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Some results on Interval-valued fuzzy BG-algebras

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Abstract- In this note the notion of interval-valued fuzzy BG-algebras (briefly, i-v fuzzy BG-algebras), the level and strong level BG-subalgebra is introduced. Then we state and prove some theorems which determine the relationship between these notions and BG-subalgebras. The images and inverse images of i-v fuzzy BG-subalgebras are defined, and how the homomorphic images and inverse images of i-v fuzzy BG-subalgebra becomes i-v fuzzy BG-algebras are studied.

Keywords- BG-algebra, fuzzy BG-subalgebra, interval-valued fuzzy set, interval-valued fuzzy BG-subalgebra.

I. Introduction

In 1966, Y. Imai and K. Iseki [5] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [9] J. Neggers and H. S. Kim introduced the notion of d-algebras, which is generalization of BCK-algebras and investigated relation between d-algebras and BCK-algebras. Also they introduced the notion of B-algebras [8]. In [6] C. B. Kim, H. S. Kim introduced the notion of BG-algebras which is a generalization of B-algebras. S. S. Ahn and H. D. Lee applied the fuzzy notions to BG-algebras and introduced the notions of fuzzy BG-algebras [1]. The concept of a fuzzy set, which was introduced in [11].

In [12], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set (i.e., a fuzzy set with an interval-valued membership function). This interval-valued fuzzy set is referred to as an i-v fuzzy set, also he constructed a method of approximate inference using his i-v fuzzy sets. Biswas [2], defined interval-valued fuzzy subgroups and S. M. Hong et. al. applied the notion of interval-valued fuzzy to BCI-algebras.

In the present paper, we using the notion of inteval-valued fuzzy set by Zadeh and introduced the concept of interval-valued fuzzy BG-subalgebras (briefly i-v fuzzy BG-subalgebras) of a BG-algebra, and study some of their properties. We prove that every BG-subalgbera of a BG-algebra X can be relized as an i-v level BG-subalgebra of an i-v fuzzy BG-sublagebra of X, then we obtain some related results which have been mentioned in the abstract.

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II. Preliminary

Definition 2.1. [6] A BG-algebra is a non-empty set X with a consonant 0 and a binary operation $*$ satisfying the following axioms:

(I) $x * x = 0$,
(II) $x * 0 = x$,
(III) $(x * y) * (0 * y) = x$,
for all $x, y \in X$.

For brevity we also call X a BG-algebra. In X we can define a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 0$.

Theorem 2.2. [6] In a BG-algebra X, we have the following properties:

(i) $0 * (0 * x) = x$,
(ii) if $x * y = 0$, then $x = y$,
(iii) if $0 * x = 0 * y$, then $x = y$,
(iv) $(x * (0 * x)) * x = x$,
For all $x, y \in X$.

A non-empty subset I of a BG-algebra X is called a subalgebra of X if $x * y \in I$ for any $x, y \in I$.

A mapping $f : X \longrightarrow Y$ of BG-algebras is called a BG-homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

We now review some fuzzy logic concept (see [11]). Let X be a set. A fuzzy set A in X is characterized by a membership function $\mu_A : X \longrightarrow [0, 1]$. Let f be a mapping from the set X to the set Y and let B be a fuzzy set in Y with membership function μ_B. The inverse image of B, denoted $f^{-1}(B)$, is the fuzzy set in X with membership function $\mu_{f^{-1}(B)}$ defined by $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ for all $x \in X$.

Conversely, let A be a fuzzy set in X with membership function μ_A. Then the image of A, denoted by $f(A)$, is the fuzzy set in Y such that:

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} \mu_A(z) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

A fuzzy set A in the BG-algebra X with the membership function μ_A is said to be have the $\sup$ property if for any subset $T \subseteq X$ there exists $x_0 \in T$ such that $\mu_A(x_0) = \sup\limits_{t \in T} \mu_A(t)$.

