results obtained from five finite elements of present model with exact solution and the results calculated using ten finite elements two nodes regular beam presented by Logan [11] and shows the good performance of the present model for isotropic beams.

TABLE I THE NATURAL FREQUENCIES OF THE CANTILEVER BEAM

Natural frequencies (rad/sec)	Exact solution	Logan, 2003, using FE (10 elements)	Present, using FE (5 elements)
ω_1	228	227.5	228
ω_2	1434	1410	1426

Case-2: Orthotropic clamped-free and pinned-pinned asymmetric laminated composite beam

Table-2 shows very good agreement between the free vibration natural frequencies results of the present laminated beam model and the results obtained by Maiti and Sinha [12] for two different slender ratios $\frac{l}{h} = 60$ and $\frac{l}{h} = 5$ with clamped free and simply supported boundary conditions. The material properties are defined as follows:

TABLE II THE NONDIMENSIONAL NATURAL FREQUENCIES FOR THE CLAMPED-FREE (C-F) AND SIMPLY SUPPORTED (S-S) LAMINATED COMPOSITE BEAMS

$E_{xx} = 129.20708$ GPa, $E_{yy} = 9.42512$ GPa,
$G_{xy} = 5.15658$ GPa, $G_{xz} = 4.30530$ GPa,
$G_{yz} = 2.54139$ GPa,
$\nu_{xy} = \nu_{xz} = 0.3$, $\nu_{yz} = 0.218837$,
$\rho = 1550.0666$ Kg/m³, $b = 0.0127$ m, $l = 0.1905$ m

lamina lay up configuration	Maiti & Sinha				Present model, using FE (5 elements)			
	C-F		S-S		C-F		S-S	
	$l/h_t = 60$	$l/h_t = 5$	$l/h_t = 60$	$l/h_t = 5$	$l/h_t = 60$	$l/h_t = 5$	$l/h_t = 60$	$l/h_t = 5$
$0/90/0/90$	8.854	3.132	26.378	17.350	8.863	3.140	26.389	17.210
$0/30/-30/0$	12.404	3.493	34.786	21.599	12.503	3.550	34.752	20.342
$0/45/-45/0$	12.271	3.227	34.403	20.092	12.312	3.225	34.289	19.788
$0/60/-60/0$	12.227	3.154	34.271	19.384	12.305	3.158	34.279	19.224

Case-3: Orthotropic clamped-clamped asymmetric laminated composite beams

In this case, the three first nondimensional natural frequencies of a clamped-clamped unsymmetrical laminated beam with $0°/45°/-45°/0$ layup configuration will be investigated. The results obtained from FE discrettized model using five elements are calculated and compared with those results presented by Loja et al. [6], using higher order shear deformation theory (HSDT). The mechanical and geometric properties of the beam are same as case-2. The results are shown in Table 3.

TABLE III THE FIRST THREE NATURAL FREQUENCIES OF ANGLE PLY LAMINATED COMPOSITE BEAM

l/h_t	Model	First mode	Second mode	Third mode
10	Loja et al. 2001, HSDT	39.89	48.99	103.8
	Present, FE (5 elements)	42.56	55.37	112.47
30	Loja et al. 2001, HSDT	72.54	119.27	183.77
	Present, FE (5 elements)	74.33	127.66	195.72

Case- 4: Stability analysis of cross ply laminated beam

The good performance of the developed beam model leads to accurate analysis and determination of the dynamic instability regions of laminated beams. Consider a cross ply $0°/90°/90°/0°$ laminated beam with equal thickness for each lamina. The stiffness and mass matrices of free vibration of the beam are calculated using the presented formulations and symbolic computations algorithm. The approximate expression for the boundaries of the principal regions of instabilities is obtained by equating to zero the determinant of the first matrix element of Eq.(22). This approximation is based upon the fact that the periodic solution of the equation of motion has a trigonometric form.

The first principal dynamic regions of instability of the shear deformable and undeformable cross ply laminated beams with ratio $\frac{l}{h} = 10$ are plotted and shown in Fig. 3. The material and geometry properties of the beam are same as previous case-2.

