International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:1, No:9, 2007

Approximate Solution of Nonlinear Fredholm Integral Equations of the First Kind via Converting to Optimization Problems

Akbar H. Borzabadi, and Omid S. Fard

Abstract**—** In this paper we introduce an approach via optimization methods to find approximate solutions for nonlinear Fredholm integral equations of the first kind. To this purpose, we consider two stages of approximation. First we convert the integral equation to a moment problem and then we modify the new problem to two classes of optimization problems, non-constraint optimization problems and optimal control problems. Finally numerical examples is proposed.

AMS Subject Classification : 45B05, 49N90.
Keywords and phrases : Fredholm integral equation, Optimization method, Optimal control, Nonlinear and linear programming.

I. Introduction

It seems that the idea of finding approximate solutions for some problems by converting them to an optimization problem will give rise to interesting results. In [5], [6] and [1], one can find the applications of this idea to find approximate solution for some ordinary and partial differential equations and some classes of integral equations, respectively, via converting these problems to an optimal control problem. The problem of finding numerical solution for integral equation is one of the oldest problems in applied mathematics and many computational methods are proposed in this area, (see [2], [3] and [4]). The standard methods for solving integral equations deal with the linear Fredholm integral equations [3]. To solve the problem will be hard where we have a nonlinear kind, as follows:

$$y(t) = \int_a^b k(s,t,x(s))ds, \quad \text{a.e. on } [a,b]. \qquad (1)$$

A.H. Borzabadi is with the Department of mathematics, Damghan University of Basic Sciences, Damghan, Iran,(e-mail: borzabadi@dubs.ac.ir).
O.S. Fard is with the Department of mathematics, Damghan University of Basic Sciences, Damghan, Iran,(e-mail: osfard@dubs.ac.ir).

where $y(\cdot) \in C([a,b])$. The conditions of existence and uniqueness of solution for the above problem can be seen in [7]. Here we assume that the integral equations which is considered have a solution. In the next section we show that we can convert the integral equation (1) to a moment problem.

II. Defining the New Problem

Suppose that $\mathcal{P}_n$ be a partition of $[a,b]$ which is obtained from dividing this interval by n subinterval with equal lengths, i.e.

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b,$$

$$t_{i+1} - t_i = h, \ \ i = 0, 1, \cdots, n-1,$$

corresponding to ith node of partition $\mathcal{P}_n$, t_i, we set

$$y(t_i) = \int_a^b k(s,t_i,x(s))ds, \quad i = 0, 1, \cdots, n. \qquad (2)$$

Now we consider a new problem which we call it moment problem and that is to obtain the function $x_n(s)$ on $[a,b]$ corresponding to partition $\mathcal{P}_n$, whose satisfies in $n+1$ equation (2) and we call this function solution of moment problem. To be taken note that corresponding to any partition $\mathcal{P}_n$ we may have a solution for moment problem. In the following proposition we will show that with some conditions, the solution of moment problem will be a good approximation of solution (1).

Proposition 1: Suppose
$i)$ $k(s,t,x(s)) \in C([a,b] \times [a,b] \times \mathbb{R})$,
$ii)$ ∇k exists and it is a bounded vector function on $[a,b] \times [a,b] \times \mathbb{R}$,
then if $\{x_n(s)\}$ be a uniformly convergent sequence of moment problem solutions to function $x^*(s)$ then the function $x^*(s)$ is solution of (1).

Proof: Consider $t \in [a,b]$. By successive refinement of partitions on $[a,b]$ we can find a sequence $\{t_n\}$ whose converges to t. Suppose $\{x_n(s)\}$ be a uniformly convergent sequence which is obtained from solution of moment problem with every refined partition. Define

$$f_n(s) = k(s, t_n, x_n(s)), \ \forall s \in [a,b].$$

Trivially $f_n(s) \in C([a,b])$. For given $\epsilon > 0$ and for each $s \in [a,b]$ we have