An interval-valued fuzzy set (briefly, i-v fuzzy set) A defined on X is given by $A = \{(x, [\mu_A^L(x), \mu_A^U(x)])\}, \ \forall x \in X$. Briefly, denoted by $A = [\mu_A^L, \mu_A^U]$ where μ_A^L and μ_A^U are any two fuzzy sets in X such that $\mu_A^L(x) \leq \mu_A^U(x)$ for all $x \in X$.

Let $\overline{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$, for all $x \in X$ and let $D[0, 1]$ denotes the family of all closed sub-intervals of $[0, 1]$. It is clear that if $\mu_A^L(x) = \mu_A^U(x) = c$, where $0 \leq c \leq 1$ then $\overline{\mu}_A(x) = [c, c]$ is in $D[0, 1]$. Thus $\overline{\mu}_A(x) \in D[0, 1]$, for all $x \in X$. Therefore the i-v fuzzy set A is given by $A = \{(x, \overline{\mu}_A(x))\}, \forall x \in X$ where $\overline{\mu}_A : X \longrightarrow D[0,1]$.

Now we define refined minimum (briefly, rmin) and order " $\leq$ " on elements $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$ of $D[0,1]$ as:

$$rmin(D_1, D_2) = [min\{a_1, a_2\}, min\{b_1, b_2\}]$$

$$D_1 \leq D_2 \Longleftrightarrow a_1 \leq a_2 \quad \wedge \quad b_1 \leq b_2$$

Similarly we can define $\geq$ and $=$.

Definition 2.3. [1] Let μ be a fuzzy set in a BG-algebra. Then μ is called a fuzzy BG-subalgebra (BG-algebra) of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

Proposition 2.4. [3] Let f be a BG-homomorphism from X into Y and G be a fuzzy BG-subalgebra of Y with the membership function μ_G. Then the inverse image $f^{-1}(G)$ of G is a fuzzy BG-subalgebra of X.

Proposition 2.5. [3] Let f be a BG-homomorphism from X onto Y and D be a fuzzy BG-subalgebra of X with the $\sup$ property. Then the image $f(D)$ of D is a fuzzy BG-subalgebra of Y.

III. Interval-valued Fuzzy BG-algebra

From now on X is a BG-algebra, unless otherwise is stated.

Definition 3.1. An i-v fuzzy set A in X is called an interval-valued fuzzy BG-subalgebras (briefly i-v fuzzy BG-subalgebra) of X if:

$$\overline{\mu}_A(x * y) \geq rmin\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}$$

for all $x, y \in X$.

Example 3.2. Let $X = \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then X is a BG-algebra. Define $\overline{\mu}_A$ as:

$$\overline{\mu}_A(x) = \begin{cases} [0.3, 0.9] & \text{if } x \in \{0, 2\} \\ [0.1, 0.6] & \text{Otherwise} \end{cases}$$

It is easy to check that A is an i-v fuzzy BG-subalgebra of X.

Lemma 3.3. If A is an i-v fuzzy BG-subalgebra of X, then for all $x \in X$

$$\overline{\mu}_A(0) \geq \overline{\mu}_A(x).$$

Proof. For all $x \in X$, we have

$$\begin{aligned} \overline{\mu}_A(0) &= \overline{\mu}_A(x * x) \geq rmin\{\overline{\mu}_A(x), \overline{\mu}_A(x)\} \\ &= rmin\{[\mu_A^L(x), \mu_A^U(x)], [\mu_A^L(x), \mu_A^U(x)]\} \\ &= [\mu_A^L(x), \mu_A^U(x)] = \overline{\mu}_A(x). \end{aligned}$$

Theorem 3.4. Let A be an i-v fuzzy BG-subalgebra of X. If there exists a sequence $\{x_n\}$ in X, such that $\lim_{n\to\infty} \overline{\mu}_A(x_n) = [1, 1]$ Then $\overline{\mu}_A(0) = [1, 1]$.