As it can be seen, for the beam without shear deformation the regions of dynamic instability trends to narrowing. The lower bound position of the shear deformable beam changes faster than upper bound. Another obvious fact is the instability region of the beam subjected to the nonconservative load doesn't intersect the axis of loading. Also, the regions of instability for nonconservative loading are enlarged in compare to conservative loading system. With damping in account, the elements ξ of the damping matrix $\mathbf{C}$ of the cross ply laminated composite beam with same material properties are calculated as described by Ni and Adams [10]. The results are depicted in Fig. 4 for different damping factors.

The stable region is enlarged when the damping ratio of the structure is increased and it has more effect on stabilizing the system. On the other hand, the greater the damping, the greater the amplitude of longitudinal forces is required to cause dynamic instability of the beam.
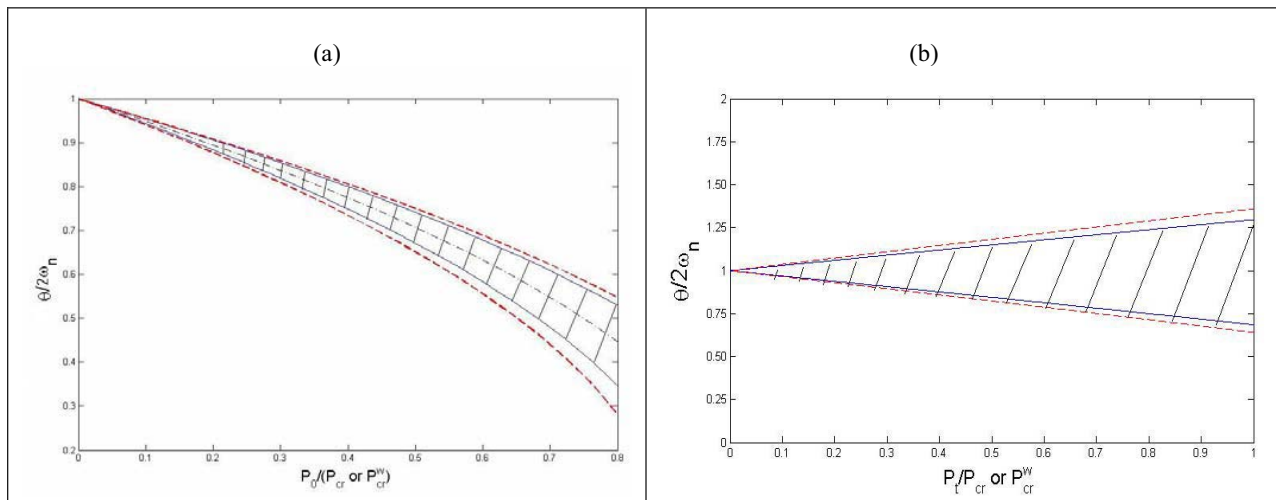
Fig. 3 Dynamic principal instability regions of a cantilever cross-ply laminated beams without shear stiffness (crosshatched region) and with shear stiffness (dash lines) subjected to (a) conservative loads (b) nonconservative loads
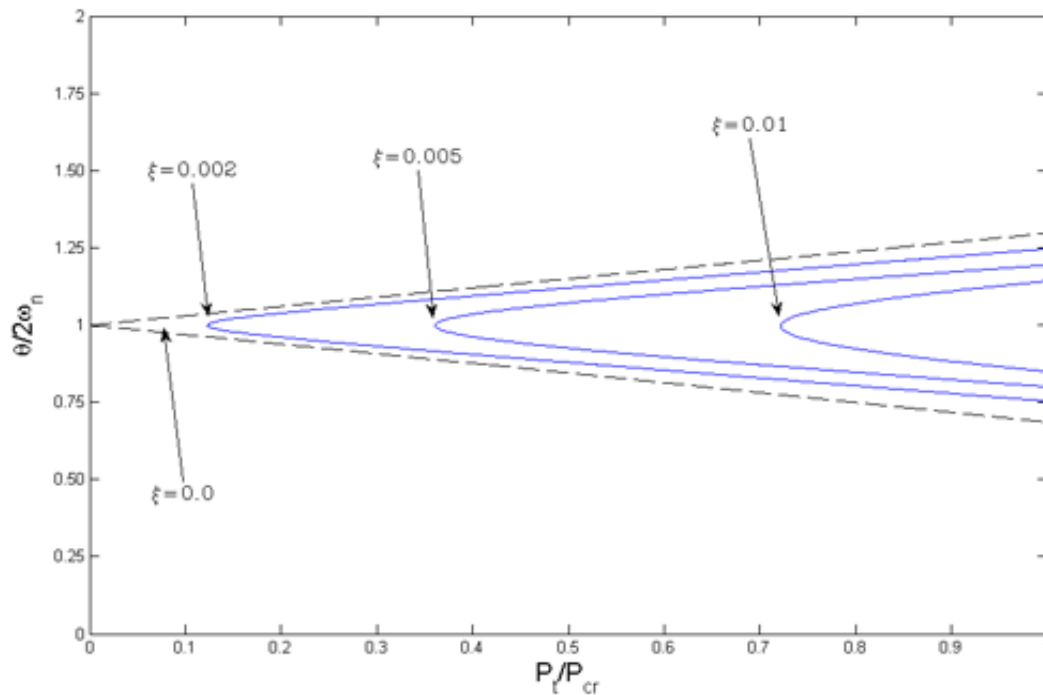