$$\begin{aligned}|f_m(s) - f_n(s)| &= |k(s,t_m,x_m(s) - k(s,t_n,x_n(s)| \\ &\leq |k(s,t_m,x_m(s) - k(s,t,x^*(s)| \\ &+|k(s,t,x^*(s) - k(s,t_n,x_n(s)| \\ &\leq |(\nabla k)_2(s,\xi_m,x_m(s)|(t-t_m)| \\ &+|(\nabla k)_3(s,t_m,\eta_m)|(x_m(s)-x^*(s))| \\ &+|(\nabla k)_2(s,\xi_n,x_n(s)|(t-t_n)|+ \\ &|(\nabla k)_3(s,t_n,\eta_n)|(x_n(s)-x^*(s))| \\ &\leq M(|t_n - t| + |(x_n(s)-x^*(s))| \\ &+|t_m - t| + |(x_m(s)-x^*(s))|)\end{aligned}$$

where M is a large positive number such that $\|\nabla k\|_2 \leq M$. Now for $\epsilon' > 0$, there is a positive integer number N such that for each $l > N$

$$|t_l - t| < \epsilon' \ \& \ |x_l(s) - x^*(s)| < \epsilon'.$$

Thus for given $\epsilon > 0$ and the above N, for each $m, n \geq N$ we have

$$|f_m(s) - f_n(s)| < 4M\epsilon',$$

and thus by choosing $\epsilon' = \frac{\epsilon}{4M}$ we have

$$|f_m(s) - f_n(s)| < \epsilon,$$

that is the sequence $\{f_n\}$ is uniformly convergence, (see [8]), and thus $f_n \to f$ where $f(s) = k(s,t,x^*(s))$. Now we can result

$$y(t) = \lim_{n\to\infty} y(t_n) = \lim_{n\to\infty} \int_a^b f_n(s)ds$$

$$= \int_a^b \lim_{n\to\infty} f_n(s)ds = \int_a^b f(s)ds = \int_a^b k(s,t,x^*(s))ds,$$

and it means that $\{x_n(s)\}$ converges to solution of (1).□

III. Modifying the Moment Problem to a Non-Constraint Optimization Problem

In general, for each partition $\mathcal{P}_n$ of $[a,b]$ the solution of moment problem (2) may not exist. Even if the solution of moment problem exists, it may be difficult to characterize it. Thus we try to approximate a solution for moment problem by converting it to an optimization problem. First we define

$$g_i(s) = k(s, t_i, x(s)), \ \forall s \in [a,b], \tag{3}$$

and then we use a numerical method of integration as Newton-Cotes and we approximate the integral

$$I_i = \int_a^b g_i(s)ds,$$

by $\mathcal{I}_i = \sum_{j=0}^m w_j g_i(s_j)$, where the nodes $s_j, \ j = 0,1,\cdots,m$ are uniformly distributed in $[a,b]$ with $s_0 = a, s_n = b$ and spacing $\lambda = \frac{b-a}{m}$ and the weights $w_j, \ j = 0,1,\cdots,m$, are depend on the order of Newton-Cotes method which is applied. To be taken note that

$$g_i(s_j) = k(s_j, t_i, x(s_j)), \ i = 0,1,\cdots,n, \ j = 0,1,\cdots,m.$$

Now we consider the following optimization problem:

$$\min_{(x_0,x_1,\cdots,x_m)} J(x_0,x_1,\cdots,x_m) = M\sum_{i=0}^n |y_i - \sum_{j=0}^m w_j g_i(s_j)|, \tag{4}$$

where $x_j = x(s_j), \ j = 0,1,\cdots,m$, $y_i = y(t_i), \ i = 0,1,\cdots,n$ and M is a huge positive number and we consider it as penalty value. The best solution of problems (4) will give rise to vanish the functional J, thus by solving (4), we can obtain an approximation for solution of moment problem (2).

We propose an algorithm to obtain an approximate solution for original equation (1), by considering Proposition 1. First we define some notation which will be used in assertion of algorithm. we consider the optimal functional value of optimization problem (4) corresponding to $\mathcal{T}_n$ as J_n^*. and we suppose that

$$(x_{n0}^*, x_{n1}^*, \cdots, x_{nm}^*) = \arg\min_{(x_0,x_1,\cdots,x_m)} J_n^*,$$

and also $\mathcal{X}_n = \max_{0\leq i\leq m} |x_{ni}^*|$.