Proof. By Lemma 3.3, we have $\overline{\mu}_A(0) \geq \overline{\mu}_A(x)$, for all $x \in X$, thus $\overline{\mu}_A(0) \geq \overline{\mu}_A(x_n)$, for every positive integer n. Consider

$$[1, 1] \geq \overline{\mu}_A(0) \geq \lim_{n\to\infty} \overline{\mu}_A(x_n) = [1, 1].$$

Hence $\overline{\mu}_A(0) = [1, 1]$.

Theorem 3.5. An i-v fuzzy set $A = [\mu_A^L, \mu_A^U]$ in X is an i-v fuzzy BG-subalgebra of X if and only if μ_A^L and μ_A^U are fuzzy BG-subalgebra of X.

Proof. Let μ_A^L and μ_A^U are fuzzy BG-subalgebra of X and $x, y \in X$, consider

$$\begin{aligned} \overline{\mu}_A(x * y) &= [\overline{\mu}_A(x * y), \overline{\mu}_A(x * y)] \\ &\geq [min\{\mu_A^L(x), \mu_A^L(y)\}, min\{\mu_A^U(x), \mu_A^U(y)\}] \\ &= rmin\{[\mu_A^L(x), \mu_A^U(x)], [\mu_A^L(y), \mu_A^U(y)]\} \\ &= rmin\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}. \end{aligned}$$

This completes the proof.

Conversely, suppose that A is an i-v fuzzyBG-subalgebras of X. For any $x, y \in X$ we have $[\mu_A^L(x * y), \mu_A^U(x * y)] = \overline{\mu}_A(x * y) \geq rmin\{\overline{\mu}_A(x), \overline{\mu}_A(y)\} = rmin\{[\mu_A^L(x), \mu_A^U(x)], [\mu_A^L(y), \mu_A^U(y)]\} = [min\{\mu_A^L(x), \mu_A^L(y)\}, min\{\mu_A^U(x), \mu_A^U(y)\}]$.

Therefore $\mu_A^L(x * y) \geq min\{\mu_A^L(x), \mu_A^L(y)\}$ and $\mu_A^U(x * y) \geq min\{\mu_A^U(x), \mu_A^U(y)\}$, hence we get that μ_A^L and μ_A^U are fuzzyBG-subalgebras of X.

Theorem 3.6. Let A_1 and A_2 are i-v fuzzy BG-subalgebras of X. Then $A_1 \cap A_2$ is an i-v fuzzy BG-subalgebras of X.

Corollary 3.7. Let $\{A_i | i \in \Lambda\}$ be a family of i-v fuzzy BG-subalgebras of X. Then $\bigcap_{i\in\Lambda} A_i$ is also an i-v fuzzy BG-subalgebras of X.

Definition 3.8. Let A be an i-v fuzzy set in X and $[\delta_1, \delta_2] \in D[0, 1]$. Then the i-v level BG-subalgebra $U(A; [\delta_1, \delta_2])$ of A and strong i-v BG-subalgebra $U(A; >, [\delta_1, \delta_2])$ of X are defined as following:

$$U(A; [\delta_1, \delta_2]) := \{x \in X \mid \overline{\mu}_A(x) \geq [\delta_1, \delta_2]\},$$

$$U(A;>,[\delta_1,\delta_2]) := \{x \in X \mid \overline{\mu}_A(x) > [\delta_1,\delta_2]\}.$$

Theorem 3.9. Let A be an i-v fuzzy BG-subalgebra of X and B be closure of image of μ_A. Then the following condition are equivalent :

(i) A is an i-v fuzzy BG-subalgebra of X.

(ii) For all $[\delta_1,\delta_2] \in Im(\mu_A)$, the nonempty level subset $U(A;[\delta_1,\delta_2])$ of A is a BG-subalgebra of X.

(iii) For all $[\delta_1,\delta_2] \in Im(\mu_A) \setminus B$, the nonempty strong level subset $U(A;>,[\delta_1,\delta_2])$ of A is a BG-subalgebra of X.