Fig. 4 Dynamic principal instability regions of a cantilever cross-ply laminated beam with different damping factors

Determination of amplitude

In this section, the vibration of parametrically excited laminated beam for the principal resonance of the system, which causes the principal instability will be studied and the amplitude of the beam will be investigated. The parametric resonance of the system occurs in the near of frequency $\theta = 2\omega_n$. The framework of the principal resonance, the parametric excitation can excite only one mode at a time, it results that for each mode, infinity of instability regions could occur. Within these instability regions, the particular mode is excited in lateral motion with exponentially growing amplitude. For $\theta = 2\omega_n$, the resulted instability region is the largest and the most significant one. It is referred to as the principal parametric

resonance. To determine the influence of loading frequency on the amplitude resonance, the first and most important instability region will be considered. The amplitude of principal resonance Λ is defined as:

$$\Lambda = \left(\mathbf{A}^2 + \mathbf{B}^2 \right)^{1/2}$$

(24)

Substituting Eq.24 in the equation 22 and solving the characteristic determinant 25, the amplitude of the vibrating a cross ply laminated composite simply supported shear deformable beam with same material and geometry properties as defined in case-2 are obtained and depicted in Fig.5.

$$\begin{vmatrix} \mathbf{K}_E + \mathbf{K}_G - \mathbf{K}_L^0 + \dfrac{\mathbf{K}_L^t}{2} - \dfrac{\mathbf{M}\theta^2}{4} & -\dfrac{1}{2}\mathbf{C}\theta\left(1+\Lambda^2\right) \\[2mm] \dfrac{1}{2}\mathbf{C}\theta\left(1+\Lambda^2\right) & \mathbf{K}_E + \mathbf{K}_G - \mathbf{K}_L^0 - \dfrac{\mathbf{K}_L^t}{2} - \dfrac{\mathbf{M}\theta^2}{4} \end{vmatrix} = 0$$
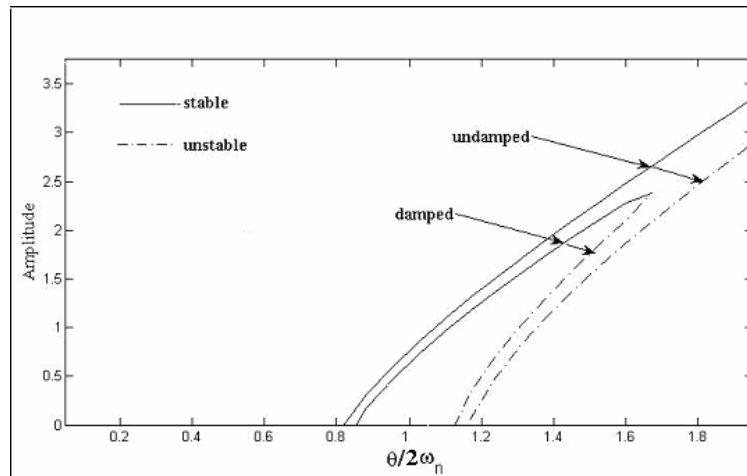
(25)

Fig. 5 The steady state resonance frequency-amplitude curve for a cross ply laminated composite beam

The resonance curves are bent toward the increasing exciting frequencies. The damping factor has important role in diminishing of the amplitude of the vibrating beam, in contrast the increasing of the nonlinear elasticity of the system does not always reduce the resonance amplitudes.