Initialization step:

Choose an error estimation ϵ and the nodes $s_j,\ j = 0, 1, \cdots, m$ which are uniformly distributed in $[a, b]$ with $s_0 = a, s_n = b$ and spacing $\lambda = \frac{b-a}{m}$. Determine weights $w_j,\ j = 0, 1, \cdots, m$ by choosing a Newton-Cotes method. Divide the interval $[a, b]$ into n subintervals $[t_{i-1}, t_i],\ i = 1, 2, \cdots, n$, each of length $h = \frac{b-a}{n}$ and also define $\mathcal{T}_n = \{t_0, t_1, \cdots, t_n\}$.

Main steps:

Step 1. Solve the problem (4) by nodes in $\mathcal{T}_n$, go to step 2.

Step 2. If $|\mathcal{X}_n - \mathcal{X}_{n-1}| < \epsilon$, stop. Otherwise go to step 3.

Step 3. Replace n by $n + 1$ and go to step 1.

Step 2 emphasizes on generation of a Cauchy sequence of approximate solutions to obtain approximate solution of problem (1).

Using Evolutionary algorithms, specially parallel genetic algorithm will increase the velocity of computations.

IV. Modifying the Moment Problem to an Optimal Control Problem

In this section we show, how we can convert problem to an optimal control problem. To this purpose we define an artificial control function $u(\cdot)$, which takes its values in a compact subset of $\mathbb{R}$, as U. We set

$$f_\circ(s, x(s), u(s)) = \sum_{i=0}^{n} |\frac{1}{b-a} y_i - k(s, t_i, x(s))|. \tag{5}$$

Now consider the following optimal control problem:

$$Minimize \quad \int_a^b f_\circ(s, x(s), u(s))ds, \tag{6}$$

$$subject\ to$$

$$\dot{x}(s) = u(s), \tag{7}$$

and the boundary conditions

$$x(a) = x_a \text{ and } x(b) = x_b, \text{ are unknown}. \tag{8}$$

One can show that if the target of the above optimal control problem vanishes then the equalities in (2) satisfies. A measure theoretical approach can be applied to solve above optimal control problem (see [1]). In fact in this manner using an embedding procedure, the problem of finding the optimal control is reduced to one consisting of the minimizing of a linear form over a set of positive measures. The resulting problem can be approximated by a finite dimensional linear programming problem. The solution of this problem is used to construct a nearly optimal control and approximate optimal control will deduce the approximate trajectory which is the approximate solution of moment problem.

V. Numerical Examples

In this section we propose our method to obtain approximate solution of some Fredholm integral equations. Before proposing of examples we define an error function as $e(t) = x^*(t) - \hat{x}(t)$ on $[a, b]$, where $x^*(t)$ and $\hat{x}(t)$ are exact and approximate solutions of the given integral equation in every example, respectively.

Example 1: In this example we apply our methods for a Fredholm integral equation of first kind as follows:

$$y(t) = \int_0^1 x(s)e^{st}ds.$$

where

$$y(t) = \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2},\ t \in [1, 2]$$

The above integral equation has analytical solution $x^*(s) = s$ on $[0, 1]$. In the Fig.1, the left diagram shows the approximate solution by applying given algorithm in Sec. 3, $\hat{x}(\cdot)$ with exact solution of problem, $x^*(\cdot)$ and right diagram shows the error function $e(s)$. In the Fig. 2, the left diagram shows approximate trajectory $\hat{x}(t)$, that is an approximate of integral equation solution and the function $x^*(s) = s$ as exact solution of integral equation. Also the right figure shows approximate optimal control. In the Fig. 3, the error function of approximate solution in Fig.2 and exact solution is given.