(iv) For all $[\delta_1,\delta_2] \in D[0,1]$, the nonempty strong level subset $U(A;>,[\delta_1,\delta_2])$ of A is a BG-subalgebra of X.

(v) For all $[\delta_1,\delta_2] \in D[0,1]$, the nonempty level subset $U(A;[\delta_1,\delta_2])$ of A is a BG-subalgebra of X.

Proof. (i $\longrightarrow$ iv) Let A be an i-v fuzzy BG-subalgebra of X, $[\delta_1,\delta_2] \in D[0,1]$ and $x,y \in U(A;<,[\delta_1,\delta_2])$, then we have $\overline{\mu}_A(x*y) \geq rmin\{\overline{\mu}_A(x),\overline{\mu}_A(y)\} > rmin\{[\delta_1,\delta_2],[\delta_1,\delta_2]\} = [\delta_1,\delta_2]$ thus $x*y \in U(A;>,[\delta_1,\delta_2])$. Hence $U(A;>,[\delta_1,\delta_2])$ is a BG-subalgebra of X.

(iv $\longrightarrow$ iii) It is clear.

(iii $\longrightarrow$ ii) Let $[\delta_1,\delta_2] \in Im(\mu_A)$. Then $U(A;[\delta_1,\delta_2])$ is a nonempty. Since $U(A;[\delta_1,\delta_2]) = \bigcap_{[\delta_1,\delta_2]>[\alpha_1,\alpha_2]} U(A;>,[\delta_1,\delta_2])$, where $[\alpha_1,\alpha_2] \in Im(\mu_A) \setminus B$. Then by (iii) and Corollary 3.8, $U(A;[\delta_1,\delta_2])$ is a BG-subalgebra of X.

(ii $\longrightarrow$ v) Let $[\delta_1,\delta_2] \in D[0,1]$ and $U(A;[\delta_1,\delta_2])$ be nonempty. Suppose $x,y \in U(A;[\delta_1,\delta_2])$. Let $[\beta_1,\beta_2] = min\{\mu_A(x),\mu_A(y)\}$, it is clear that $[\beta_1,\beta_2] = min\{\mu_A(x),\mu_A(y)\} \geq \{[\delta_1,\delta_2],[\delta_1,\delta_2]\} = [\delta_1,\delta_2]$. Thus $x,y \in U(A;[\beta_1,\beta_2])$ and $[\beta_1,\beta_2] \in Im(\mu_A)$, by (ii) $U(A;[\beta_1,\beta_2])$ is a BG-subalgebra of X, hence $x*y \in U(A;[\beta_1,\beta_2])$. Then we have $\overline{\mu}_A(x*y) \geq rmin\{\mu_A(x),\mu_A(y)\} \geq \{[\beta_1,\beta_2],[\beta_1,\beta_2]\} = [\beta_1,\beta_2] \geq [\delta_1,\delta_2]$. Therefore $x*y \in U(A;[\delta_1,\delta_2])$. Then $U(A;[\delta_1,\delta_2])$ is a BG-subalgebra of X.

(v $\longrightarrow$ i) Assume that the nonempty set $U(A;[\delta_1,\delta_2])$ is a BG-subalgebra of X, for every $[\delta_1,\delta_2] \in D[0,1]$. In contrary, let $x_0,y_0 \in X$ be such that

$$\overline{\mu}_A(x_0*y_0) < rmin\{\overline{\mu}_A(x_0),\overline{\mu}_A(y_0)\}.$$

Let $\overline{\mu}_A(x_0) = [\gamma_1,\gamma_2]$, $\overline{\mu}_A(y_0) = [\gamma_3,\gamma_4]$ and $\overline{\mu}_A(x_0*y_0) = [\delta_1,\delta_2]$. Then

$$[\delta_1,\delta_2] < rmin\{[\gamma_1,\gamma_2],[\gamma_3,\gamma_4]\} = [min\{\gamma_1,\gamma_3\},min\{\gamma_2,\gamma_4\}].$$

So $\delta_1 < min\{\gamma_1,\gamma_3\}$ and $\delta_2 < min\{\gamma_2,\gamma_4\}$.