Response of the system

The last section of this paper proceeds to dynamic response of the laminated beams subjected to varying time loading. The response of the structure will be investigated in stable and unstable regions and the result will be determined and plotted. The periodic solution of the Mathieu type equations of motion of the beam will be established using the Floquet's theory.

With two parameters R and Z as defined as follows:

$$\mathbf{R} = \mathbf{M}^{-1}\left(\left(\mathbf{K}_E + \mathbf{K}_G\right) - \mathbf{K}_L^0\right), \quad \mathbf{Z} = \mathbf{M}^{-1}\mathbf{K}_L^t \quad (26)$$

Equation 22 can be expressed as:

$$\ddot{\mathbf{U}}_t + \left\{\mathbf{R} - \mathbf{Z}\cos\theta t\right\}\mathbf{U}_t = 0 \qquad (27)$$

The Floquet solutions of the above Mathieu type equation can be expressed in Fourier series as:

$$\mathbf{U}_t = e^{i\lambda t} \sum_{n=-\infty}^{\infty} b_n e^{in\theta t} \qquad (28)$$

Substitute Eq. (28) into Eq. (27) and regrouping, the following solution is obtained.

$$\sum_{n=-\infty}^{\infty} \left\{ \frac{1}{2}b_{n-1}|\mathbf{Z}| + b_n\left[-\left(\lambda+n\theta\right)^2 + |\mathbf{R}|\right] + \frac{1}{2}b_{n+1}|\mathbf{Z}| \right\} e^{i(\lambda+n\theta)t} = 0$$

(29)

This is a homogeneous set of equations, and to get a nontrivial solution the determinant is set to zero. This then specifies the characteristic value λ for a given set of material and geometry properties of the beam, $\mathbf{R}$ and $\mathbf{Z}$. With λ so determined then b_n in terms of b_0 can be determined. Finally b_0 can be determined from the initial conditions at $t=0$. The first three term approximation is used for investigating the motion of the beam subjected to varying time load with loading frequency θ. Approximate solution of the system with

just the three terms, which leads to the set of equations 30.

$$\begin{bmatrix} -(\lambda-\theta)^2+|\mathbf{R}| & \frac{1}{2}|\mathbf{Z}| & 0 \\ \frac{1}{2}|\mathbf{Z}| & -\lambda^2+|\mathbf{R}| & \frac{1}{2}|\mathbf{Z}| \\ 0 & \frac{1}{2}|\mathbf{Z}| & -(\lambda+\theta)^2+|\mathbf{R}| \end{bmatrix} \begin{Bmatrix} b_{-1} \\ b_0 \\ b_1 \end{Bmatrix} = 0$$

(30)

For real value of λ and imposing the initial condition that $\mathbf{U}(t=0)=\mathbf{U}_0$ gives:

$$b_0 = \mathbf{U}_0 / \left[\frac{-\frac{1}{2}|\mathbf{Z}|b_0}{\left[-(\lambda-\theta)^2+|\mathbf{R}|\right]} + 1 + \frac{-\frac{1}{2}|\mathbf{Z}|b_0}{\left[-(\lambda+\theta)^2+|\mathbf{R}|\right]} \right]$$

(31)

and final solution becomes:

$$\mathbf{U}_t = b_0 \left\{ \frac{-\frac{1}{2}|\mathbf{Z}|b_0}{\left[-(\lambda-\theta)^2+|\mathbf{R}|\right]} e^{i(\lambda-\theta)t} + e^{i\lambda t} + \frac{-\frac{1}{2}|\mathbf{Z}|b_0}{\left[-(\lambda+\theta)^2+|\mathbf{R}|\right]} e^{i(\lambda+\theta)t} \right\}$$

(32)

Now the presented formulation is examined to find the response of the nonconservative cross ply laminated beam in the regions of dynamic instability and stability. The material and geometry properties are same as defined in Table-1. Three points from Fig.5 are chosen to investigate the response of the middle of a simply supported beam:

1- *Stable state*: the response of the system for the first point in stable region with nondimensional parameters $\theta/2\omega_n = 1.75$, $P_t/P_{cr} = 0.8$ is calculated and plotted in Fig. 6a. Also the periodic loading is depicted to compare the frequency of the system to loading frequency. As it can be seen the response of the beam is pure periodic and follow the loading frequency history.