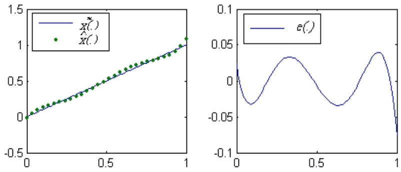

Fig.1 The approximate solution and error function

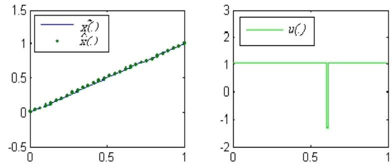

Fig.2 The approximate solution and the approximate optimal control

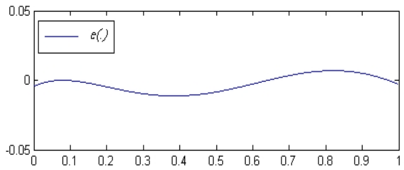

Fig. 3 The error function of approximate solution in Fig. 2

Example 2: In this example we apply our methods for a nonlinear Fredholm integral equation of first kind as follows:

$$y(t) = \int_0^1 s e^{x(s)t} ds.$$

where

$$y(t) = \frac{1}{2(t+1)}(e^{t+1} - 1), \ t \in [0, 1]$$

The above integral equation has an analytical solution as $x^*(s) = s^2$ on $[0, 1]$. In the Fig.4, the left diagram shows the approximate solution which is obtained by applying given algorithm in Sec. 3, $\hat{x}(\cdot)$ with exact solution of problem, $x^*(\cdot)$ and right diagram shows the error function $e(s)$. In the Fig. 5, the left diagram shows approximate trajectory $\hat{x}(t)$, that is an approximate of integral equation solution and the function $x^*(s) = s^2$ as one of exact solution of integral equation. Also the right figure shows approximate optimal control. In the Fig. 6, the error function of approximate solution in Fig.5 and exact solution is given.

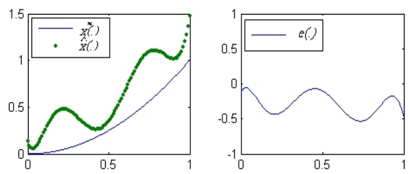

Fig. 4The approximate solution and error function

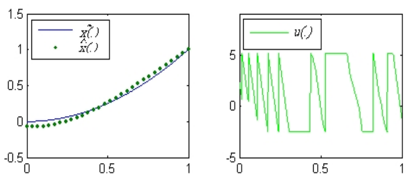

Fig. 5 The approximate solution and the approximate optimal control

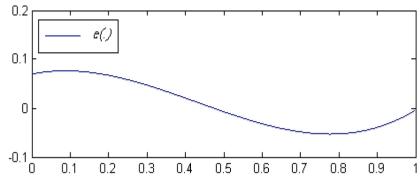

Fig. 6 The error function of approximate solution in Fig. 5.

REFERENCES

[1] A. H. Borzabadi, A. V. Kamyad, and H. H. Mehne, ”A different approach for solving the nonlinear Fredholm integral equations of the second kind”, *Applied Mathematics and Computation*, No. 173, 724735,2006.

[2] L. M. Delves, ”A fast method for the solution of fredholm integral equations”, *J. Inst. Math. Appl.*, Vol. 20, 173-182, 1977.

[3] L. M. Delves and J. L. Mohamed, ”*Computational Methods for Integral Equations*”, Cambridge Univ. Press, 1985.

[4] L. M. Delves and J. Walsh, ”*Numerical Solution of Integral Equations*”, Oxford Univ. Press, 1974.

[5] S. Effati and A. V. Kamyad, ”Solution of boundary value problems for linear second orderODE’s by using measure theory”, *J. Analysis*, Vol. 6, pp.139-149, 1998.

[6] M. Gachpazan, A. Kerachian and A. V. Kamyad, ”A new Method for solving nonlinear second order differential equations”, *Korean J. Comput. Appl. Math.*, Vol. 7, No. 2, pp. 333-345, 2000.

[7] A. J. Jerri, ”*Introduction to Integral Equations with Applications*”, London: Wiley, 1999.

[8] W. Rudin, ”*Principles of Mathematical Analysis*”, 3rd ed, McGraw-Hill, New York, 1976.