Consider

$$[\lambda_1,\lambda_2] = \frac{1}{2}\overline{\mu}_A(x_0*y_0) + rmin\{\overline{\mu}_A(x_0),\overline{\mu}_A(y_0)\}$$

We get that

$$\begin{aligned}[\lambda_1,\lambda_2] &= \frac{1}{2}([\delta_1,\delta_2] + min\{\gamma_1,\gamma_3\}, min\{\gamma_2,\gamma_4\}]) \\ &= [\frac{1}{2}(\delta_1 + min\{\gamma_1,\gamma_3\}), \frac{1}{2}(\delta_2 + min\{\gamma_2,\gamma_4\})]\end{aligned}$$

Therefore

$$min\{\gamma_1,\gamma_3\} > \lambda_1 = \frac{1}{2}(\delta_1 + min\{\gamma_1,\gamma_3\}) > \delta_1$$

$$min\{\gamma_2,\gamma_4\} > \lambda_2 = \frac{1}{2}(\delta_2 + min\{\gamma_2,\gamma_4\}) > \delta_2$$

Hence

$$[min\{\gamma_1,\gamma_3\},min\{\gamma_2,\gamma_4\}] > [\lambda_1,\lambda_2] > [\delta_1,\delta_2] = \overline{\mu}_A(x_0*y_0)$$

so that $x_0*y_0 \notin U(A;[\delta_1,\delta_2])$

which is a contradiction, since

$$\overline{\mu}_A(x_0) = [\gamma_1,\gamma_2] \geq [min\{\gamma_1,\gamma_3\},min\{\gamma_2,\gamma_4\}] > [\lambda_1,\lambda_2]$$

$$\overline{\mu}_A(y_0) = [\gamma_3,\gamma_4] \geq [min\{\gamma_1,\gamma_3\},min\{\gamma_2,\gamma_4\}] > [\lambda_1,\lambda_2]$$

imply that $x_0,y_0 \in U(A;[\delta_1,\delta_2])$. Thus $\overline{\mu}_A(x*y) \geq rmin\{\overline{\mu}_A(x),\overline{\mu}_A(y)\}$ for all $x,y \in X$. Which completes the proof.

Theorem 3.10. Each BG-subalgebra of X is an i-v level BG-subalgebra of an i-v fuzzy BG-subalgebra of X.

Proof. Let Y be a BG-subalgebra of X, and A be an i-v fuzzy set on X defined by

$$\overline{\mu}_A(x) = \begin{cases} [\alpha_1,\alpha_2] & \text{if } x \in Y \\ [0,0] & \text{Otherwise} \end{cases}$$

where $\alpha_1,\alpha_2 \in [0,1]$ with $\alpha_1 <,\alpha_2$. It is clear that $U(A;[\alpha_1,\alpha_2]) = Y$. Let $x,y \in X$. We consider the following cases:

case 1) If $x,y \in Y$, then $x*y \in Y$ therefore $\overline{\mu}_A(x*y) = [\alpha_1,\alpha_2] = rmin\{[\alpha_1,\alpha_2],[\alpha_1,\alpha_2]\} = rmin\{\overline{\mu}_A(x),\overline{\mu}_A(y)\}$.

case 2) If $x,y \notin Y$, then $\overline{\mu}_A(x) = [0,0] = \overline{\mu}_A(y)$ and so $\overline{\mu}_A(x*y) \geq [0,0] = rmin\{[0,0],[0,0]\} = rmin\{\overline{\mu}_A(x),\overline{\mu}_A(y)\}$.

case 3) If $x \in Y$ and $y \notin Y$, then $\overline{\mu}_A(x) = [\alpha_1,\alpha_2]$ and $\overline{\mu}_A(y) = [0,0]$. Thus $\overline{\mu}_A(x*y) \geq [0,0] = rmin\{[\alpha_1,\alpha_2],[0,0]\} = rmin\{\overline{\mu}_A(x),\overline{\mu}_A(y)\}$.

case 4) If $y \in Y$ and $x \notin Y$, then by the same argument as in case 3, we can conclude that $\overline{\mu}_A(x*y) \geq rmin\{\overline{\mu}_A(x),\overline{\mu}_A(y)\}$.