2- *Asymptotically stable or dynamically critical*: the response of the system for the second point is on the curve of instability region with nondimensional parameters $\theta/2\omega_n = 0.75$, $P_t/P_{cr} = 0.8$ is calculated and plotted in Fig.6b. As it can be seen the response of the beam is aperiodic and does not follow the loading frequency history.

3- *Unstable state*: the response of the system for the third point in instable region with nondimensional

parameters $\theta/2\omega_n = 1$, $P_t/P_{cr} = 0.8$ is calculated and plotted in Fig. 6c. The values of the characteristic λ are complex in this region and leading to unstable solution. As it can be seen the displacement shows an increasing due to the compressive periodic load, which is %80 of the lowest critical load. Another fact that it is obvious from the response curve, the beam frequency is higher than the loading frequency. It is clear that load parameters carrying the structure in unstable state is unreliable and hazardous and causes the structure failure. For this reason structure designer try to eliminate the instability of the structure with load control and adding damping in the structure. Response of the forced system in unstable region depends on the excitation parameters and signature varies due to these parameters values. For example the amplitudes of the beam corresponded to substantial excitation loading parameters $P/P_{cr} \gg 1$ increase in a typical nonlinear manner accompanied by beats as shown in Fig. 6d.

With damping in account, the equation of motion of the beam Eq.(27) becomes;

$$\ddot{\mathbf{U}}_t + \mathbf{C}\dot{\mathbf{U}}_t + \{\mathbf{R} - \mathbf{Z}\cos\theta t\}\mathbf{U}_t = 0$$

(33)

The first three terms of approximate solution leads to the set of equations

$$\begin{bmatrix} -(\lambda-\theta)^2-(\lambda-\theta)+|\mathbf{R}|+|\mathbf{C}| & \frac{1}{2}|\mathbf{Z}| & 0 \\ \frac{1}{2}|\mathbf{Z}| & -\lambda^2-\lambda+|\mathbf{R}|+|\mathbf{C}| & \frac{1}{2}|\mathbf{Z}| \\ 0 & \frac{1}{2}|\mathbf{Z}| & -(\lambda+\theta)^2-(\lambda+\theta)+|\mathbf{R}|+|\mathbf{C}| \end{bmatrix} \begin{Bmatrix} b_{-1} \\ b_0 \\ b_1 \end{Bmatrix} = 0$$

(34)

The response of the beam for small amount of damping and large excitation changes as shown in Fig.7a and for large and very large amount of damping amplitude of the vibration will be back to zero quicker as shown in Fig.7b and 7c. This fact can be determined that using appropriate damping in structures subjected to periodic loading can reduce the violation and unpredictable motion of the system in instability state.

VIII. CONCLUSION

In this paper, dynamic stability analysis of laminated composite beams under varying time loading was studied. The equations of motion were established based on the dynamic version of virtual work principle formulation. A five node twenty degrees of freedom beam model was developed to descritize the governing equations. This model considers axial bending, stretching, bending-stretching and twisting couplings for general lay-ups and for the most used cross ply lamina. The matrix form of the equations of motion was solved using

the symbolic computation to determine the principal instability regions of the beam
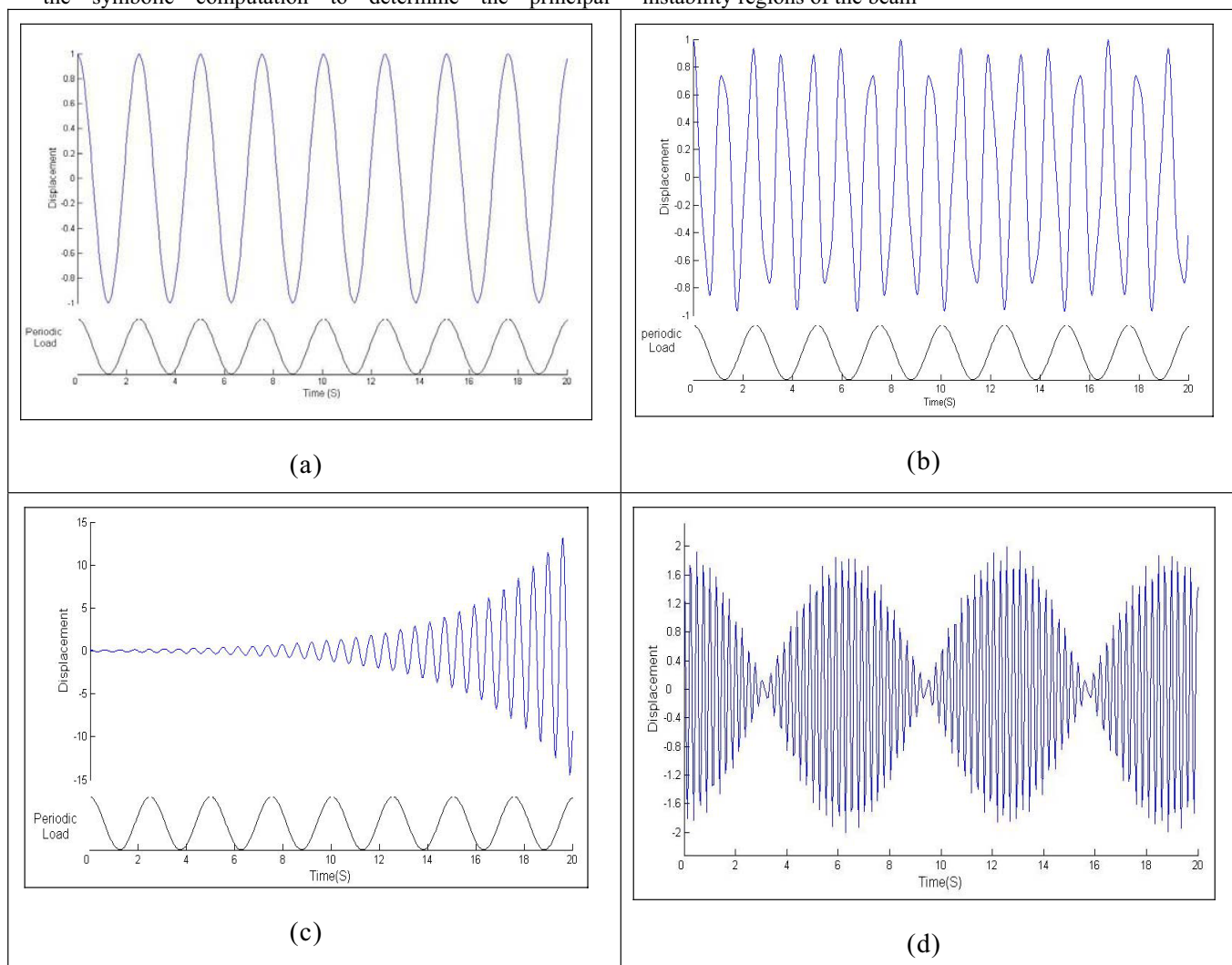

(a)

(b)

(c)

(d)

Fig. 8 Responses of a cross ply simply supported laminated composite beam subjected to a periodic loading
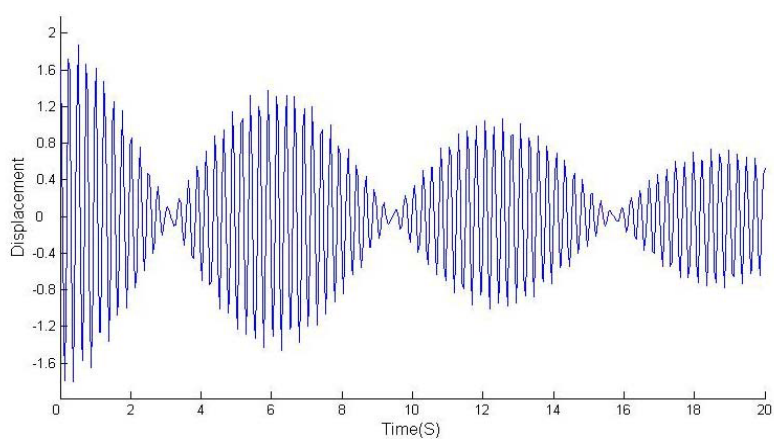