Therefore A is an i-v fuzzy BG-subalgebra of X.

Theorem 3.11. If A is an i-v fuzzy BG-subalgebra of X, then the set

$$X_{\overline{\mu}_A} := \{x \in X \mid \overline{\mu}_A(x) = \overline{\mu}_A(0)\}$$

is a BG-subalgebra of X.

Definition 3.12. [2] Let f be a mapping from the set X into a set Y. Let B be an i-v fuzzy set in Y. Then the inverse image of B, denoted by $f^{-1}[B]$, is the i-v fuzzy set in X with the membership function given by $\overline{\mu}_{f^{-1}[B]}(x) = \overline{\mu}_B(f(x))$, for all $x \in X$.

Lemma 3.13. [2] Let f be a mapping from the set X into a set Y. Let $m = [m^L, m^U]$ and $n = [n^L, n^U]$ be i-v fuzzy sets in X and Y respectively. Then

(i) $f^{-1}(n) = [f^{-1}(n^L), f^{-1}(n^U)]$,
(ii) $f(m) = [f(m^L), f(m^U)]$,

Proposition 3.14. Let f be a BG-homomorphism from X into Y and G be an i-v fuzzy BG-subalgebra of Y with the membership function μ_G. Then the inverse image $f^{-1}[G]$ of G is an i-v fuzzy BG-subalgebra of X.

Proof. Since $B = [\mu_B^L, \mu_B^U]$ is an i-v fuzzy BG-subalgebra of Y, by Theorem 3.5, we get that μ_B^L and μ_B^U are fuzzy BG-subalgebra of Y. By Proposition 2.4, $f^{-1}[\mu_B^L]$ and $f^{-1}[\mu_B^U]$ are fuzzy BG-subalgebra of X, by above lemma and Theorem 3.5, we can conclude that $f^{-1}(B) = [f^{-1}(\mu_B^L), f^{-1}(\mu_B^U)]$ is an i-v fuzzy BG-subalgebra of X.

Definition 3.15. [2] Let f be a mapping from the set X into a set Y, and A be an i-v fuzzy set in X with membership function μ_A. Then the image of A, denoted by $f[A]$, is the i-v fuzzy set in Y with membership function defined by:

$$\overline{\mu}_{f[A]}(y) = \begin{cases} rsup_{z \in f^{-1}(y)} \overline{\mu}_A(z) & \text{if } f^{-1}(y) \neq \emptyset \\ [0,0] & \text{otherwise} \end{cases}$$

Where $f^{-1}(y) = \{x \mid f(x) = y\}$.

Theorem 3.16. Let f be a BG-homomorphism from X onto Y. If A is an i-v fuzzy BG-subalgebra of X, then the image $f[A]$ of A is an i-v fuzzy BG-subalgebra of Y.

Proof. Assume that A is an i-v fuzzy BG-subalgebra of X, then $A = [\mu_A^L, \mu_A^U]$ is an i-v fuzzy BG-subalgebra of X if and only if μ_B^L and μ_B^U are fuzzy BG-subalgebra of X. By Proposition 2.5, $f[\mu_A^L]$ and $f[\mu_A^U]$ are fuzzy BG-subalgebra of Y, by Lemma 3.13, and Theorem 3.5, we can conclude that $f[A] = [f[\mu_A^L], f[\mu_A^U]]$ is an i-v fuzzy BG-subalgebra of Y.

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