Fig. 7 (a) Response of a cross ply simply supported laminated composite beam subjected to a large periodic loading in unstable region with small damping ratio
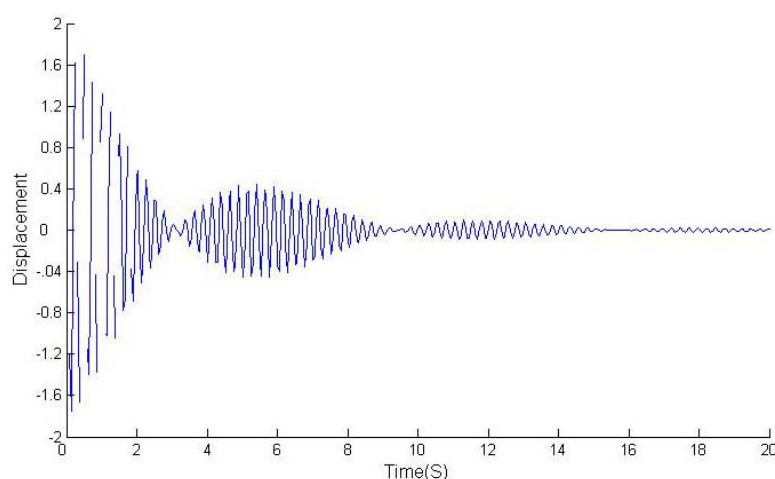
Fig. 7 (b) Response of a cross ply simply supported laminated composite beam subjected to a large periodic loading in unstable region with large damping ratio
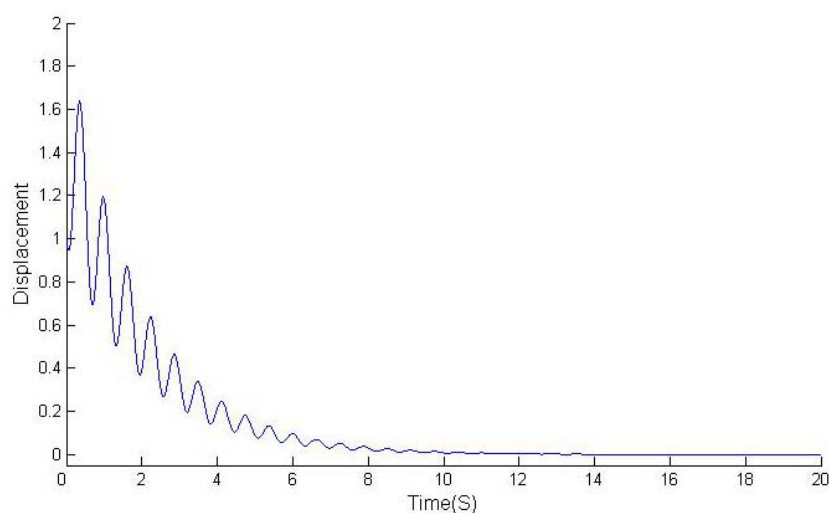

Fig. 7 (b) Response of a cross ply simply supported laminated composite beam subjected to a large periodic loading in unstable region with very large damping ratio

subjected to conservative and nonconservative loading with and without damping in account. The results show the important roles of the damping in decreasing the dynamic instability regions and eliminating perturbed behavior of the system. The insertion of transverse shear deformation results in higher amplitudes of the response curves. The stationary amplitude response curves have a right-hand overhang, which is generally due to geometrical non-linearities. Considering shear deformation results in a considerable decrease in size of the stability regions and in a shift of instability zones towards lower excitation frequencies. The numerical results are compared with those of the other studies and the finite elements available in the literature. The comparison shows that the developed finite element model predicts the natural frequencies of the beams better than the other models and the results are obtained with less computation time and faster convergence.

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